Volterra Integro-Differential Equations
and Infinite Systems
of Ordinary Differential Equations

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Abstract—We establish a connection between finite-dimensional systems of integro-differential
equations with the Hilbert-Schmidt kernel and ordinary differential equations in $\ell_2$ (countable systems
of differential equations). Such a reduction allows use of results obtained earlier for the countable
systems of differential equations in study of integro-differential equations. In particular, it can be
employed for study of stability and solutions' constructions for integro-differential equations. © 2005
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1. INTRODUCTION

About a century ago the notion of “aftereffect” was introduced in physics [1]. It was discovered
that to model processes with aftereffect, it is not sufficient to employ ordinary or partial differential
equations. An approach to resolve the problem was to use integral or integro-differential
equations, and, as well, equations with delay. One of the first to study the integro-differential
equations was Volterra. In 1913, he published Lectures on Integral and Integro-Differential Equations
[2]. Later, in 1931, in [3, Chapter 5], a classification of integro-differential equations is
provided. Among the equations are

(a) equations of oscillations of the wire:

$$m(t) - \mu \frac{d^2 w}{dt^2} = hw(t) + \int_0^t \varphi(t, \tau) w(\tau) \, d\tau,$$

(b) partial derivative equations of elliptic type occurring in questions of hereditary phenomena
in physics:

$$\Delta u(t) + \int_0^t \left[ \frac{\partial^2 u(\tau)}{\partial x^2} f(t, \tau) + \frac{\partial^2 u(\tau)}{\partial y^2} \varphi(t, \tau) + \frac{\partial^2 u(\tau)}{\partial z^2} \psi(t, \tau) \right] \, d\tau = 0,$$

$$\Delta u(t) = \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2}.$$
(c) partial derivative integro-differential equations of hyperbolic type:
\[
\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{\partial u(z,t)}{\partial z^2} + \int_0^t \frac{\partial^2 u(z,t)}{\partial z^2} \psi(t,\tau) \, d\tau,
\]

(d) equations of parabolic type:
\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \int_{t_0}^t A(t,\tau) \frac{\partial^2 u(x,\tau)}{\partial x^2} \, d\tau = 0.
\]

In the quoted paper, it is also marked out that application of the Fourier method to equations of type (c) and (d) leads to study of equations of type (a) in case (c), and in the case of (d) it is necessary to consider equations of the type
\[
\frac{dx}{dt} = -k^2 x - \int_{t_0}^t k^2 A(t,\tau) x(\tau) \, d\tau.
\]

As yet, another example of integro-differential equation is the Gurtin-Pipkin type energy balance equation with phase relaxation equation (see [4,5]):
\[
\begin{align*}
\frac{d}{dt} u'(t, x) + \frac{\ell}{2} u'(t, x) &= \int_0^t K(t-s) u''(t-s) \, ds, \\
\frac{\tau}{2} y''(t, x) &= y(t, x) + \varepsilon^2 y''(t, s) + u(t, x) - \varepsilon y^3(t, x),
\end{align*}
\]
\[x \in [0, u], \quad t \in [0, \infty).\]

Here \( u \) represents the temperature, \( y \) the nonconserved phase variable, \( K \) a kernel connected with thermal diffusivity, \( \tau \) the relaxation time, \( \xi \) typical interaction distance, \( \varepsilon \) a parameter.

Integral PDEs appear in mathematical models of many processes, see, e.g., [6]. A possibility of reducing of problems in dynamics of viscoelastic systems to systems of integro-differential equations was demonstrated, e.g., in [7]. There it was shown that the averaging method can be applied in this situation. The averaging method for integro-differential equations was further developed in [8,9].

In many problems described by integro-differential equations in partial derivatives an important role is devoted to study of ordinary integro-differential equations
\[
\frac{dx}{dt} = X \left( t, x, \mu, \int_0^t \varphi(t, s, x(s)) \, ds \right),
\]
which are, of course, of independent interest (see, e.g., equations (a),(e)). Study of equations of type (g), linear as well as nonlinear, attracted a lot of attention recently. Let us describe just a method of reduction for systems of type (g). Let us have, in (g),
\[
\varphi(t, s, x(s)) = K(t, s) g(s, x(s)).
\]

If the kernel \( K(t, s) \) can be represented as
\[
K(t, s) = \sum_{j=1}^n F_j(t) G_j(s),
\]
then the kernel is called degenerate.
In [10–12], a reduction method was developed such that we were able to reduce an integro-differential equation to a finite-dimensional system of ordinary differential equations (of higher dimension). It allowed us to solve for integro-differential equations problems of stability, oscillation and bifurcation. In [13,14], a method of reduction is used along with the Fourier method and applied to the Gurtin-Pipkin model (equation (f)).

In the current paper, we develop the reduction method for systems of type (g) and (h) without assumption on the kernel to be degenerate. We establish a rigorous connection between finite-dimensional integro-differential systems with the Hilbert-Schmidt kernel and ordinary differential equations in $\ell_2$ (countable systems of differential equations). Such a reduction allows use of results obtained earlier for the countable systems of differential equations (see, e.g., [15–17]) in study of integro-differential equations. In particular, it can be employed for study of stability and solutions’ constructions for integro-differential equations.

Let us notice that though the description of the reduction method is given here for a system of type (g), it can be applied to a wider class of the systems. For instance, this can be done for systems not solvable with respect to the derivative,

$$A(t)\dot{x} = f\left(\mu, t, x, \int_0^t K(\mu, t, s)\xi(\mu, s, x(s))\,ds\right),$$

and for singularly perturbed systems of integro-differential equations,

$$\mu\dot{x} = f\left(\mu, t, x, \int_0^t K(\mu, t, s)\xi(\mu, s, x(s))\,ds\right).$$

The mentioned equations for $\mu = 0$ become integral ones. As well, the method can be applied to equations incorporating delay and integration terms,

$$\dot{x} = f\left(\mu, t, x(t-h(t)), \int_0^t K(\mu, t, s)\xi(\mu, s, x(s))\,ds\right).$$

In the last case, results from the theory of functional-differential equations (e.g., [18,19]) are relevant.

Let us also mention that different approaches to study of stability of linear and nonlinear integro-differential equations based on the direct Lyapunov method, Laplace transform, and integral estimates can be found in, e.g., [20–24].

2. PRELIMINARIES

We start with summarizing several results from the theory of integro-differential equations (IDE) and the theory of countable systems of ordinary differential equations (ODE). The main sources are [8,15,16], where many other references can be found.

2.1. IDE

Let us consider a system of equations in $\mathbb{R}^n$,

$$\frac{dx}{dt} = X\left(t, x, \int_0^t \varphi(t, s, x(s))\,ds\right),$$

with boundary condition

$$x(0) = x_0$$

and functions $X(t, x, y), x, y \in \mathbb{R}^n, \varphi(t, s, x)$. Let

$$Q : \{t \geq 0, \ s \geq 0, \ |x - x_0| \leq a, \ y \in \mathbb{R}^n\}. $$
THEOREM 1. (See [8].) Let $X(t, x, y)$ and $\varphi(t, s, x)$ be continuous on $Q$, and let over $Q$

(1) $\|X(t, x, y)\| \leq M, \|X(t, x', y') - X(t, x'', y'')\| \leq \lambda(t)\{\|x' - x''\| + \|y' - y''\|\}$,

(2) $\|\varphi(t, s, x') - \varphi(t, s, x'')\| \leq \mu(t, s)\|x' - x''\|$.

Then system (1),(2) possesses a unique solution. This solution is continuously differentiable on

$I = \{0 < t < a/M\}$.

It is known that if boundary Condition (2) has form

$$x(t_0) = x_0, \quad t_0 > 0,$$

then the solution to system (1)–(3) could be nonunique, and $t_0$ is a singular point for the system. Conditions are known [8] for the lack of singularity, i.e., when (1)–(3) has a unique solution for any $t_0$ satisfying $0 < t_0 < T \leq \infty$, and any given $x_0$ from a prescribed area $D \subseteq \mathbb{R}^n$. For instance, this is the case when the conditions of Theorem 1 are valid on

$$Q' = \{t \geq 0, s \geq 0, x \in \mathbb{R}^n, y \in \mathbb{R}^n\},$$

and, moreover,

$$q = \int_0^\infty \lambda(t) \left[1 + \int_0^t \mu(t, s) \, ds\right] \, dt < 1,$$

$$\int_0^\infty \left\|X\left(t, 0, \int_0^t \varphi(t, 0, 0) \, ds\right)\right\| \, dt \leq P < \infty.$$          \hfill (4) \hfill (5)

Let us consider in what follows (1) such that

$$\varphi(t, s, x(s)) = K(t, s)g(s, x(s)).$$

It is easy to modify Condition (2) of Theorem 1 for the case of (6). Indeed, if on $Q$ we have

$$\|K(t, s)\| \leq \mu_1(t, s),$$

$$\|g(s, x') - g(s, x'')\| \leq \mu_2(s)\|x' - x''\|,$$

then (2) holds true for $\mu(t, s) = \mu_1(t, s)\mu_2(t, s)$.  \hfill (6)

2.2. Countable Systems of ODE

Consider a countable system of ordinary differential equations

$$\frac{dx_s}{dt} = \omega_s(t, x_1, x_2, \ldots, x_n, \ldots),$$

where $\omega_s$ are defined on

$$\mathcal{H} = \{0 \leq t \leq r, \ |x_s| \leq R, \ s = 1, 2, \ldots\}.$$  \hfill (7)

Let the following conditions be valid on $\mathcal{H}$.

(1) The functions $\omega_s$ are continuous in $t$ at any point of $\mathcal{H}$.

(2) The functions $\omega_s$ satisfy

$$|\omega_s(t, x'_1, x'_2, \ldots) - \omega_s(t, x''_1, x''_2, \ldots)| \leq \alpha(t)\delta x,$$

where $\delta x = \sup|\delta_1, |x'_2 - x''_2|, \ldots|, \alpha(t)$ is continuous in $t$, $0 \leq t \leq r$.

(3) $|\omega_s(t, 0, 0, \ldots)| \leq \beta(t), \ s = 1, 2, \ldots$, where $\beta(t)$ is continuous in $t$, $0 \leq t \leq r$.  \hfill (8)
Conditions (1)-(3) yield that \( \omega_s \)'s also satisfy the following extra conditions.

(4) In any point of \( \mathcal{H} \)

\[
|\omega_s(t, x_1, x_2, \ldots)| \leq \alpha(t) \sup(|x_1|, |x_2|, \ldots) + \beta(t).
\]

(5) In any point of \( \mathcal{H} \) the functions \( \omega_s \) are continuous, i.e.,

\[
|\omega_s(t + \delta t, x_1 + \delta x_1, x_2 + \delta x_2, \ldots) - \omega_s(t, x_1, x_2, \ldots)| \to 0,
\]

whenever \( |\delta t| + |\delta x| \to 0 \) and \( |\delta x| = \sup(|\delta x_1|, |\delta x_2|, \ldots) \).

(6) If for \( t \in [0, r] \) the functions \( x_1(t), x_2(t), \ldots \) are equicontinuous and belong to \( \mathcal{H} \) (i.e., \( |x_s(t)| < R \)), then \( \omega_s(t, x_1(t), x_2(t), \ldots), s = 1, 2, \ldots \) are continuous.

The functions \( \omega_s(t, x_1, x_2, \ldots) \) always can be extended in such a way that Conditions (1)-(6) be fulfilled on

\[
\mathcal{H} = \{ |t| < \infty, x = \sup(|x_1|, |x_2|, \ldots) < \infty \}.
\]

For set

\[
y_s = \begin{cases} 
  x_s, & |x_s| \leq R, \\
  \frac{x_s}{|x_s|}, & |x_s| \geq R,
\end{cases}
\]

\[
f_s(t, x_1, x_2, \ldots) = \omega_s(t, y_1, y_2, \ldots), \quad s = 1, 2, \ldots,
\]

then the system

\[
\frac{dx_s}{dt} = f_s(t, x_1, x_2, \ldots), \quad s = 1, 2, \ldots
\]

satisfies Conditions (1)-(6) on \( \mathcal{H} \) and coincides with (7) on \( \mathcal{H} \).

**Theorem 2.** (See [15].) Let functions \( f_s \) satisfy Conditions (1)-(6) on \( \mathcal{H} \). Then for every point \( (t_0, x_1^0, x_2^0, \ldots) \) in \( \mathcal{H} \) there exists a unique solution to (8) passing through it. This solution is bounded, equicontinuous, and exists for every finite \( t \).

This theorem in a straightforward way yields that there exists a unique bounded, equicontinuous solution to (7) on \( \mathcal{H} \) passing through every, internal in \( \mathcal{H} \), point \( (t_0, x_1^0, x_2^0, \ldots) \).

Apart from bounded solutions, countable systems may possess as well unbounded ones.

**Example.** (See [15].) Let the linear system

\[
\frac{dx_s}{dt} = x_{s+1}
\]

satisfy Conditions (1)-(6) on \( \mathcal{H} \). Its unique bounded, equicontinuous solution is

\[
x_s(t) = x_s^0 + (t - t_0)x_s^0 + \frac{(t - t_0)^2}{2!} x_{s+2}^0 + \cdots.
\]

Set

\[
\psi(t) = e^{-1/(t-t_0)^2}, \quad \psi(t_0) = 0.
\]

Then

\[
x_1 = \psi(t), \quad x_2 = \psi''(t), \ldots
\]

provides a solution to (9) under boundary conditions

\[
x_1(t_0) = \psi'(t_0) = 0, \quad x_2(t_0) = \psi''(t_0) = 0, \ldots
\]

Moreover, the solution (11) is not bounded,

\[
\sup(|f'(t)|, |f''(t)|, \ldots) = \infty, \quad t \neq t_0,
\]

and the sum of (10) and (11) gives an unbounded solution to (9) passing through \( (t_0, x_1^0, x_2^0, \ldots) \).
Let us return to consideration of (7). Assume that the functions $\omega_s(t, x_1, x_2, \ldots)$ satisfy on $\mathcal{H}$ the strong Cauchy-Lipschitz conditions:

\begin{equation}
|\omega_s(t, x_1, x_2, \ldots, x_m, x'_{m+1}, \ldots, x''_{m+k}, \ldots) - \omega_s(t, x_1, x_2, \ldots, x_m, x'_{m+1}, \ldots, x''_{m+k}, \ldots)| \\
\leq \alpha(t) \varepsilon_s(m) \delta x,
\end{equation}

where $\delta x = \sup |x'_{m+1} - x'_{m+1}|, \ldots, \alpha(t)$ is continuous, and $\varepsilon_s(m) \to 0$ when $m \to \infty$. The right-hand sides of (8) also satisfy (7) on $\mathcal{H}$.

Consider the punctured system

\begin{equation}
\frac{du_{sm}}{dt} = f_s(t, u_{1m}, \ldots, u_{mm}, 0, \ldots), \\
u_{sm}(t_0) = x^0_s, \quad s = 1, \ldots, m.
\end{equation}

**Theorem 3.** (See [15].) Let $f(t, x_1, x_2, \ldots)$ satisfy (1)-(7). Then the solutions of (8) and (12) satisfy

\begin{equation}
\lim_{m \to \infty} u_{sm}(t) = x_s(t), \quad s = 1, 2, \ldots,
\end{equation}

so that for any $g$ and $\varepsilon > 0$, there exists a number $N(g, \varepsilon)$ such that for $m > N(g, \varepsilon)$,

\begin{equation}
\max_{t} |x_1(t) - u_{1m}(t)|, \ldots, |x_g(t) - u_{sm}(t)| < \varepsilon.
\end{equation}

Consider a homogeneous linear countable system

\begin{equation}
\frac{dx_s}{dt} = p_{s1} x_1 + p_{s2} x_2 + \cdots + p_{sn} x_n + \cdots, \quad s = 1, 2, \ldots.
\end{equation}

Let $\sigma$ be a segment $[a, b]$, or semi-interval $[0, \infty)$, or interval $(-\infty, \infty)$, and let $p_{sn}(t)$ satisfy

(a) $p_{sn}(t)$ are continuous and

\begin{equation}
p_s(t) = \sum_{n=1}^{\infty} |p_{sn}(t)| \leq \alpha(t),
\end{equation}

where $\alpha(t)$ is continuous.

Condition (a) guarantees that (1)-(6) hold true on $H_1: t \in \sigma, \sup |x_1|, |x_2|, \ldots < \infty$.

Let the norm of the solution be

\begin{equation}
x(t) = \sup |x_1(t)|, |x_2(t)|, \ldots.
\end{equation}

**Theorem 4.** Under Condition (a) there is a unique solution to (13) passing through any point $(t_0, x^0_1, x^0_2, \ldots)$ of $H_1$. This solution is bounded, equicontinuous and exists for all $t \in \sigma$. The norm $x(t)$ of the solution for any $t \in \sigma$ satisfies

\begin{equation}
x(t_0) \exp \left\{ - \int_{t_0}^{t} \alpha(\tau) \, d\tau \right\} \leq x(t) \leq x(t_0) \exp \left\{ \int_{t_0}^{t} \alpha(\tau) \, d\tau \right\}.
\end{equation}

Estimate (14) allows introducing the notion of characteristic number of a solution

\begin{equation}
x_1(t), x_2(t), \ldots
\end{equation}

to (13).

Let $\sigma = [0, \infty)$, and assume that under Condition (a), the function $\alpha(t)$ for every $t \geq 0$ satisfies

\begin{equation}
\frac{1}{t} \int_{t_0}^{t} \alpha(\tau) \, d\tau \leq \alpha < \infty.
\end{equation}

Let $\lambda$ be the characteristic (in Lyapunov sense) number of the function $x(t)$, the norm of the solution (15). In what follows, we term the characteristic number of the solution (15) the characteristic number of its norm.
THEOREM 5. (See [15].) Modulo condition (16), the characteristic number \( \lambda \) of any nonzero solution to system (13) is finite, and \( \lambda \in [-\alpha, \alpha] \).

Clearly, if \( \lambda > 0 \), \( x(t) \to 0 \) when \( t \to \infty \). If \( \lambda < 0 \), \( x(t) \) is unbounded when \( t \) grows. The set of the characteristic numbers of the solutions constitutes the spectrum of (13). There are options for the spectrum to be finite, countable and uncountable. For instance, the spectrum of (9) is \([-1, 1]\).

THEOREM 6. (See [15].) If the spectrum of (13) does not intersect interval \((-\infty, \alpha]\), then there exists a finite number \( B \) such that the norm of any solution (15) to (13) satisfies

\[
x(t) \leq x(0)Be^{-t\alpha}.
\]

Notice that the last inequality could be written in a more general form:

\[
x(t) \leq x(t_0)De^{-r(t-t_0)}, \quad t \geq 0,
\]

where it is possible to set

\[
D = Be^{(\alpha-r)t_0}.
\]

The value of \( D \) in (18) depends in general on \( t_0 \). In [15], it is shown that if the coefficients of \( p_{sk}(t) \) are constant or periodic functions in \( t \) with common period, then \( D \) may be chosen as independent of \( t_0 \). This fact is essentially used in the study of stability and uniform stability. We state now a result about uniform stability in the first approximation.

THEOREM 7. Let system (1) be defined on \( H_1 \), and satisfy Conditions (1),(2). Moreover, let

\[
(3') \omega_s(t,0,0,\ldots) = 0, \quad s = 1, 2, \ldots,
\]

Assume that

\[
\omega_s(t,x_1,x_2,\ldots) = \sum_{j=1}^{\infty} p_{sj}x_j + L_s(t,x_1,x_2,\ldots),
\]

where the function \( L_s(t,x_1,x_2,\ldots) \) satisfies

\[
|L_s(t,x_1,x_2,\ldots)| \leq \gamma(x),
\]

and \( \gamma(x) \to 0 \) when \( x \to 0 \), \( x = \sup\{|x_1|, |x_2|, \ldots\} \). For the solution \( x_1 = x_2 = \cdots = 0 \) to the system (7),(19) to be uniformly stable in the first approximation, with arbitrary higher-order terms satisfying (20), it is necessary and sufficient that the solutions to the first approximation system (13) satisfy

\[
x(t) \leq x_0Be^{-\alpha(t-t_0)},
\]

where \( B \geq 1, \alpha > 0 \), are constants independent of \( x_0 \) and \( t_0 \). Moreover, the stability is asymptotical here. If the coefficients are either not dependent on \( t \) or are periodical functions in \( t \), it is necessary and sufficient that there exists \( \alpha_0 > 0 \) such that the spectrum of (13) does not intersect \((-\infty, \alpha_0]\).

3. REDUCTION OF IDE TO COUNTABLE SYSTEMS OF ODE

Consider a system of IDE in \( \mathbb{R}^n \)

\[
\frac{dx}{dt} = X \left( t, x, \int_0^t K(t,s)g(s,x(s)) \, ds \right), \quad x(0) = x_0.
\]

Assume that the kernel \( K(t,s) \) can be represented as

\[
K(t,s) = \sum_{j=1}^{\infty} C_j(t)K_j(t,s), \quad K_j(t,s) = F_j(t)G_j(s),
\]
where \( C_j(t), F_j(t), G_j(t) \) are \( n \times n \) matrices. Let \( c_{kj}^{(j)}(t) \) denote the elements of a matrix \( C_j(t) \). Assume that
\[
\|C(t)\| = \sup_{k} \sum_{j=1}^{\infty} \sum_{t=1}^{n} |c_{kj}^{(j)}(t)| \leq \alpha(t) < \infty, \tag{24}
\]
where \( C(t) = (C_1(t), C_2(t), \ldots) \) is an infinite block row matrix.

Let us show that the system of IDE (22),(23) is possible to reduce to a countable system of ODE. Assume that the matrices \( F_j(t), j = 1, 2, \ldots, \) are invertible. Let
\[
K_j(t, s) = F_j(t)F_j^{-1}(s)K_j(s, s), \quad K_j(s, s) = F_j(s)G_j(s). \tag{25}
\]
Set
\[
P_j(t) = \frac{dF_j(t)}{dt}F_j^{-1}(t), \tag{26}
\]
and consider the following system of ODE in \( \mathbb{R}^n \):
\[
\frac{dz_j}{dt} = P_j(t)z_j, \quad j = 1, 2, \ldots. \tag{27}
\]
It is known that if \( Z_j(t) \) is an integral matrix of the system (27), then it satisfies the matrix system
\[
\frac{dZ_j}{dt} = P(t)Z_j, \tag{28}
\]
and any solution to system (28) can be represented as
\[
Z_j(t) = Z_j(t, t_0)C_j, \tag{29}
\]
where \( Z_j(t, t_0) \) is the matriciant of the system (28), i.e., \( Z_j(t_0, t_0) = E_n, \) \( E_n \) being the identity \( n \times n \) matrix, and \( C_j \) is a constant \( n \times n \) matrix. The integral matrix \( Z_j(t) \) has a multiplicative derivative
\[
D_t Z_j(t) = \frac{dZ_j(t)}{dt}Z_j^{-1}(t). \tag{30}
\]
Moreover,
\[
D_t Z_j(t) = P_j(t). \tag{31}
\]
For a nonhomogeneous system with the same \( P_j(t) \),
\[
\frac{dz_j}{dt} = P_j(t)z_j + f_j(t), \tag{31}
\]
we have the following solution to the Cauchy problem:
\[
z_j(t) = Z_j(t, t_0)z_j(t_0) + \int_{t_0}^{t} Z_j(t, t_0)Z_j^{-1}(s, t_0)f_j(s) \, ds. \tag{32}
\]
At this stage, let us introduce new variables
\[
y_j(t) = \int_{0}^{t} K_j(t, s)g(s, x(s)) \, ds \tag{33}
\]
\[
y_j(t) = \int_{0}^{t} F_j(t)F_j^{-1}(s)g_j(s, x(s)) \, ds, \quad y_j(0) = 0,
\]
where \( g_j(s, x(s)) = K_j(s, s)g(s, x(s)) \).

From (32) with \( t_0 = 0, \) (26) and (25), it follows that the vector-functions \( y_j(t) \) satisfy the following system of ODE:
\[
\frac{dy_j}{dt} = P_j(t)y_j + g_j(t, x(t)), \tag{34}
\]
with boundary condition \( y(0) = 0 \). This way, the system of IDE (22),(23) is put in correspondence to the following countable system of ODE:

\[
\begin{align*}
\frac{dx}{dt} &= X \left( t, x, \sum_{j=1}^{\infty} C_j(t)y_j \right), \\
\frac{dy_j}{dt} &= \Phi_j(t)y_j + g_j(t, x), \quad j = 1, 2, \ldots,
\end{align*}
\]  

where \( x(0) = x_0, \ y_j(0) = 0, \ j = 1, 2, \ldots \) and all variables \( x, y_j, j = 1, 2, \ldots \), are \( n \)-dimensional vectors.

From the method of construction of (35) and results of Section 2 follows that if \((x(t), y_1(t), \ldots, y_j(t), \ldots)\) is a bounded, equicontinuous solution to (35), then \(x(t)\) is the solution of the IDE system (22),(23). The announced relation between the systems (22),(23) and (35) reduces study of systems of IDE in \( \mathbb{R}^n \) to study of countable systems of ODE. The “puncturing” theorem (see Section 1) demonstrates also that for construction of solutions to the IDE system (22),(23), it is possible to use methods known for finite-dimensional systems of ODE.

Consider a linear IDE equation \((x \in \mathbb{R}^n)\)

\[
\frac{dx}{dt} = A(t)x + \int_0^t K(t, s)x(s) \, ds
\]  

with kernel (23). In this case, system (35) becomes a linear countable system of ODE,

\[
\begin{align*}
\frac{dx}{dt} &= A(t)x + \sum_{j=1}^{\infty} C_j(t)y_j, \\
\frac{dy_j}{dt} &= P_j(t)y_j + B_j(t)x, \quad x(0) = x_0, \ y_j(0) = 0,
\end{align*}
\]  

where

\[
P_j(t) = \frac{dF_j(t)}{dt}F_j^{-1}(t), \quad B_j(t) = F_j(t)G_j(t), \quad j = 1, 2, \ldots.
\]

Countable system (37) corresponds to the following infinite block matrix:

\[
R(t) = 
\begin{pmatrix}
A(t) & C_1(t) & C_2(t) & \ldots & C_j(t) & C_{j+1}(t) & \ldots \\
B_1(t) & P_1(t) & 0 & \ldots & 0 & 0 & \ldots \\
B_2(t) & 0 & P_2(t) & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
B_j(t) & 0 & 0 & \ldots & P_j(t) & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]  

Every block in the matrix is an \( n \times n \) matrix. All the rows corresponding to the variables \( y_j, \ j = 1, 2, \ldots, \) contain only a finite number of nonzero entries.

Let \( R(t) = (r_{ij}(t)), i, j = 1, 2, \ldots \) and

\[
\|R(t)\| = \sup_k \sum_{\ell=1}^{\infty} |r_{k\ell}(t)|.
\]

Assume that on \( H_1 \),

\[
H_1 := \{ t \in \sigma, \sup \{ \|x\|, \|y_1\|, \|y_2\|, \ldots \} < \infty \},
\]

all the functions \( r_{k\ell}(t) \) are continuous, and there exists a continuous function \( \alpha(t) \) such that on \( \sigma \)

\[
\|R(t)\| \leq \alpha(t) < \infty.
\]  

(39)
In particular, it follows then that (24) is also valid. Thus, under condition of uniform boundedness of all the norms of the finite $n \times n$ matrices, $\|A(t)\|, \|B_j(t)\|, \|P_j(t)\|, j = 1, 2, \ldots$, condition (39) holds true (perhaps with another function $\alpha(t)$). The structure of matrices $P_j$ and $B_j$ yields that for (39) to be valid, is sufficient to require that (24) is satisfied and the kernel elements $F_j(t), G_j(t), F_j^{-1}(t)$, $\frac{df_j}{dt}, j = 1, 2, \ldots$ are uniformly bounded on $\sigma$.

From the convergence of series (23), it also follows that the right-hand side of (37) satisfies also the strong Cauchy-Lipschitz condition (see (7) of Section 2).

All aforementioned allows us to apply to (37) the "puncturing" theorem, which can be put in the following form.

Let us introduce the following matrices for $m = 1, 2, \ldots$,

$$R_m(t) = \begin{pmatrix}
A(t) & C_1(t) & C_2(t) & \ldots & C_m(t) \\
B_1(t) & P_1(t) & 0 & \ldots & 0 \\
B_2(t) & 0 & P_2(t) & \ldots & 0 \\
\vdots & \ldots & \ldots & \ldots & \ldots \\
B_m(t) & 0 & 0 & \ldots & P_m(t)
\end{pmatrix}, \quad (40)$$

and the following systems of equations:

$$\frac{du_m}{dt} = R_m(t)u_m, \quad u_m = \begin{pmatrix}
x^{(m)}(t) \\
y_1^{(m)}(t) \\
y_2^{(m)}(t) \\
\vdots \\
y_m^{(m)}(t)
\end{pmatrix}, \quad u_m(0) = u_m^{(0)}. \quad (41)$$

**Theorem 8.** Under condition (39) in $H_1$, the solutions to system (37), (41) satisfy

$$\lim_{m \to \infty} x^{(m)}(t) = x(t),$$

$$\lim_{m \to \infty} y_s^{(m)}(t) = y_s(t), \quad s = 1, 2, \ldots, \quad (42)$$

where $x^{(m)}(0) = x_0, y_s^{(m)}(0) = y_s^{(0)}$.

Theorem 8 yields that the solution to a system of IDE, (36) can be approximated on $\sigma$ if the boundary conditions

$$x^{(m)}(0) = x_0, \quad y_s(0) = 0, \quad s = 1, 2, \ldots,$$

are chosen in solving (41).

Let us require additionally that $\sigma$ is defined on $H_1$ as $\sigma = [0, \infty)$, and moreover assume that

$$\frac{1}{t} \int_0^t \alpha(\tau) d\tau \leq \alpha < \infty. \quad (43)$$

Then if (43) is satisfied, every solution has characteristic number $\lambda$. Let $\Sigma$ stand for the spectrum of (36).

**Theorem 9.** Let conditions (39) and (43) be satisfied. If the spectrum $\Sigma$ does not intersect $(-\infty, r]$ where $r > 0$, then the zero solution to (37), and, therefore, IDE (36) are asymptotically stable.

4. LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS IN $\ell_2$

The previous results are based on the reduction method and employed theorems on countable systems of differential equations obtained by Persidsky et al. [15]. In this section, we describe the reduction method for IDE systems in conjunction with the theory of completely continuous operators in $\ell_2$.

Let us restrict ourselves to the system of IDE equations

$$\frac{dx}{dt} = Ax + \int_0^t K(t, s)x(s) ds \quad (44)$$
under assumption that the kernel \( K(t, s) \) can be represented as follows:

\[
K(t, s) = \sum_{j=1}^{\infty} c_j K_j(t, s), \quad K_j(t, s) = F_j(t)G_j(s).
\]  

(45)

Let us assume that

(a) the matrices \( A_j, B_j \), defined by

\[
B_j = K_j(s, s) = F_j(s)G_j(s), \quad A_j = \frac{dF_j(t)}{dt}F_j^{-1}(t)
\]

(46)

are invertible constant \( n \times n \) matrices;

(b) the matrices \( c_j \) are representable as

\[
c_j = \alpha_j \beta_j,
\]

where \( \alpha_j, \beta_j \) are invertible matrices, and

\[
\sum_{j=1}^{\infty} \left( \sum_{k=1}^{n} (\alpha_{j,k})^2 \right) < \infty, \quad k = 1, 2, \ldots, n;
\]

\[
\sum_{j=1}^{\infty} \left( \sum_{k=1}^{n} (\beta_{j,k})^2 \right) < \infty, \quad k = 1, 2, \ldots, n,
\]

(47)

where \( \alpha_{j,k}, \beta_{j,k} \) are entries of the matrices \( \alpha_j, \beta_j \).

Let us introduce variables (cp. to (33))

\[
y_j = \int_0^t \beta_j K(t, s)x(s) \, ds, \quad j = 1, 2, \ldots.
\]

(48)

By the reduction method, we obtain the following linear system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= Ax + \sum_{j=1}^{\infty} \alpha_j y_j, \\
\frac{dy_j}{dt} &= \beta_j P_j \beta_j^{-1} y_j + \beta_j B_j x, \quad y_j(0) = 0,
\end{align*}
\]

(49)

where

\[
P_j = \frac{dF_j}{dt}F_j^{-1}, \quad B_j = F_j G_j, \quad j = 1, 2, \ldots,
\]

with constant coefficients (see Condition (a)).

The following infinite block matrix (cp. with (38)) corresponds to (49):

\[
R = \begin{pmatrix}
A & \alpha_1 & \alpha_2 & \ldots & \alpha_j & \alpha_{j+1} & \ldots \\
\beta_1 B_1 & \beta_1 P_1 \beta_1^{-1} & 0 & \ldots & 0 & 0 & \ldots \\
\beta_2 B_2 & 0 & \beta_2 P_2 \beta_2^{-1} & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\beta_j B_j & 0 & 0 & \ldots & \beta_j P_j \beta_j^{-1} & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix},
\]

(50)

Let the following condition be valid:

\[
\|B_j\| < b < \infty, \quad j = 1, 2, \ldots,
\]

\[
\|P_j\| < p < \infty, \quad j = 1, 2, \ldots.
\]

(51)
It is easy to see that
\[
\sum_{k, \ell=1}^{\infty} r_{k, \ell}^2 < \infty, \tag{52}
\]
where \( r_{k, \ell} \) are entries of the matrix \( R \). Thus, the matrix operator \( R \) considered in \( \ell_2 \) under conditions \( \alpha, \beta, \gamma \) is a completely continuous operator. Searching for solutions of system (49) in the form
\[
z = z_0 e^{-\lambda t}, \quad z = (x, y_1, y_2, \ldots),
\]
we arrive at the classical problem about solving
\[
(R + \lambda I)z_0 = 0, \tag{53}
\]
where \( R \) is a completely continuous operator [25,26].

Since every completely continuous operator is the limit in norm of a sequence of finite-dimensional operators, we may use it in finding eigenvalues \( \lambda \) of \( R \) by the truncating method.

**Theorem 10.** Let system (44),(45) satisfy conditions (\( \alpha \)), (\( \beta \)), (\( \gamma \)). Then, if the operator \( R \) has a negative eigenvalue, then the zero solution of (44) is unstable.

**References**

1. Picard, 1907.