Stability of Stochastic Delay Hybrid Systems with Jumps

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This study investigates sufficient conditions for stability of delay jump diffusion processes in the sense of almost sure stability, stability in distribution, and exponential stability in mean square. The Lyapunov function method and the Razumikhin argument play an important role in this study.

Keywords: Almost sure stability, Exponential stability in mean square, Stability in distribution, Razumikhin argument, Poisson random measure

1. Introduction

The jump diffusion process has come to play an important role in many branches of science and industry. In their book [25], Øksendal and Sulem have studied the optimal control, optimal stopping and impulse control for jump diffusion processes. In mathematical finance theory, many researchers have developed option pricing theory, for example, Merton [24] was the first to use the jump process to describe the stock dynamics, Bardhan and Chao [2] were amongst the first authors to consider market completeness in a discontinuous model. The jump diffusion models have been discussed by Chan [6], Föllmer and Schweizer [9], El Karoui and Quenez [13], Henderson and Hobson [12], and Merculio and Runggaldier [23], to name a few.

In this study, we shall investigate the delay hybrid jump diffusion process. First, the importance of delay equations derives from the fact that many phenomena witnessed around us do not have an immediate effect from the moment of their occurrence. It is clear that to incorporate time delays in the feedback control problem is important and useful. For instance, scenarios arise from queuing networks, customers wait in a line before departing from the service; schedulers decide their random delay before they start to move [7]. Applications also arise in adaptive control of linear delay systems [8], control of movement for robot arms [1], and so on. To generalize the memoryless model for deterministic systems, efforts have been made to the development of functional differential equations; see [10]. From a control engineering point of view, time delays are common in practical systems and are often the cause of instability and/or poor performance. Moreover, it is usually difficult to obtain accurate values for the delay and conservative estimates often have to be used. The importance of time delay has already motivated several studies on the stability of switching diffusions with time delay; see, for example, [14, 18, 20, 26, 27] among others.

Second, since many physical systems are subject to frequent unpredictable structural changes, such as random failures, sudden environment disturbances, abrupt variation of the operating point on a nonlinear plant, and so on. Markovian switching systems are often used to describe such systems and are important systems. Loosely speaking, a Markovian switching system is a hybrid system with state vector that has two components $X(t)$ and $\alpha(t)$. The first one is in general referred to as the state, and the second one is regarded as the mode. In its operation, the system will switch from one mode to another in a random way, based on a Markov chain with finite state space.

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$S = \{1, 2, \ldots, N\}$. For example, option pricing has been considered in [15], where the underlying price process switches among a finite number of states. Finally, it is interesting and beneficial to treat differential delay systems with jumps under regime switching. The presence of the delay together with jumps and switching poses new challenges and difficulties.

Throughout this study, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $| \cdot |$ denote the Euclidean norm as well as the matrix trace norm. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^d)$ denote the family of continuous function $\phi$ from $[-\tau, 0]$ to $\mathbb{R}^d$ with the norm $\| \phi \| = \sup_{-\tau \leq \theta \leq 0} | \phi(\theta) |$. Denote by $C^\gamma([-\tau, 0]; \mathbb{R}^d)$ the family of all bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; \mathbb{R}^d)$-valued random variables. $L^1(\mathbb{R}_+; \mathbb{R}^d)$ denote the family of all function such that $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\int_0^\infty g(s)ds < \infty$. Let $B(t)$ be $m$-dimensional Brownian motion and $N(t, z)$ be a $n$-dimensional Poisson process and denote the compensated Poisson process by

$$\tilde{N}(dt, dz) = (\tilde{N}_1(dt, dz_1), \ldots, \tilde{N}_n(dt, dz_n))$$

$$= (N_1(dt, dz_1) - v_1(dz_1)dt, \ldots, N_n(dt, dz_n) - v_n(dz_2)dt),$$

where $\{N_j, j = 1, \ldots, n\}$ are independent $1$-dimensional Poisson random processes with characteristic measure $\{v_j, j = 1, \ldots, n\}$ coming from $n$ independent $1$-dimensional Poisson point processes.

Let $\alpha(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (q_{ij})_{N \times N}$ given by

$$P[\alpha(t + \Delta) = j | \alpha(t) = i] = \begin{cases} 
q_{ij}\Delta + o(\Delta): & \text{if } i \neq j \\
1 + q_{ii}\Delta + o(\Delta): & \text{if } i = j
\end{cases}$$

where $\Delta > 0$. Here $q_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ while

$$q_{ii} = -\sum_{j \neq i} q_{ij}.$$ 

We assume that the Markov chain $\alpha(\cdot)$ is independent of the Brownian motion $B(\cdot)$ and the Poisson process $N(t, z)$. It is well known that almost every sample path of $\alpha(t)$ is a right continuous step function.

Consider the $d$-dimensional stochastic delay hybrid system with jumps

$$dX(t) = b(X(t), X(t - \tau), t, \alpha(t))dt$$

$$+ \sigma(X(t), X(t - \tau), t, \alpha(t))dB(t)$$

$$+ \int_{\mathbb{R}^n} \gamma(X(t^-), X((t - \tau)^-), t, \alpha(t), z)\tilde{N}(dt, dz)$$

(1.1)

for $t \in [0, T]$ with initial data

$$[X(t) : -\tau \leq t \leq 0] = \{\xi(t) : -\tau \leq t \leq 0\} \in C^b([-\tau, 0]; \mathbb{R}^d), \alpha(0) = \ell_0,$$

where $X(t^-) = \lim_{t \rightarrow \tau^+} X(s), b : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times S \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times S \rightarrow \mathbb{R}^{d \times m}$, $\gamma : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times S \times \mathbb{R}^n \rightarrow \mathbb{R}^{d \times n}$. We note that each column $\gamma^{(k)}$ of the $d \times n$ matrix $\gamma = [\gamma_{ij}]$ depends on $z$ only through the $k$th coordinate $z_k$, i.e.

$$\gamma^{(k)}(x, y, t, \ell, z) = \gamma^{(k)}(x, y, t, \ell, z_k);$$

$$z = (z_1, \ldots, z_n) \in \mathbb{R}^n, \ell \in S,$$

we refer to [25] where this type of dependence is discussed and investigated for SDEs.

**Remark 1.1**: If we add the left limit to the drift and diffusion terms, then the equation is equivalent to the Eq. (1.1). That is the following equation

$$dX(t) = b(X(t^-), X((t - \tau)^-), t, \alpha(t))dt$$

$$+ \sigma(X(t^-), X((t - \tau)^-), t, \alpha(t))dB(t)$$

$$+ \int_{\mathbb{R}^n} \gamma(X(t^-), X((t - \tau)^-), t, \alpha(t), z)\tilde{N}(dt, dz)$$

is equivalent to the Eq. (1.1), since integration of both equations are the same. Moreover, the integral on the right hand side of (1.1) is just a shorthand matrix expression. Using the notation above, we can rewrite out in detail component $X_i(t), 1 \leq i \leq d$, in (1.1), that is

$$dX_i(t)$$

$$= b_i(X(t), X(t - \tau), t, \alpha(t))dt$$

$$+ \sum_{j=1}^m \sigma_{ij}(X(t), X(t - \tau), t, \alpha(t))dB_j(t)$$

$$+ \sum_{j=1}^n \int_{\mathbb{R}} \gamma_{ij}(X(t^-), X((t - \tau)^-), t, \alpha(t), z_j)\tilde{N}_j(dt, dz_j).$$

(1.2)
Remark 1.2: Let \( 0 = t_0 < t_1 < t_2 < \ldots \), and \( \theta(t) = i + 1, t \in [t_i, t_{i+1}) \). Let's consider the following hybrid delay systems (cf. [5]):

\[
\begin{align*}
dX(t) &= b(X(t^-), X((t - \tau)^-), t, \theta(t))dt \\
&\quad + \sigma(X(t^-), X((t - \tau)^-), t, \theta(t))dB(t), \\
&i \in [t_i, t_{i+1}), \\
X(t_{i+1}) &= J_{i+1}(X(t_i^-)), i = 0, 1, \ldots
\end{align*}
\]

Generally speaking, this process is discontinuous, there is a jump at each \( t_i, i \geq 1 \). However, if \( t_i, i \geq 1 \) are random, and the waiting time of jumps are independent, identical distributed as a negative exponential distribution, then the hybrid systems above should be in the form of Eq. (1.1).

One of the main purposes of this work is to obtain the sufficient criteria for almost sure stability (see [16] and [31]). Many studies assume that there is equilibrium position, that is, \( b(0, 0, t, \ell) \equiv 0, \sigma(0, 0, t, \ell) \equiv 0 \) and \( \gamma(0, 0, t, \ell, z) \equiv 0 \), so Eq. (1.1) has the solution \( X(t) \equiv 0 \) corresponding to the initial data \( \xi(s) = 0, -\tau \leq s \leq 0, \alpha(0) = \ell_0 \). This solution is called the trivial solution or equilibrium position. We shall investigate almost sure stability without the equilibrium position assumption in section 2. Moreover, the proof of Theorem 2.2 tells us how to obtain the exponential stability and polynomial stability. The second purpose is to discuss the stability in distribution for Eq. (1.1). For nondegenerate multidimensional diffusion processes without jump, Bhattacharya [4] and Has'minski [11] have given the sufficient conditions for positive recurrence, null recurrence, and existence of invariant measure. Furthermore, for a class of degenerate diffusion processes without jumps, sufficient conditions for stability in distribution and existence of invariant measure are proved in Basak and Bhattacharya [3]. These results are generalized by Zhu and Yin [32], Yuan and Mao [29]. Moreover, stability in distribution for delay diffusion processes without jump has been studied in [30]. Recently, Wee [28] obtained the sufficient condition of the stability in distribution for jump diffusion processes without delay. In this article, we shall extend the previous result to delay hybrid diffusion processes with jumps, the technical used here are different from [28]. Another purpose of this study is to investigate the exponential stability for Eq. (1.1) by using Razumikhin argument. We should point out that Razumikhin-type theorem for diffusion without jump was first introduce by Mao [19].

Given \( V \in C^2([\mathbb{R}^d \times \mathbb{R}_+ \times S; \mathbb{R}_+]) \), we define the operator \( \mathcal{L}V \) by

\[
\begin{align*}
\mathcal{L}V(x, y, t, \ell) &= V_t(x, t, \ell) + V_x(x, t, \ell)b(x, y, t, \ell) \\
&\quad + \frac{1}{2} \text{trace} \left[ \sigma^T(x, y, t, \ell)V_x(x, t, \ell)\sigma(x, y, t, \ell) \right] \\
&\quad + \int_{\mathbb{R}} \sum_{k=1}^n \left\{ V(x + \gamma_k(x, y, \ell, \ell), t, \ell) - V(x, t, \ell) - V_k(x, t, \ell, \ell) \gamma_k(x, y, \ell, \ell) \right\} \nu_k(dz_k) \\
&\quad + \sum_{j=1}^n q_{ij} V(x, t, j), \quad (1.3)
\end{align*}
\]

where

\[
\begin{align*}
V_t(x, t, \ell) &= \left( \frac{\partial V(x, t, \ell)}{\partial x_1}, \ldots, \frac{\partial V(x, t, \ell)}{\partial x_d} \right), \\
V_{xx}(x, t, \ell) &= \left( \frac{\partial^2 V(x, t, \ell)}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,d}.
\end{align*}
\]

For future use, we cite Itô’s formula (see [25] and [22]).

\[
V(X(t), t, \alpha(t)) = V(X(0), 0, \alpha(0)) \\
+ \int_0^t \mathcal{L}V(X(t), X(t - \tau), s, \alpha(s))ds \\
+ \int_0^t V_x(X(s), s, \alpha(s))\sigma(X(s), X(s - \tau), \\
&\quad s, \alpha(s))dB(s) \\
+ \sum_{k=1}^n \int_0^t \int_{\mathbb{R}} \left\{ V(X(s^-) + \gamma_k(s^-), X((s - \tau)^-), \\
&\quad s, \alpha(s), z_k), s, \alpha(s) \right\} \tilde{N}(ds, dz_k) \\
+ \int_0^t \int_{\mathbb{R}} \left\{ V(X(s^-), s, \ell_0 + h(\alpha(s), \ell)), - V(X(s^-), \\
&\quad s, \alpha(s))\mu(ds, d\ell), \quad (1.4)
\end{align*}
\]

the details of the function \( h \) and the measure \( \mu(ds, d\ell) \) see [22, p. 46–48].
We will need

**Assumption (L.L.)** (Local Lipschitz) for each $R \in \mathbb{N}$, there exists a constant $L_R$ such that

$$
|\sigma(x, y, t, \ell) - \sigma(\bar{x}, \bar{y}, t, \ell)|^2 + |b(x, y, t, \ell) - b(\bar{x}, \bar{y}, t, \ell)|^2 \\
+ \sum_{j=1}^d \sum_{j=1}^d \int_{\mathbb{R}} \gamma(j)(x, y, t, \ell, z_j) - \gamma(j)(\bar{x}, \bar{y}, t, \ell, z_j)^2 \nu_j(dz_j)
$$

$$
\leq L_R(|x - \bar{x}|^2 + |y - \bar{y}|^2),
$$

(1.5)

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$, and $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R, \ell \in S$. Moreover,

$$
\sup \left\{ |\sigma(0, 0, t, \ell)| \vee |b(0, 0, t, \ell)| \right\}
\leq \infty.
$$

(1.6)

This condition is called local Lipschitz condition. If $L_R$ is independent of $R$, it is called Lipschitz or global Lipschitz condition. There is a unique solution to Eq. (1.1) under global Lipschitz condition in infinite horizontal time (see [25]). In general, the Assumption (L.L.) will only guarantee a unique maximal local solution to Eq. (1.1) for any given initial data $\xi, \ell_0$. However, the addition conditions imposed in our main result, Theorem 2.2, will guarantee that this maximal local solution is in fact a unique global solution (see Lemma 2.3), which is denoted by $X(t; \xi, \ell_0)$ and $X(t; \xi, \ell_0)$, where these notations emphasize the dependence on the initial data $\xi, \ell_0$. It could be proved that $(X, \alpha(t))$ is a strong Markov process.

The rest of the study is arranged as follows: in section 2, we study almost sure stability for stochastic delay hybrid systems; in section 3, we investigate exponential stability in mean square; stability in distribution will be discussed in section 4; a short summary will be presented in the last section.

## 2. Almost Surely Stability

**Definition 2.1:** The solution of Eq. (1.1) is said to be almost surely asymptotically stable if for all $\xi \in C_{\mathcal{F}_0}^N([-\tau, 0]; \mathbb{R}^d), \ell_0 \in S$

$$
P \left( \lim_{t \to \infty} |X(t; \xi, \ell_0)| = 0 \right) = 1.
$$

**Theorem 2.2:** Under Assumption (L.L.), Assume that there are functions $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^d \times S; \mathbb{R}_+), g \in L^1(\mathbb{R}^d \times \mathbb{R}_+), w_1, w_2 \in C(\mathbb{R}^d \times \mathbb{R}_+)$ such that

$$
LV(x, y, t, \ell) \leq g(t) - w_1(x) + w_2(y),
$$

(2.1)

$$
\forall (x, y, t, \ell) \in \mathbb{R}^d \times \mathbb{R}_+ \times S, w_1(0) = w_2(0) = 0, w_1(x) > w_2(x), \forall x \neq 0
$$

(2.2)

and

$$
\lim_{|x| \to \infty} \inf_{0 \leq \tau < \infty \ell \in S} V(x, \tau, \ell) = \infty.
$$

(2.3)

Then the solution of Eq. (1.1) is almost surely asymptotically stable.

To prove this theorem, let us present two lemmas.

**Lemma 2.3:** Under the conditions of Theorem 2.2, for any initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C_{\mathcal{F}_0}^N([-\tau, 0]; \mathbb{R}^d), \ell_0 \in S$, Eq. (1.1) has a unique global solution.

**Proof:** The method used here is the standard truncated technique (see e.g. [20, p. 48–59]) so we only outline the proof. Fix any initial data $\xi$ and let $\Xi$ be the bound for $\xi$. For each integer $i \geq \Xi$, define

$$
b_i(x, y, t, \ell) = b_i \left( \frac{|x| \wedge i}{|x|}, \frac{|y| \wedge i}{|y|}, y, \ell \right).
$$

where we set $(|x| \wedge n)/|x| = 0$ when $x = 0$. Define $\sigma_i(x, y, t, \ell, z)$ and $\gamma_i(x, y, t, \ell)$ similarly. By (LL), we observe that $b_i(x, y, t, \ell), \sigma_i(x, y, t, \ell)$, and $\gamma_i(x, y, t, \ell)$ satisfy the global Lipschitz condition. Therefore, there exists a unique global solution $X_i(t)$ on $t \in [-\tau, \infty)$ to the equation

$$
dX_i(t) = b_i(X_i(t), X_i(t - \tau), t, \alpha(t))dt
+ \sigma_i(X_i(t), X_i(t - \tau), t, \alpha(t))d\beta(t)
+ \int_{\mathbb{R}^d} \gamma_i(X_i(t)^-), X_i(t - \tau^-), t, \alpha(t), z)\bar{N}(dt, dz)
$$

with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi, \ell_0 \in S$. Define the stopping time

$$
\rho_i = \inf \{t \geq 0 : |X_i(t)| \geq i\},
$$

where we set $\inf \emptyset = \infty$ as usual. It is not difficult to show that

$$
X_i(t) = X_{i+1}(t) \text{ if } 0 \leq t \leq \rho_i.
$$
This implies that $\rho_t$ is increasing in $t$. Let $\rho = \lim_{t \to \infty} \rho_t$. The property mentioned earlier also enables us to define $X(t)$ for $t \in [-\tau, \rho]$ as follows

$$X(t) = X_\tau(t) \quad \text{if} \quad -\tau \leq t \leq \rho_t.$$ 

It is clear that $X(t)$ is a unique solution to Eq. (1.1) for $t \in [-\tau, \rho)$. To complete the proof, we only need to show $P(\rho = \infty) = 1$. Using the Itô formula (1.4) and taking the expectation, we have that for any $t > 0$,

$$E V(X(t \wedge \rho_t), t \wedge \rho_t, \alpha(t \wedge \rho_t))$$

where operator $L^{(k)} \mathcal{V}$ is defined similarly as $L \mathcal{V}$ was defined by (1.3) except $b, \sigma$, and $\gamma$ there are replaced by $\tilde{b}, \tilde{\sigma}$, and $\tilde{\gamma}$, respectively. By the definitions of $\tilde{b}, \tilde{\sigma}$, and $\tilde{\gamma}$, we hence observe that

$$L^{(k)} \mathcal{V}(X(s), X(s - \tau), s, \alpha(s))$$

$$= \mathcal{L} \mathcal{V}(X(s), X(s - \tau), s, \alpha(s)) \quad \text{if} \quad 0 \leq s \leq t \wedge \rho_t.$$ 

Using (2.1) and (2.2), we then derive from (2.4) that

$$E V(X(t \wedge \rho_t), t \wedge \rho_t, \alpha(t \wedge \rho_t))$$

$$\leq E V(\xi(0), 0, \xi_0) + \int_0^t g(s)ds + E \int_{-\tau}^0 w_2(\xi(\theta))d\theta.$$ 

This yields

$$P(\rho \leq t)$$

$$\leq \frac{E V(\xi(0), 0, \xi_0) + \int_0^t g(s)ds + E \int_{-\tau}^0 w_2(\xi(\theta))d\theta}{\inf_{\xi_0 \geq 0, \xi(t) \in \mathcal{S}} V(x, t, \xi)}.$$ 

Letting $t \to \infty$ and using (2.3), we obtain $P(\rho \leq t) = 0$. Since $t$ is arbitrary, we must have

$$P(\rho = \infty) = 1.$$ 

The proof is therefore complete. \hfill \Box

We will also need the useful convergence theorem of nonnegative semimartingales (see, e.g., Lipster and Shiryaev [17]) which we cite here as a lemma.

**Lemma 2.4:** Let $A_1(t)$ and $A_2(t)$ be two adapted increasing processes on $t \geq 0$ with $A_1(0) = A_2(0) = 0$ a.s. Let $M(t)$ be a real-valued local martingale with $M(0) = 0$ a.s. Let $\zeta$ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $E\zeta < \infty$. Define

$$X(t) = \zeta + A_1(t) - A_2(t) + M(t) \quad \text{for} \quad t \geq 0.$$ 

If $X(t)$ is nonnegative, then

$$\lim_{t \to \infty} A_1(t) < \infty \quad \text{a.s.}$$

where $C \subset D$ a.s. means $P(C \cap D^c) = 0$. In particular, if

$$\lim_{t \to \infty} A_1(t) < \infty \quad \text{a.s., then, with probability one,}$$

$$\lim_{t \to \infty} X(t) < \infty \quad \text{a.s.}$$

and

$$-\infty < \lim_{t \to \infty} M(t) < \infty.$$ 

That is, all of the three processes $X(t), A_1(t),$ and $M(t)$ converge to finite random variables.

Let us now begin to prove our main result.

**Proof of Theorem 2.2:** We divide the proof into three steps.

**Step 1.** By Itô’s formula, if $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$, then for any $t \geq 0$

$$V(X(t), t, \alpha(t))$$

$$= V(\xi(0), 0, \xi_0) + \int_0^t \mathcal{L} V(X(s), X(s - \tau), s, \alpha(s))ds$$

$$+ \int_0^t V_\alpha(X(s), s, \alpha(s))\sigma(X(s), X(s - \tau), s, \alpha(s))dB(s)$$

$$+ \sum_{k=1}^n \int_0^t \int_{\mathcal{R}} [V(X(s))$$

$$+ \gamma^{(k)}(X(s), X(s - \tau), s, \alpha(s), z_k), s, \alpha(s))]$$

$$- V(X(s), s, \alpha(s))\tilde{N}_k(ds, dz_k)$$

$$+ \int_0^t \int_{\mathcal{R}} [V(X(s^-), s, \xi_0 + h(\alpha(s), \xi))]$$

$$- V(X(s^-), s, \alpha(s))\mu(ds, d\xi),$$

(2.5)
Using conditions (2.1) and (2.2) we derive that
\[
\int_0^t L V(X(s), X(s - \tau), s, \alpha(s)) ds \\
\leq \int_0^t g(s) ds + \int_{-\tau}^0 w_2(X(s)) ds \\
- \int_0^t (w_1(X(s)) - w_2(X(s))) ds.
\]
Therefore,
\[
V(X(t), t, \alpha(t)) \\
\leq V(\xi(0), 0, \ell_0) + \int_0^t g(s) ds + \int_{-\tau}^0 w_2(X(s)) ds \\
- \int_0^t (w_1(X(s)) - w_2(X(s))) ds \\
+ \int_0^t V_+(x(s), s, \alpha(s)) \sigma(X(s), X(s - \tau), s, \alpha(s)) d\mathcal{B}(s) \\
+ \sum_{k=1}^n \int_0^t \int_{\mathbb{R}} |V(X(s)) + \gamma(k)(X(s), X(s - \tau), s, \alpha(s), z_k, s, \alpha(s)) \\
- V(X(s), \alpha(s))) N_0(ds, d\zeta_k) \\
+ \int_0^t \int_{\mathbb{R}} [V(X(s^-), s, \ell_0 + h(\alpha(s), \ell)) \\
- V(X(s^-), s, \alpha(s)) \mu(ds, d\ell)].
\]
(2.6)
Applying Lemma 2.4 we immediately obtain that
\[
\lim_{t \to \infty} \sup_{0 \leq t < \infty} V(X(t), t, \alpha(t)) < \infty \quad \text{a.s.}
\] (2.7)
Moreover, taking the expectations on both sides of (2.6) and letting \( t \to \infty \), we obtain that
\[
E \int_0^\infty (w_1(X(s)) - w_2(X(s))) ds < \infty
\] (2.8)
This implies
\[
\int_0^\infty (w_1(X(s)) - w_2(X(s))) ds < \infty \quad \text{a.s.}
\] (2.9)
Step 2. Set \( w = w_1 - w_2 \). Clearly, \( w \in \mathcal{C}(\mathbb{R}^2; \mathbb{R}_+) \). It is straightforward to see from (2.9) that
\[
\lim_{t \to \infty} \inf_{0 \leq t < \infty} w(X(t)) = 0 \quad \text{a.s.}
\] (2.10)
We now claim that
\[
\lim_{t \to \infty} w(X(t)) = 0 \quad \text{a.s.}
\] (2.11)
If this is false, then
\[
P \left( \lim_{t \to \infty} \sup_{0 \leq t < \infty} w(X(t)) > 0 \right) > 0.
\]
Hence, there is a number \( \varepsilon > 0 \) such that
\[
P(\Omega_1) \geq 3\varepsilon,
\] (2.12)
where
\[
\Omega_1 = \left\{ \lim_{t \to \infty} \sup_{0 \leq t < \infty} w(X(t)) > 2\varepsilon \right\}.
\]
It is easy to observe from (2.7)
\[
\sup_{0 \leq t < \infty} V(X(t), t, \alpha(t)) < \infty \quad \text{a.s.}
\]
Define \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) by
\[
\rho(r) = \inf_{0 \leq t < \infty} V(x(t, \ell)).
\]
Obviously,
\[
\sup_{0 \leq t < \infty} \rho(|X(t)|) \leq \sup_{0 \leq t < \infty} V(X(t), t, \alpha(t)) < \infty \quad \text{a.s.}
\]
On the other hand, by (2.3) we have
\[
\lim_{t \to \infty} \rho(r) = \infty.
\]
Therefore,
\[
\sup_{0 \leq t < \infty} |X(t)| < \infty \quad \text{a.s.}
\] (2.13)
Recalling the boundedness of the initial data we can then find a positive number \( h \), which depends on \( \varepsilon \), such that large for \( |\xi(\theta)| < h \) for all \( -\tau \leq \theta \leq 0 \) almost surely while
\[
P(\Omega_2) \geq 1 - \varepsilon,
\] (2.14)
where
\[
\Omega_2 = \left\{ \sup_{-\tau \leq t < \infty} |X(t)| < h \right\}.
\]
It is easy to see from (2.12) and (2.14) that
\[
P(\Omega_1 \cap \Omega_2) \geq 2\varepsilon.
\] (2.15)
We now define a sequence of stopping times,
\[ 
\tau_h = \inf\{t \geq 0 : |X(t)| > h\}, \\
\theta_1 = \inf\{t \geq 0 : w(X(t)) > 2\varepsilon\}, \\
\theta_{2k} = \inf\{t \geq \theta_{2k-1} : w(X(t)) \leq \varepsilon\}, \quad k = 1, 2, \ldots, \\
\theta_{2k+1} = \inf\{t \geq \theta_{2k} : w(X(t)) > 2\varepsilon\}, \quad k = 1, 2, \ldots,
\]
where throughout this study we set \(\inf \emptyset = \infty\). Note from (2.10) and the definition of \(\Phi_1\) and \(\Phi_2\) that if \(\omega \in \Phi_1 \cap \Phi_2\), then
\[ 
\tau_h = \infty \quad \text{and} \quad \theta_k < \infty, \quad \forall k \geq 1. \tag{2.16}
\]
Let \(I_A\) be denoted as the indicator function of set \(A\), using the fact \(\theta_{2k} < \infty\) whenever \(\theta_{2k-1} < \infty\) and (2.8), we can get
\[
\begin{align*}
\infty > & E\int_0^{\infty} w(X(t))dt \\
\geq & \sum_{k=1}^{\infty} E \left[ I_{[\theta_{2k-1}, \infty]}(\theta_{2k} - \theta_{2k-1}) \right] \\
\geq & \varepsilon \sum_{k=1}^{\infty} E[I_{[\theta_{2k-1}, \infty]}(\theta_{2k} - \theta_{2k-1})]. \tag{2.17}
\end{align*}
\]
On the other hand, by Assumption (LL), there exists a constant \(K_h > 0\) such that
\[
|b(x, y, t, \ell)|^2 \vee |\sigma(x, y, t, \ell)|^2 \\
\vee \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} |f^{(k)}(x, y, t, \ell, z)|^2 \nu_k(dz) \leq K_h \tag{2.18}
\]
whenever \(|x| \vee |y| \leq h, \ell \in S\) and \(t \in \mathbb{R}_+\). Compute
\[
E \left[ I_{[\theta_{2k-1}, \infty]} \sup_{0 \leq t \leq T} |X(\tau_h \wedge (\theta_{2k-1} + t)) - X(\tau_h \wedge \theta_{2k-1})|^2 \right]
\]
By using Hölder inequality, Burkholder-Davis-Gundy inequality, Doob martingale inequality and Itô isometry, and (2.18), it is not easy to show
\[
E \left[ I_{[\theta_{2k-1}, \infty]} \sup_{0 \leq t \leq T} |X(\tau_h \wedge (\theta_{2k-1} + t)) \right.
\]
\[
- X(\tau_h \wedge \theta_{2k-1})|^2 \left. \right] \leq 6K_h(T + 4)T. \tag{2.19}
\]
Since \(w(.)\) is continuous in \(\mathbb{R}^d\), it must be uniformly continuous in the closed ball \(S_h = \{x \in \mathbb{R}^d : |x| \leq h\}\). We can therefore choose \(\delta = \delta(\varepsilon) > 0\) so small such that
\[
|w(x) - w(y)| < \varepsilon/2 \quad \text{whenever} \ x, y \in S_h, |x - y| < \delta. \tag{2.20}
\]
We furthermore choose \(T = T(\varepsilon, \delta, h) > 0\) sufficiently small for
\[
\frac{6K_h(T + 4)T}{\delta^2} < \varepsilon.
\]
It then follows from (2.19) that
\[
P \left( [\theta_{2k-1} \wedge \tau_h < \infty] \cap \left\{ \sup_{0 \leq t \leq T} |X(\tau_h \wedge (\theta_{2k-1} + t)) \right. \right.
\]
\[
- X(\tau_h \wedge \theta_{2k-1})| \geq \delta \left. \right\} \right) \leq \frac{6K_h(T + 4)T}{\delta^2} < \varepsilon.
\]
This together with (2.15) and (2.16) yields
\[
P\left(\left\{ \sigma_{2k-1} < \infty, \tau_h = \infty \right\} \cap \left\{ \sup_{0 \leq s \leq t} |X(\theta_{2k-1} + t) - X(\theta_{2k-1})| \geq \delta \right\} \right) \leq \varepsilon
\]
and
\[
P\left(\left\{ \theta_{2k-1} < \infty, \tau_h = \infty \right\} \cap \left\{ \sup_{0 \leq s \leq t} |X(\theta_{2k-1} + t) - X(\theta_{2k-1})| \geq \delta \right\} \right) \geq \varepsilon. \tag{2.21}
\]
Using (2.20) we derive that
\[
P\left(\left\{ \theta_{2k-1} < \infty, \tau_h = \infty \right\} \cap \left\{ \sup_{0 \leq s \leq t} |w(X(\theta_{2k-1} + t)) - w(X(\theta_{2k-1}))| < \varepsilon \right\} \right) \geq \varepsilon. \tag{2.22}
\]
Set
\[
\bar{\Omega}_k = \left\{ \sup_{0 \leq s \leq T} |w(X(\theta_{2k-1} + t))) - w(X(\theta_{2k-1}))| < \varepsilon \right\}.
\]
Noting that
\[
\theta_{2k}(\omega) - \theta_{2k-1}(\omega) \geq T
\]
if \( \omega \in \{ \theta_{2k-1} < \infty, \tau_h = \infty \} \cap \bar{\Omega}_k, \)
we derive from (2.17) and (2.22) that
\[
\infty > \varepsilon \sum_{k=1}^{\infty} E \left[ 1_{\{ \theta_{2k-1} < \infty, \tau_h = \infty \}} (\theta_{2k} - \theta_{2k-1}) \right] \\
\geq \varepsilon \sum_{k=1}^{\infty} E \left[ 1_{\{ \theta_{2k-1} < \infty, \tau_h = \infty \}} \cap \bar{\Omega}_k (\theta_{2k} - \theta_{2k-1}) \right] \\
\geq \varepsilon T \sum_{k=1}^{\infty} P \left( \left\{ \theta_{2k-1} < \infty, \tau_h = \infty \right\} \cap \bar{\Omega}_k \right) \\
\geq \varepsilon T \sum_{k=1}^{\infty} \varepsilon = \infty,
\]
which is a contradiction. So (2.11) must hold.

**Step 3.** From (2.11) and (2.13) there is an \( \Omega_0 \subset \Omega \) with \( P(\Omega_0) = 1 \) such that
\[
\lim_{t \to \infty} w(X(t, \omega)) = 0 \quad \text{and} \quad \sup_{0 \leq t < \infty} |X(t, \omega)| < \infty
\]
for all \( \omega \in \Omega_0. \tag{2.23} \]
We shall now show that
\[
\lim_{t \to \infty} X(t, \omega) = 0 \quad \forall \omega \in \Omega_0. \tag{2.24}
\]
If this is false, then there is some \( \tilde{\omega} \in \Omega_0 \) such that
\[
\lim_{t \to \infty} |X(t, \tilde{\omega})| > 0,
\]
hence there is a subsequence \( \{X(t_k, \tilde{\omega})\}_{k \geq 1} \) of \( \{X(t, \tilde{\omega})\}_{t \geq 0} \) such that
\[
|X(t_k, \tilde{\omega})| \geq c, \quad \forall k \geq 1,
\]
for some \( c > 0 \). Since \( \{X(t_k, \tilde{\omega})\}_{k \geq 1} \) is bounded so there must be an increasing subsequence \( \{t_k\}_{k \geq 1} \) such that \( \{X(t_k, \tilde{\omega})\}_{k \geq 1} \) converges to some \( z \in \mathbb{R}^d \) with \( |z| \geq \alpha \).
Hence
\[
w(z) = \lim_{k \to \infty} w(X(t_k, \omega)) > 0.
\]
However, by (2.23), \( w(z) = 0 \), which is a contradiction. So (2.24) must hold. This implies that the trivial solution of Eq. (1.1) is almost surely asymptotically stable and the proof is therefore complete. \( \square \)

## 3. Stability of Trivial Solution

In this section we shall assume that
\[
b(0, 0, t, \ell) \equiv 0, \quad \sigma(0, 0, t, \ell) \equiv 0 \quad \text{and} \quad \gamma(0, 0, t, \ell, z) \equiv 0, \tag{3.1}
\]
so equation (1.1) has the solution \( X(t) \equiv 0 \) corresponding to the initial data \( \xi(s) = 0, -\tau \leq s \leq 0 \), for which \( \alpha(0) = \ell_0 \). This solution is called the trivial solution or equilibrium position. In this section, we shall assume that there is a unique solution to Eq. (1.1) with initial data \( \xi \in C^2_{\mathcal{F}_0}([0, \infty) \times \mathbb{R}^d), \alpha(0) = \ell_0 \), moreover, the solution satisfies \( \mathbb{E} (\sup_{-\tau \leq s \leq 0} |X(s)|^2) < \infty \) for all \( t \geq 0 \) (denoted by \( L^2_{\mathcal{F}_0}(\Omega, \mathbb{R}^d) \)). The main result of section is in Theorem 3.1, which is called exponential stability in mean square. The method used in the section is called Razumikhin argument.

**Theorem 3.1:** Let \( \lambda_1, c_1, c_2 \) all be positive constants and \( q > 1 \). Assume that there exists a function \( V \in \mathbb{R}^d \times [-\tau, \infty) \times S; \mathbb{R}_+ \) such that
\[
c_1 |x|^2 \leq V(x, t, \ell) \leq c_2 |x|^2
\]
for all \( (x, t, \ell) \in \mathbb{R}^d \times [-\tau, \infty) \times S \) \( \tag{3.2} \)
and, moreover,
\[
\mathbb{E} L V(X(t), X(t - \tau), t, \alpha(t)) \leq -\lambda_1 \mathbb{E} V(X(t), t, \alpha(t)) \tag{3.3}
\]

provided \( X \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}^d) \) satisfying
\[
\mathbb{E}V(X(t + \theta), t + \theta, \alpha(t + \theta)) < q \mathbb{E}V(X(t), t, \alpha(t))
\]
(3.4)
for all \(-\tau \leq \theta \leq 0\). Then for all \( \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d) \), \( \ell_0 \in S \)
\[
\mathbb{E}|X^{\xi, \ell_0}(t)|^2 \leq \frac{c_2}{c_1} \mathbb{E}\|\xi\|^2 e^{-\kappa t}, \text{ on } t > 0,
\]
where \( \kappa = \min\{\lambda_1, \ln(q)/\tau\} \).

**Proof:** Fix any initial data \( \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d) \), \( \ell_0 \in S \) and write \( X^{\xi, \ell_0}(t) = X(t) \) simply. Recalling the fact that \( X(t) \) is right continuous with left limit and \( \mathbb{E}(\sup_{-\tau \leq \theta \leq 0} |X(s)|^2 < \infty \) for all \( t \geq 0 \), we see that \( \mathbb{E}V(X(t), t, \alpha(t)) \) is right continuous with left limit. Let \( \varepsilon \in (0, \kappa) \) be arbitrary and set \( \bar{\kappa} = \kappa - \varepsilon \). Define
\[
H(t) = \sup_{-\tau \leq \theta \leq 0} \left[ e^{\bar{\kappa}t} \mathbb{E}V(X(t + \theta), t + \theta, \alpha(t + \theta)) \right],
\]
for \( t \geq 0 \).

We claim that
\[
D_+H(t) = \limsup_{h \to 0^+} \frac{H(t + h) - H(t)}{h} \leq 0,
\]
for all \( t \geq 0 \). (3.5)
Note that for each \( t \geq 0 \), either \( H(t) > e^{\bar{\kappa}t} \mathbb{E}V(X(t), t, \alpha(t)) \) or \( H(t) = e^{\bar{\kappa}t} \mathbb{E}V(X(t), t, \alpha(t)) \). In the case of former, it follows from the right continuity of \( \mathbb{E}V(X(\cdot), t, \alpha(\cdot)) \) that for all \( h > 0 \) sufficiently small
\[
H(t) > e^{\bar{\kappa}(t+h)} \mathbb{E}V(X(t + h), t + h, \alpha(t + h)),
\]
and hence
\[
H(t + h) \leq H(t) \text{ and } DH_+(t) \geq 0.
\]
In the other case, i.e., \( H(t) = e^{\bar{\kappa}t} \mathbb{E}V(X(t), t, \alpha(t)) \), we have
\[
e^{\bar{\kappa}(t+h)} \mathbb{E}V(X(t + \theta), t + \theta, \alpha(t + \theta)) \leq e^{\bar{\kappa}t} \mathbb{E}V(X(t), t, \alpha(t))
\]
for all \(-\tau \leq \theta \leq 0\). So
\[
\mathbb{E}V(X(t + \theta), t + \theta, \alpha(t + \theta)) \leq e^{-\bar{\kappa}\theta} \mathbb{E}V(X(t), t, \alpha(t))
\]
\[
\leq e^{\bar{\kappa}t} \mathbb{E}V(X(t), t, \alpha(t)),
\]
(3.6)
for all \(-\tau \leq \theta \leq 0\). Note that either \( \mathbb{E}V(X(t), t, \alpha(t)) = 0 \) or \( \mathbb{E}V(X(t), t, \alpha(t)) > 0 \). In the former case, (3.6) and (3.2) yield that \( X(t + \theta) = 0 \) a.s. for all \(-\tau \leq \theta \leq 0\).

By the uniqueness and the assumption (3.1), one sees that \( X(t) \equiv 0 \) for all \( t \geq 0 \), and hence \( D_+H(t) = 0 \). On the other hand, in case of \( \mathbb{E}V(X(t), t, \alpha(t)) > 0 \), (3.6) implies
\[
\mathbb{E}V(X(t + \theta), t + \theta, \alpha(t + \theta)) \leq q \mathbb{E}V(X(t), t, \alpha(t)),
\]
for all \(-\tau \leq \theta \leq 0\), since \( e^{\bar{\kappa}t} < q \). In other words, \( X_t \in L^2_{\mathcal{F}_T}([-\tau, 0]; \mathbb{R}^d) \) satisfies (3.4). Thus, by condition (3.3), we have
\[
\mathbb{E}L^2V(X(t), X(t - \tau), t, \alpha(t)) \leq -\lambda_1 \mathbb{E}V(X(t), t, \alpha(t)).
\]
This gives
\[
\bar{\kappa} \mathbb{E}V(X(t), t, \alpha(t)) + \mathbb{E}L^2V(X(t), X(t - \tau), t, \alpha(t))
\]
\[
\leq -(\lambda_1 - \bar{\kappa}) \mathbb{E}V(X(t), t, \alpha(t)).
\]
By the right continuity for all \( h > 0 \) sufficiently small
\[
\bar{\kappa} \mathbb{E}V(X(s), s, \alpha(s)) + \mathbb{E}L^2V(X(s), X(s - \tau), s, \alpha(s)) \leq 0,
\]
if \( t \leq s \leq t + h \).
By Itô’s formula, we can then derive that
\[
e^{\bar{\kappa}(t+h)} \mathbb{E}V(X(t + h), t + h, \alpha(t + h))
\]
\[
- e^{\bar{\kappa}t} \mathbb{E}V(X(t), t, \alpha(t))
\]
\[
\leq \int_t^{t+h} e^{\bar{\kappa}s} \mathbb{E}|\tilde{a}| \mathbb{E}V(X(s), s, \alpha(s))
\]
\[
+ \mathbb{E}L^2V(X(s), X(s - \tau), s, \alpha(s)) \|ds \leq 0.
\]
That is
\[
e^{\bar{\kappa}(t+h)} \mathbb{E}V(X(t+h), t+h, \alpha(t+h)) \leq e^{\bar{\kappa}t} \mathbb{E}V(X(t), t, \alpha(t)).
\]
This implies \( H(t + h) = H(t) \) for all \( h > 0 \) sufficiently small, and hence \( D_+H(t) = 0 \). Inequality (3.5) has therefore been proved. It now follows from (3.5)
\[
H(t) \leq H(0) \text{ for all } t \geq 0.
\]
By condition (3.2), we obtain
\[
\mathbb{E}|X(t)|^2 \leq \frac{c_2}{c_1} \mathbb{E}\|\xi\|^2 e^{-\bar{\kappa}t} = \frac{c_2}{c_1} \mathbb{E}\|\xi\|^2 e^{-(\bar{\kappa}-\varepsilon)t}.
\]
Since \( \varepsilon \) is arbitrary, the required inequality must hold. The proof is therefore complete. \( \square \)

**Remark 3.2:** Under the conditions of Theorem 3.1, moreover assume
\[
|\sigma(x, y, t, \ell)| + |b(x, y, t, \ell)|
\]
\[
+ \sum_{i=1}^d \sum_{j=1}^n \int_{\mathbb{R}} |\gamma_i(x, y, t, \ell)| \eta_j(d\ell) \leq C|\alpha| + |\beta|,
\]
where \( C \) is a constant. Then similar to that of [21, Theorem 5.1], we can prove that the solution is almost sure exponential stable.

4. Stability in Distribution

In this section, we shall assume that Eq. (1.1) is autonomous, that is, the coefficients \( \alpha, \sigma \), and \( \gamma \) are all independent of \( t \). This implies that \( Y^x_{t, t_0} := (X^x_{t, t_0}, \alpha_{t_0}(t)) \) is a homogeneous strong Markov process.

We begin this section by defining the stability in distribution of system (1.1).

**Definition 4.1:** The process \( Y^x_{t, t_0} \) is said to be stable in distribution if there exists a probability measure \( \pi (\cdot \times \cdot) \) on \( D ([−\tau, 0]; \mathbb{R}^n) \times S \) such that \( P(Y^x_{t, t_0} \in dt \times \{ \xi \}) \) converges weakly to \( \pi (dt \times \{ \xi \}) \) as \( t \to \infty \) for any \( \xi \in C^b_{F_0} ([−\tau, 0]; \mathbb{R}^n) \), \( t_0 \in S \). In this case, system (1.1) is said to be stable in distribution.

**Lemma 4.2:** Let Assumption (LL) hold and \( c_3, \beta \) be positive numbers and \( \lambda_2 > \lambda_3 \geq 0 \). Assume that there exist function \( V(x, \ell) \in C^2 (\mathbb{R}^n \times S; \mathbb{R}_+) \) and \( w_3(x) \in C (\mathbb{R}^n; \mathbb{R}_+) \) such that

\[
\begin{align*}
  c_3 |x|^2 &\leq V(x, \ell) \leq w_3(x) & \tag{4.1} \\
  \mathcal{L} V(x, \ell) &\leq -\lambda_2 w_3(x) + \lambda_3 w_3(y) + \beta & \tag{4.2} \\
  \sup_{0 \leq t < \infty} \mathbb{E}|X^x_{t, t_0}(t)|^2 &< \infty, \forall \xi \in C^b_{F_0} ([−\tau, 0]; \mathbb{R}^n), \ell_0 \in S. & \tag{4.3}
\end{align*}
\]

In the same way as the proof of Lemma 2.3, we can prove the following lemma for the existence and uniqueness for equation (1.1).

**Lemma 4.3:** Under the conditions of Lemma 4.2, for any initial data \( \{ x(\theta) : −\tau \leq \theta \leq 0 \} = \xi \in C^b_{F_0} ([−\tau, 0]; \mathbb{R}^n), \ell_0 \in S \), Eq. (1.1) has a unique global solution.

We are now ready to prove Lemma 4.2.

**Proof of Lemma 4.2:** Fix any \( \xi \in C^b_{F_0} ([−\tau, 0]; \mathbb{R}^n), \ell_0 \in S \) and write \( X^x_{t, t_0}(t) = X(t) \) and \( X^x_{t, t_0} = X_t \). Define the stopping time

\[
\tau_R = \inf \{ t \geq 0, |X(t)| > R \}.
\]

By Itô’s formula, we have

\[
\begin{align*}
  e^{(t - \lambda_3)lt_0} &\int_{0}^{t\wedge \tau_R} V(X(t \wedge \tau_R), \alpha(t \wedge \tau_R)) \\
  = V(\xi(0), \ell_0) + &\int_{0}^{t\wedge \tau_R} \alpha(t \wedge \tau_R) ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 V_\alpha(X(s), \alpha(s)) ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 V_x(X(s), \alpha(s)) ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 \sigma(X(s), \alpha(s)) \sigma(X(s)) ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 \nu(X(s), \alpha(s)) dB(s) \\
  &+ \frac{1}{2} \int_{0}^{t\wedge \tau_R} \lambda_3 \text{trace}[\sigma^T(X(s), \alpha(s)) \nu(X(s), \alpha(s)) \nu(X(s), \alpha(s)) ds]
\end{align*}
\]

\[
\begin{align*}
  + \sum_{k=1}^{n} \int_{0}^{t\wedge \tau_R} &\int_{0}^{t\wedge \tau_R} \lambda_3 \left| V(X(s^\wedge \tau_R), \gamma^k(X(s^\wedge \tau_R)), \nu(X(s), \alpha(s))) V_\alpha(X(s), \alpha(s)) \sigma(X(s), \alpha(s)) \right| ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 \sigma(X(s), \alpha(s)) \sigma(X(s)) ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 \nu(X(s), \alpha(s)) \nu(X(s), \alpha(s)) dB(s) \\
  &+ \frac{1}{2} \int_{0}^{t\wedge \tau_R} \lambda_3 \text{trace}[\sigma^T(X(s), \alpha(s)) \nu(X(s), \alpha(s)) \nu(X(s), \alpha(s)) ds]
\end{align*}
\]

\[
\begin{align*}
  + \sum_{k=1}^{n} \int_{0}^{t\wedge \tau_R} &\int_{0}^{t\wedge \tau_R} \lambda_3 \left| V(X(s^\wedge \tau_R), \gamma^k(X(s^\wedge \tau_R)), \nu(X(s), \alpha(s))) V_\alpha(X(s), \alpha(s)) \sigma(X(s), \alpha(s)) \right| ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 \sigma(X(s), \alpha(s)) \sigma(X(s)) ds \\
  &+ \int_{0}^{t\wedge \tau_R} \lambda_3 \nu(X(s), \alpha(s)) \nu(X(s), \alpha(s)) dB(s) \\
  &+ \frac{1}{2} \int_{0}^{t\wedge \tau_R} \lambda_3 \text{trace}[\sigma^T(X(s), \alpha(s)) \nu(X(s), \alpha(s)) \nu(X(s), \alpha(s)) ds]
\end{align*}
\]
Noting the last two terms are martingales, taking the expectation and using the conditions (4.2) and (4.1), we obtain

\[
e^{(\lambda_2 - \lambda_3)\tau} \mathbb{E} V(X(t \land \tau), \alpha(t \land \tau)) \leq V(\xi(0), \ell_0) + \mathbb{E} \int_0^{\tau \wedge \theta} e^{(\lambda_2 - \lambda_3)s} \left( (\lambda_2 - \lambda_3) V(X(s), \alpha(s)) - \lambda_2 \omega(X(s)) + \lambda_3 \omega(X(s - \tau)) + \beta \right) ds \leq V(\xi(0), \ell_0) + \lambda_3 \int_{-\tau}^0 w_3(\xi(s)) ds + \frac{\beta}{\lambda_2 - \lambda_3} \left( e^{(\lambda_2 - \lambda_3)\tau} - 1 \right) .
\]

Since \( \tau_R \to \infty \) as \( R \to \infty \), by condition (4.1) we get

\[
\mathbb{E} |X(t)|^2 \leq e^{-(\lambda_2 - \lambda_3)\tau} \frac{1}{C_3} \left( V(\xi(0), \ell_0) + \lambda_3 \int_{-\tau}^0 w_3(\xi(s)) ds + \frac{\beta}{\lambda_2 - \lambda_3} \left( e^{(\lambda_2 - \lambda_3)\tau} - 1 \right) \right) < \infty , \quad (4.4)
\]

as required. The proof is therefore completed.

\[\square\]

For the future use, we need the following linear growth condition

**Assumption (LG) (Linear Growth)** there exists a constant \( L > 0 \) such that for all \( x, y \in \mathbb{R}^d, \ell \in S \)

\[
|b(x, y, \ell)|^2 + |\sigma(x, y, \ell)|^2 + \int_{\mathbb{R}} \sum_{k=1}^n |y^{(k)}(x, y, \ell, z_k)|^2 v_k(dz_k) \leq L(1 + |x|^2 + |y|^2) , \quad (4.5)
\]

**Lemma 4.4:** Let Assumption (LG) hold. Under the conditions of Lemma 4.2, for any initial data \( \{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C_{\mathbb{P}_0}^2(1 - \tau, 0], \mathbb{R}^d) \), \( \ell_0 \in S \) we have

\[
\mathbb{E} \|X^{\xi, \ell_0}_t\| < \infty . \quad (4.6)
\]

**Proof:** For any \( t \geq \tau \) and \( \theta \in [0, \tau] \), by linear growth condition, Itô’s formula, Burkholder-Davis-Gundy inequality, Doob martingale inequality, Hölder inequality and Itô isometry, we have that

\[
\mathbb{E} \sup_{0 \leq \theta \leq \tau} |X(t - \theta)|^2 \leq 5\mathbb{E} |\xi(0)|^2 + 5\mathbb{E} \mathbb{E} \sup_{0 \leq \theta \leq \tau} \left| \int_0^{t - \theta} b(X(s), X(s - \tau), \alpha(s)) ds \right|^2 + 5\mathbb{E} \mathbb{E} \sup_{0 \leq \theta \leq \tau} \left| \int_0^{t - \theta} \sigma(X(s), X(s - \tau), \alpha(s)) dB(s) \right|^2 + 5\mathbb{E} \mathbb{E} \sup_{0 \leq \theta \leq \tau} \left| \int_{t - \tau}^{t - \theta} \gamma(X(s), X(s - \tau), \alpha(s), z) ds \right|^2 \leq C + C\mathbb{E} \int_{t - \tau}^t (1 + |X(s)|^2 + |X(s - \tau)|^2) ds \leq C + C\mathbb{E} \int_{t - \tau}^t (1 + |X(s)|^2) ds ,
\]

where \( C \) is a constant. This, together with (4.4), yields

\[
\mathbb{E} \|X_t\|^2 < \infty . \quad (4.7)
\]

In what follows we need to consider the difference between two solutions of Eq. (1.1) starting from different initial values, namely

\[
X^{\xi, \ell_0}(t) - X^{\eta, \ell_0}(t) = \xi(0) - \eta(0) + \int_0^t \left[ b(X^{\xi, \ell_0}(s), X^{\xi, \ell_0}(s - \tau), \alpha(s)) - b(X^{\eta, \ell_0}(s), X^{\eta, \ell_0}(s - \tau), \alpha(s)) \right] ds + \int_0^t \left[ \sigma(X^{\xi, \ell_0}(s), X^{\xi, \ell_0}(s - \tau), \alpha(s)) - \sigma(X^{\eta, \ell_0}(s), X^{\eta, \ell_0}(s - \tau), \alpha(s)) \right] dB(s) + \int_0^t \int_{\mathbb{R}^d} \left[ \gamma(X^{\xi, \ell_0}(s), X^{\xi, \ell_0}(s - \tau), \alpha(s), z) - \gamma(X^{\eta, \ell_0}(s), X^{\eta, \ell_0}(s - \tau), \alpha(s), z) \right] \tilde{N}(dt, dz) . \quad (4.8)
\]

For a given function \( U \in C^2(\mathbb{R}^d \times S, \mathbb{R}_+) \), we define an operator \( \mathbb{I} U : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times S \to \mathbb{R} \) associated
with Eq. (4.8) by
\[
\mathbb{L}U(x, y, u_1, u_2, \ell) = U_x(x - y, \ell)[b(x, u_1, \ell) - b(y, u_2, \ell)] + \frac{1}{2}\text{trace}\left((\sigma(x, u_1, \ell) - \sigma(y, u_2, \ell))^T U_{xx}(x - y, \ell) \times (\sigma(x, u_1, \ell) - \sigma(y, u_2, \ell))\right) + \int_\mathbb{R} \sum_{k=1}^n U(x - y + \gamma(k)(x, u_1, \ell, z_k) - \gamma(k)(y, u_2, \ell, z_k)) - U(x - y, \ell) - U_x(x - y, \ell)[\gamma(k)(x, u_1, \ell, z_k) - \gamma(k)(y, u_2, \ell, z_k)] \right) v_k(dz_k).
\]
\[(4.9)\]

**Lemma 4.5:** Let Assumption (LG) hold. Assume that there exist positive numbers \(c_4\) and \(\lambda_4 > \lambda_5 \geq 0\) and \(U(x, \ell) \in C^2(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}_+)\), \(w_4(x) \in C(\mathbb{R}^d; \mathbb{R})\) such that
\[
c_4|x|^2 \leq w_4(x) \wedge U(x, \ell)
\]
\[(4.10)\]
and
\[
\mathbb{L}U(x, y, u_1, u_2, \ell) \leq -\lambda_4 w_4(x - y) + \lambda_5 w_4(u_1 - u_2)
\]
\[(4.11)\]
for all \(x, y, u_1, u_2 \in \mathbb{R}^d, \ell \in S\). Then
\[
\lim_{t \to \infty} \mathbb{E}|X_t^{\xi, \ell_0} - X_t^{\eta, \ell_0}|^2 = 0
\]
uniformly in \(\xi, \eta \in K, \ell_0 \in S\),
\[(4.12)\]
for any compact subset \(K\) of \(C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)\).

**Proof:** Let \(N\) be positive number and define the stopping time
\[
\tau_N = \inf\{t > 0 : |X_t^{\xi, \ell_0} - X_t^{\eta, \ell_0}| > N\}.
\]
Setting \(T_N = \tau_N \wedge t\) and applying the Itô formula to (4.8) we can show that
\[
\mathbb{E}U(X_t^{\xi, \ell_0}(T_N) - X_t^{\eta, \ell_0}(T_N), \alpha(T_N))
\]
\[
\leq U(\xi(0) - \eta(0), \ell_0) + \lambda_5 \int_{-\tau}^0 w_2(\xi(s) - \eta(s))ds - (\lambda_4 - \lambda_5)E \int_0^\tau |X_s^{\xi, \ell_0} - X_s^{\eta, \ell_0}|^2 ds.
\]
\[(4.13)\]
This implies
\[
\int_0^\infty \mathbb{E}|X_t^{\xi, \ell_0} - X_t^{\eta, \ell_0}|^2 ds \leq \frac{U(\xi(0) - \eta(0), \ell_0)}{\lambda_4 - \lambda_5} + \frac{\lambda_5}{\lambda_4 - \lambda_5} \int_{-\tau}^0 w_2(\xi(s) - \eta(s))ds < \infty.
\]
\[(4.14)\]
If (4.12) is false, then there exists a constant \(L > 0\) such that
\[
\lim_{t \to \infty} \mathbb{E}|X_t^{\xi, \ell_0}(t) - X_t^{\eta, \ell_0}(t)|^2 = L.
\]
\[(4.15)\]
So there is a positive number \(\epsilon (\leq L/6)\) and a sequence \(\{t_n, n = 1, 2, \ldots\}, t_n \to \infty\) as \(n \to \infty\) such that
\[
\mathbb{E}|X_t^{\xi, \ell_0}(t_n) - X_t^{\eta, \ell_0}(t_n)|^2 \geq L - \epsilon.
\]
\[(4.16)\]
Let \(t > t_n, |t - t_n| < 1\), by (4.8)
\[
X_t^{\xi, \ell_0}(t) = X_t^{\xi, \ell_0}(t_n)
\]
\[
= X_t^{\xi, \ell_0}(t_n) + \int_{t_n}^t [b(X_s^{\xi, \ell_0}(s), X_s^{\xi, \ell_0}(s - \tau), \alpha(s))
\]
\[
- b(X_s^{\eta, \ell_0}(s), X_s^{\eta, \ell_0}(s - \tau), \alpha(s))]ds
\]
\[
+ \int_{t_n}^t [\sigma(X_s^{\xi, \ell_0}(s), X_s^{\xi, \ell_0}(s - \tau), \alpha(s))
\]
\[
- \sigma(X_s^{\eta, \ell_0}(s), X_s^{\eta, \ell_0}(s - \tau), \alpha(s))]dB(s)
\]
\[
+ \int_{t_n}^t [\gamma(X_s^{\xi, \ell_0}(s), X_s^{\xi, \ell_0}(s - \tau), \alpha(s))
\]
\[
- \gamma(X_s^{\eta, \ell_0}(s), X_s^{\eta, \ell_0}(s - \tau), \alpha(s), z)]d\tilde{N}(dt, dz).
\]
Therefore
\[ |X^t_{\xi_0} - X^{\eta_0}_{\xi_0}(t)|^2 \]
\[ \geq \frac{1}{3} \left| X^t_{\xi_0}(t_n) - X^{\eta_0}_{\xi_0}(t_n) \right|^2 - \left| \int_{t_n}^t \left[ b(X^t_{\xi_0}(s),X^t_{\xi_0}(s-\tau),\alpha(s)) + \sigma(X^t_{\xi_0}(s),X^t_{\xi_0}(s-\tau),\alpha(s)) \right] ds \right|^2 \]
\[ = \sigma(X^{\eta_0}_{\xi_0}(s),X^{\eta_0}_{\xi_0}(s-\tau),\alpha(s))dB(s) \]
\[ = \int_{t_n}^t \left[ \gamma(X^t_{\xi_0}(s^-),X^t_{\xi_0}(s-\tau)^-),\alpha(s),z \right] \tilde{N}(dt,dz) \right|^2 \]

Using the linear growth condition and the Itô isometry, we derive
\[ \mathbb{E}[|X^t_{\xi_0} - X^{\eta_0}_{\xi_0}(t)|^2] \]
\[ \geq \frac{1}{3} \mathbb{E}[|X^t_{\xi_0}(t_n) - X^{\eta_0}_{\xi_0}(t_n)|^2 - C\mathbb{E} \int_{t_n}^t (4 + |X^t_{\xi_0}(s)|^2 + |X^t_{\eta_0}(s)|^2) + |X^t_{\xi_0}(s-\tau)|^2 + |X^t_{\eta_0}(s-\tau)|^2 | ds, \]

where \( C \) is a constant, independent of \( n \) and initial data etc., which may change line by line. \( > \)From Lemma 4.2 we know that there is \( 0 < \delta < 1 \) such that
\[ CE \int_{t_n}^t (4 + |X^t_{\xi_0}(s)|^2 + |X^t_{\eta_0}(s)|^2) + |X^t_{\xi_0}(s-\tau)|^2 + |X^t_{\eta_0}(s-\tau)|^2 \leq \epsilon, \]
whenever \( |t - t_n| < \delta. \) This, together with (4.17), yields
\[ \mathbb{E}[|X^t_{\xi_0} - X^{\eta_0}_{\xi_0}(t)|^2] \geq L/6, \]
whenever \( |t - t_n| < \delta. \) It then follows that
\[ \int_0^\infty \mathbb{E}[|X^t_{\xi_0}(s) - X^{\eta_0}_{\xi_0}(s)|^2] ds = \infty. \]

Since this contradicts with (4.14), we must have (4.12).

Similar to the proof of Lemma 4.4, we have

**Lemma 4.6:** Under the conditions of Lemma 4.5,
\[ \lim_{t \to \infty} \mathbb{E}[|X^t_{\xi_0} - X^t_{\eta_0}_{\xi_0}|^2] = 0 \]
uniformly in \( \xi, \eta \in K, t_0 \in S. \) (4.19)

Since \( Y_t \) has properties (4.6) and (4.19) along the lines of the proof of Theorem 3.1 in [30], we can obtain the following theorem:

**Theorem 4.7:** Under the conditions of Lemma 4.4 and Lemma 4.5, the process \( Y_t = (X_t,\alpha(t)) \) is stable in distribution.

5. Summary

We have discussed almost sure stability of the solution of stochastic delay hybrid systems by using supermartingale convergence and Lyapunov method. The exponential stability in mean square has been investigated by applying Razumikhin-type theorem. Moreover, we have studied the stability in distribution. An interesting question is can we obtain almost sure stability for stochastic delay systems as in deterministic ODE by using Razumikhin argument? This question has been challenged for more than 20 years since Mao [19] presented stability in mean square for stochastic delay systems by using Razumikhin argument.

References

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