Exponential stability of equidistant Euler–Maruyama approximations of stochastic differential delay equations

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Abstract

Our aim is to study under what conditions the exact and numerical solution (based on equidistant nonrandom partitions of integration time-intervals) to a stochastic differential delay equation (SDDE) share the property of mean-square exponential stability. Our approach is trying to avoid the use of Lyapunov functions or functionals. We show that under a global Lipschitz assumption an SDDE is exponentially stable in mean square if and only if for some sufficiently small stepsize \( h \) the Euler–Maruyama (EM) method is exponentially stable in mean square. We then replace the global Lipschitz condition with a finite-time convergence condition and establish the same “if and only if” result. The important feature of this result is that it transfers the asymptotic problem into a finite-time convergence problem. Replacing the exact and EM numerical solution with stochastic processes, we also discuss whether a family of stochastic processes share the stability property. This new approach allows us to discuss (i) whether a family of SDDEs share the stability property, and (ii) whether an SDDE with variable time lag shares stability property with the modified EM method. As another application of this general approach we consider a linear SDDE with variable time lag and establish an “if and only if” result. It should also be mentioned that the questions addressed, results proved, as well as style of analysis borrow heavily from [14] but the computations involved to cope with time delay are nontrivial.

Keywords: Mean-square stability; Brownian motion; Euler–Maruyama’s method; Stochastic flow; Itô’s formula; Exponential stability

1. Introduction

Stochastic modelling has come to play an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stability in stochastic models. There is an extensive literature on stochastic stability, for example, Arnold [1,2], Elworthy [9], Freidlin and Wentzell [10], Friedman [11], Friedman and Pinsky [12], Khasminskii [15], Kolmanovskii and Myshkis [17], Ladde and Lakshmikantham [19], Mao [20–22] and Mohammed [26] to name but a few.

In this paper we consider the following problems. First of all, suppose that we are given a stochastic differential delay equation (SDDE)

\[
dy(t) = f(y(t), y(t - \tau)) \, dt + g(y(t), y(t - \tau)) \, dw(t)
\]
with initial data \( \xi \in L^2_{\mathbb{F}}([-\tau, 0]; \mathbb{R}^n) \), and we are required to find out whether or not it is exponentially stable in mean square. Given that we fail to find an appropriate Lyapunov function or functional to show the exponential stability, we can carry out careful numerical simulations, say by the Euler–Maruyama (EM) method (see e.g. [16,22,24,25,34]). That is, for a stepsize \( \Delta = \tau/N \) (\( N \) is an integer), set

\[ x(k\Delta) = \xi(k\Delta), \quad -N \leq k \leq 0 \]

and compute the discrete approximations

\[ x((k+1)\Delta) = x(k\Delta) + f(x(k\Delta), x((k-N)\Delta))\Delta + g(x(k\Delta), x((k-N)\Delta))\Delta w_k, \quad k \geq 1, \tag{1.1} \]

where \( \Delta w_k = w((k+1)\Delta) - w(k\Delta) \). The question is:

(Q1) If for a sufficiently small \( \Delta \) the EM numerical solution is exponentially stable in mean square, can we confidently infer that the underlying SDDE is exponentially stable in mean square?

The answer to (Q1) is not so obvious since most existing results are on the finite-time convergence for numerical methods (see e.g. [6,8,23,24,32,33,35,36]) while the exponential stability is clearly an asymptotic property. If we can establish a positive result, it will certainly have important practical implications.

We next consider the converse problem. Traditional stability analysis of numerical methods for ordinary differential equations (ODEs) is motivated by the question “for what choices of stepsize does the numerical method reproduce the characteristics of the test equation?” Suppose that we are now given an SDDE which is exponentially stable in mean square. The question is:

(Q2) For what choices of stepsize does the EM numerical method reproduce the mean-square exponential stability of the underlying SDDE?

In the case when the underlying equation is a stochastic differential equation (SDE), results that answer (Q1) and (Q2) can be found in [3–5,18,27–30,33]. In particular, the “if and only if” results for linear SDEs can be founded in [13,31] while for general nonlinear SDEs in [14]. Baker and Buckwar [7] consider \( p \)th mean stability of numerical methods for scalar SDDEs under assumptions of global Lipschitz coefficients and the existence of a Lyapunov function. Our aim here is to give very positive answers to both (Q1) and (Q2) without the existence of a Lyapunov function or functional.

In Section 2, we describe SDDEs and the EM method along with the definitions of the exponential stability in mean square for SDDEs and the EM method. Although our analysis will be carried out with a continuous EM method, we point out that the exponentially stable in mean square of the discrete EM method is equivalent to that of the continuous one. In Section 3 we show that under a global Lipschitz assumption an SDDE is exponentially stable in mean square if and only if for some sufficiently small stepsize \( \Delta \) the EM method is exponentially stable in mean square. Considering that many important SDDE models do not satisfy the global Lipschitz condition, we introduce in Section 4 Property (P1) which is a finite-time convergence property and then show that the EM method shares mean-square exponential stability with the SDDE as long as they have Property (P1). The important feature of this result is that it transfers the asymptotic problem into a finite-time convergence problem. That is, in order to reveal that the EM method shares mean-square exponential stability with the SDDE, it is enough to show that the EM method and the underlying SDDE have Property (P1).

In Section 5 we turn our attention to establishing even more general results. Replacing the exact and EM numerical solution with stochastic processes, we can discuss whether a family of stochastic processes share the stability property. This new approach allows us to discuss (i) whether a family of SDDEs share the stability property, and (ii) whether an SDDE with variable time lag shares stability property with the modified EM method. New approach also allows us to discuss whether an SDDE shares stability property with other numerical methods e.g. the theta method, though this will not be done in this paper due to the page limit (it will be reported elsewhere). As another application of this general approach we consider linear SDDEs in Section 6 and establish an “if and only if” result for them.

It should be pointed out that the reason why we organize our paper in the way above, rather than begin with the general approach, is because we would like to develop our theory step by step so that it is more understandable.
It should also be mentioned that the questions addressed, results proved, as well as style of analysis borrow heavily from [14] but the computations involved to cope with time delay are nontrivial.

2. SDDEs and Euler–Maruyama’s method

Throughout this paper we use the following notations. Let $| \cdot |$ be the Euclidean norm in $\mathbb{R}^n$ and $\langle x, y \rangle$ be the inner product of vectors $x, y \in \mathbb{R}^n$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. If $x$ is a real number, its integer part is denoted by $\text{In}[x]$. Let $\mathbb{R}_+ = [0, \infty)$ and $\tau > 0$. Let $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of continuous functions $\varphi$ from $[-\tau, 0]$ to $\mathbb{R}^n$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space. Denote by $L^2_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ the family of $\mathcal{F}_t$-measurable, $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(u) : -\tau \leq u \leq 0\}$ such that

$$
\|\xi\|_E^2 := \sup_{-\tau \leq u \leq 0} \mathbb{E}|\xi(u)|^2 < \infty.
$$

If $x(t)$ is a continuous $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t + u) : -\tau \leq u \leq 0\}$ for $t \geq 0$ which is regarded as a $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process.

Let us consider the $n$-dimensional autonomous SDDE

$$
dy(t) = f(y(t), y(t - \tau)) \, dt + g(y(t), y(t - \tau)) \, dw(t),
$$

where $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}$. We impose the following hypotheses:

(H1) Assume that both $f$ and $g$ are globally Lipschitz continuous, that is

$$
|f(x, y) - f(x, \bar{y})|^2 \leq K_1(|x - \bar{x}|^2 + |y - \bar{y}|^2)
$$

and

$$
|g(x, y) - g(x, \bar{y})|^2 \leq K_2(|x - \bar{x}|^2 + |y - \bar{y}|^2)
$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, where $K_1$ and $K_2$ are constants. Assume also, for the purpose of stability study in this paper, that $f(0, 0) = 0$ and $g(0, 0) = 0$.

It is well-known that under (H1), for any initial data $y_0 = \xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ given at time $t = 0$, the SDDE (2.1) has a unique continuous solution on $t \geq -\tau$, see, for example, [22,26]. We shall denote this solution by $y(t; 0, \xi)$. In this paper we consider exponential stability in mean square of the origin, which we define as follows (see e.g. [17,20,21]).

**Definition 2.1.** The SDDE (2.1) is said to be exponentially stable in mean square if there is a pair of positive constants $\lambda$ and $M$ such that for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$

$$
\mathbb{E}|y(t; 0, \xi)|^2 \leq M\|\xi\|_E^2 e^{-\lambda t}, \quad \forall t \geq 0.
$$

We refer to $\lambda$ as the rate constant and $M$ as the growth constant.

In this paper we often need to introduce the solution to the SDDE (2.1) for initial data $y_s = \xi \in L^2_{\mathcal{F}_s}([-\tau, 0]; \mathbb{R}^n)$ given at time $t = s$. Hypotheses (H1) guarantee the existence and uniqueness of this solution which is denoted by $y(t; s, \xi)$ on $t \geq s - \tau$. It is easy to observe that the solutions to the SDDE (2.1) have the following flow property:

$$
y(t; 0, \xi) = y(t; s, y_s) \quad 0 \leq s < t < \infty.
$$

Moreover, due to the autonomous property of the SDDE (2.1) (i.e. both $f$ and $g$ are independent of $t$), the exponential stability (2.2) implies

$$
\mathbb{E}|y(t; s, \xi)|^2 \leq M\|\xi\|_E^2 e^{-\lambda(t-s)}, \quad \forall t \geq s.
$$

(2.3)
Let us now recall the discrete EM approximate solution (1.1). To highlight the initial data \( z \) given at time \( t = 0 \) we write this discrete EM solution as \( x(k \Delta; 0, z) \). Following Definition 2.1, we may now define exponential stability in mean square for the discrete EM method.

**Definition 2.2.** Given a stepsize \( \Delta = \tau / N \) for some positive integer \( N \), the discrete EM method is said to be exponentially stable in mean square on the SDDE (2.1) if there is a pair of positive constants \( \gamma \) and \( H \) such that for any initial data \( \xi \in L^2_{\mathcal{F}_0}([t \tau, 0); \mathbb{R}^n) \)

\[
E|x(k \Delta; 0, \xi)|^2 \leq H \| \xi \|_E^2 e^{-\gamma k \Delta}, \quad \forall k \geq 0.
\]

(2.4)

In our analysis we find it convenient to work with continuous-time approximations and hence we define

\[
x(t) = x(0) + \int_0^t f(z(r), z(r - \tau)) \, dr + \int_0^t g(z(r), z(r - \tau)) \, dw(r),
\]

(2.5)

where

\[
z(t) = \sum_{k=-N}^{\infty} x(k \Delta) \mathbf{1}_{[k \Delta, (k+1) \Delta)}(t)
\]

with \( \mathbf{1}_G \) denoting the indicator function for the set \( G \). Again we will write this continuous EM solution as \( x(t; 0, \xi) \) to highlight the initial data \( \xi \) given at time \( t = 0 \) while the corresponding \( z(t) \) is written as \( z(t; 0, \xi) \). We may similarly define exponential stability in mean square for the continuous EM method.

**Definition 2.3.** Given a stepsize \( \Delta = \tau / N \) for some positive integer \( N \), the continuous EM method is said to be exponentially stable in mean square on the SDDE (2.1) if there is a pair of positive constants \( \gamma \) and \( H \) such that for any initial data \( \xi \in L^2_{\mathcal{F}_0}([t \tau, 0); \mathbb{R}^n) \)

\[
E|x(t; 0, \xi)|^2 \leq H \| \xi \|_E^2 e^{-\gamma t}, \quad \forall t \geq 0.
\]

(2.6)

The following proposition shows that the exponential stability in mean square of the discrete EM method is equivalent to that of the continuous EM method.

**Proposition 2.1.** Under (H1) the discrete EM method on the SDDE (2.1) is exponentially stable in mean square with rate constant \( \gamma \) and growth constant \( \bar{H} \) if and only if the continuous EM method is exponentially stable in mean square with the same rate constant \( \gamma \) but may be a different growth constant \( H \). Moreover, their growth constants \( \bar{H} \) and \( H \) can be made arbitrarily close by taking \( \Delta \) sufficiently small.

**Proof.** Obviously (2.6) implies (2.4) and in this case we even have \( \bar{H} = H \) so we need only to show (2.6) from (2.4). Fix \( z \) and write \( x(t; 0, z) = x(t) \). For any \( t \geq 0 \) choose \( k \geq 0 \) for \( t \in [k \Delta, (k+1) \Delta) \). Note that

\[
x(t) = x(k \Delta) + f(x(k \Delta), x((k-N) \Delta)(t-k \Delta) + g(x(k \Delta), x((k-N) \Delta))(w(t) - w(k \Delta)).
\]

Let \( \varepsilon > 0 \) be arbitrary. By (H1), it is straightforward to show that

\[
E|x(t)|^2 \leq (1 + \varepsilon) E|x(k \Delta)|^2 + 2(1 + \varepsilon^{-1}) \Delta(K_1 + K_2)[E|x(k \Delta)|^2 + E|x((k-N) \Delta)|^2].
\]

Using (2.4) we have

\[
E|x(t)|^2 \leq \bar{H} \| \xi \|_E^2 e^{-\gamma k \Delta}|(1 + \varepsilon) + 2(1 + \varepsilon^{-1}) \Delta(K_1 + K_2)e^{\gamma \tau}|.
\]

Consequently, (2.6) follows by setting

\[
H = \bar{H} e^{\gamma \Delta}|(1 + \varepsilon) + 2(1 + \varepsilon^{-1}) \Delta(K_1 + K_2)e^{\gamma \tau}.
\]

which can be made arbitrarily close to \( \bar{H} \) by choosing \( \varepsilon \) and then \( \Delta \) sufficiently small. \( \square \)
From now on, our analysis will be based on the continuous EM method, but the proposition above shows that in practice one needs only work on the discrete EM method.

In what follows we shall often need to use the continuous EM solution \( x(t; s, \zeta) \) to the SDDE (2.1) for initial data \( \zeta \in L^2_{\mathbb{F}^0}([-\tau, 0]; \mathbb{R}^n) \) given at time \( s \geq 0 \). This \( x(t; s, \zeta) \) can be defined in the same way as \( x(t; 0, \zeta) \). That is, compute the discrete approximations \( x(s + k\Delta; s, \zeta) \), form the step process \( z(t; s, \zeta) = z(t) \) and then define

\[
  x(t; s, \zeta) = \xi(0) + \int_s^t f(z(r), z(r - \tau)) \, dr + \int_s^t g(z(r), z(r - \tau)) \, dw(r).
\]

As for the exact solution \( y(t; 0, \zeta) \), the EM solutions have the following flow property too:

\[
  x(t; 0, \zeta) = x(t; s, x_s) \quad \forall 0 \leq s < t < \infty
\]

provided \( s \) is the multiple of \( \Delta \).

3. EM method shares stability with SDDEs

In this section by the EM method we mean the continuous EM method. The EM approximate solution \( x(t; 0, \zeta) \) depends clearly on the stepsize \( \Delta \) so we should have written it as \( x_{\Delta}(t; 0, \zeta) \), as we will do so in Section 4 below, but for the sake of simplicity we shall still use notation \( x(t; 0, \zeta) \) in this section.

It is of interest to ask whether the EM method shares exponential mean-square stability with the SDDEs. The results below answer this question positively.

**Theorem 3.1.** Let (H1) hold. Assume that the SDDE (2.1) is exponentially stable in mean square, namely

\[
  \mathbb{E}|y(t; 0, \zeta)|^2 \leq M \|\zeta\|^2 e^{-\gamma t}, \quad \forall t \geq 0
\]

for all \( \zeta \in L^2_{\mathbb{F}^0}([-\tau, 0]; \mathbb{R}^n) \). Then there exists a \( \Delta^* > 0 \) such that for every \( \Delta < \Delta^* \), the EM method is exponentially stable in mean square on the SDDE with rate constant \( \gamma \) and growth constant \( H \), both of which are independent of \( \Delta \). More precisely,

\[
  \mathbb{E}|x(t; 0, \zeta)|^2 \leq H \|\zeta\|^2 e^{-\gamma t}, \quad \forall t \geq 0
\]

with \( \gamma = \frac{1}{2} \lambda \) and \( H = 2MC_1e^{1/2\lambda T} \), where

\[
  C_1 = 3[1 + \tau(K_1 + K_2)]e^{3[\tau(K_1 + K_2)]} \quad \text{and} \quad T = \tau(9 + \ln[4\log(M)/\lambda T]).
\]

Let us emphasize that the rate constant \( \gamma \) and the growth constant \( H \) obtained in this Theorem are independent of \( \Delta \). To prove this theorem, let us present a number of lemmas.

**Lemma 3.1.** Under (H1),

\[
  \sup_{-\tau \leq t \leq \tau} \mathbb{E}|x(t; 0, \zeta)|^2 \leq C_1 \|\zeta\|^2,
\]

where \( C_1 \) was defined in Theorem 3.1.

**Proof.** Write \( x(t; 0, \zeta) = x(t) \). Using the Hölder inequality, the Itô isometry and hypotheses (H1), we compute from (2.5) that

\[
  \mathbb{E}|x(t)|^2 \leq 3\mathbb{E}|\xi(0)|^2 + 3\tau\mathbb{E} \int_0^t |f(z(s), z(s - \tau))|^2 \, ds + 3\mathbb{E} \int_0^t |g(z(s), z(s - \tau))|^2 \, ds
\]

\[
  \leq 3\mathbb{E}|\xi(0)|^2 + 3[1 + \tau(K_1 + K_2)] \int_0^t (\mathbb{E}|z(s)|^2 + \mathbb{E}|z(s - \tau)|^2) \, ds
\]

\[
  \leq 3[1 + \tau(K_1 + K_2)] \|\zeta\|^2 + 3(\tau(K_1 + K_2)) \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}|x(r)|^2 \, dr.
\]
Now for any $t_1 \in [0, \tau]$, by taking the supremum for $t \in [0, t_1]$ on both sides of the inequality above we obtain

$$
\sup_{0 \leq r \leq t} \mathbb{E}|x(t)|^2 \leq 3[1 + \tau(K_1 + K_2)]\|\xi\|_E^2 + 3(\tau K_1 + K_2) \int_0^{t_1} \sup_{0 \leq r \leq s} \mathbb{E}|x(r)|^2 \, ds.
$$

So the assertion follows from the well-known Gronwall inequality. □

**Lemma 3.2.** Under (H1),

$$
\sup_{0 \leq t \leq \tau + T} \mathbb{E}|x(t; 0, \xi)|^2 \leq C_2 \|\xi\|_E^2 \quad \text{for } \forall T > 0,
$$

where $C_2 = C_2(T) = 3C_1 e^{6T (K_1 + K_2)}$ and $C_1$ was defined in Theorem 3.1.

**Proof.** Again write $x(t; 0, \xi) = x(t)$. In the same way as in the proof of Lemma 3.1 we can show that for $t \in [\tau, \tau + T]$, $x(t) = x(t; 0, \xi)$,

$$
\mathbb{E}|x(t)|^2 \leq 3\mathbb{E}|x(\tau)|^2 + 3(T K_1 + K_2) \int_{\tau}^{t} (\mathbb{E}|z(s)|^2 + \mathbb{E}|z(s - \tau)|^2) \, ds
$$

$$
\leq 3C_1 \|\xi\|_E^2 + 6(T K_1 + K_2) \int_{\tau}^{t} \sup_{0 \leq r \leq s} \mathbb{E}|x(r)|^2 \, ds.
$$

Using Lemma 3.1 and noting that the right-hand side term of the inequality above is increasing in $t$, we obtain

$$
\sup_{0 \leq r \leq t} \mathbb{E}|x(r)|^2 \leq 3C_1 \|\xi\|_E^2 + 6(T K_1 + K_2) \int_{\tau}^{t} \sup_{0 \leq r \leq s} \mathbb{E}|x(r)|^2 \, ds
$$

and the assertion follows from the well-known Gronwall inequality. □

**Lemma 3.3.** Under (H1), for any $T > 0$,

$$
\mathbb{E}|x(t; 0, \xi) - z(t; 0, \xi)|^2 \leq C_3 \|\xi\|_E^2 A \quad \text{for } \forall t \in [0, \tau + T],
$$

where $C_3 = C_3(T) = 4(\tau K_1 + K_2)C_2(T)$ and $C_2(T)$ was defined in Lemma 3.2.

**Proof.** Write $x(t; 0, \xi) = x(t)$ and $z(t; 0, \xi) = z(t)$. For any $t \in [0, \tau + T]$, choose $k$ for $t \in [kA, (k + 1)A)$. Note that

$$
x(t) - z(t) = x(t) - x(kA) = \int_{kA}^{t} f(z(s), z(s - \tau)) \, ds + \int_{kA}^{t} g(z(s), z(s - \tau)) \, dw(s).
$$

Hence

$$
\mathbb{E}|x(t) - z(t)|^2 \leq 2A \mathbb{E} \int_{kA}^{t} |f(z(s), z(s - \tau))|^2 \, ds + 2A \mathbb{E} \int_{kA}^{t} |g(z(r), z(r - \tau))|^2 \, dr
$$

$$
\leq 2(\tau K_1 + K_2) \int_{kA}^{(k+1)A} (\mathbb{E}|z(s)|^2 + \mathbb{E}|z(s - \tau)|^2) \, ds.
$$

Applying Lemma 3.2 and noting $C_2 \geq 1$ we have

$$
\mathbb{E}|x(t) - z(t)|^2 \leq 4(\tau K_1 + K_2)C_2(T) \|\xi\|_E^2 A
$$

as required. □

**Lemma 3.4.** Write $x(t; 0, \xi) = x(t)$ and recall the definition $x_\tau = \{x(u) : 0 \leq u \leq \tau\}$. Define $y(t) = y(t; \tau, x_\tau)$ which is the solution to the SDDE (2.1) with initial data $x_\tau$ given at $t = \tau$. Then

$$
\sup_{\tau \leq t \leq \tau + T} \mathbb{E}|x(t) - y(t)|^2 \leq C \|\xi\|_E^2 A \quad \text{for } \forall T > 0,
$$

(3.2)
Thus
\[ C = C(T) = 2(TK_1 + K_2)(4T + \tau)C_3(T)e^{8T(TK_1 + K_2)} \]
and \( C_3(T) \) is the same as defined in Lemma 3.3.

**Proof.** Using the Hölder inequality, the Itô isometry and hypotheses (H1), we can show that for \( \tau \leq t \leq \tau + T \),
\[
\mathbb{E}|x(t) - y(t)|^2 \leq 2(TK_1 + K_2) \int_t^\tau \left[ \mathbb{E}|z(s) - y(s)|^2 + \mathbb{E}|z(s - \tau) - y(s - \tau)|^2 \right] ds
\]
\[
\leq 4(TK_1 + K_2) \int_\tau^\tau \mathbb{E}|z(s) - y(s)|^2 ds + 2(TK_1 + K_2) \int_0^\tau \mathbb{E}|z(s) - y(s)|^2 ds.
\]

But, by Lemma 3.3, if \( 0 \leq s \leq \tau \)
\[
\mathbb{E}|z(s) - y(s)|^2 = \mathbb{E}|z(s) - x(s)|^2 \leq C_3(T)\|\xi\|_E^2 A
\]
while if \( \tau \leq s \leq \tau + T \)
\[
\mathbb{E}|z(s) - y(s)|^2 \leq 2\mathbb{E}|z(s) - x(s)|^2 + 2\mathbb{E}|x(s) - y(s)|^2
\]
\[
\leq 2C_3(T)\|\xi\|_E^2 A + 2\mathbb{E}|x(s) - y(s)|^2.
\]
Thus
\[
\mathbb{E}|x(t) - y(t)|^2 \leq 2(TK_1 + K_2)(4T + \tau)C_3(T)\|\xi\|_E^2 A + 8(TK_1 + K_2) \int_\tau^\tau \mathbb{E}|x(s) - y(s)|^2 ds
\]
whence the desired result (3.2) follows from the Gronwall inequality. \( \square \)

We can now begin to prove Theorem 3.1. Let us draw the reader’s attention to the fact that only properties (3.1) and (3.2) rather than hypotheses (H1) themselves are used in the following proof. This is particularly useful when we generalize our results in Section 4 below.

**Proof of Theorem 3.1.** Fix any initial data \( \xi \), write \( x(t; 0, \xi) = x(t) \) and define \( y(t) = y(t; \tau, x_\tau) \). The exponential stability in mean square of the SDDE (2.1) shows
\[
\mathbb{E}|y(t)|^2 \leq M\|x_\tau\|_E^2 e^{-\lambda(t - \tau)} \quad \text{for } \forall t \geq \tau.
\]  
(3.3)

It is straightforward to see from the definition of \( T \) that
\[
Me^{-\lambda(T - 2\tau)} \leq e^{-3/4T}.
\]  
(3.4)

Now, for any \( \alpha > 0 \),
\[
\mathbb{E}|x(t)|^2 \leq (1 + \alpha)\mathbb{E}|x(t) - y(t)|^2 + (1 + \alpha^{-1})\mathbb{E}|y(t)|^2.
\]  
(3.5)

By (3.2) and (3.3)
\[
\sup_{T - \tau \leq t \leq 2T - \tau} \mathbb{E}|x(t)|^2 \leq (1 + \alpha)\beta_1 A\|\xi\|_E^2 + (1 + \alpha^{-1})M\|x_\tau\|_E^2 e^{-\lambda(T - 2\tau)}
\]
\[
\leq [(1 + \alpha)\beta_1 A + (1 + \alpha^{-1})Me^{-\lambda(T - 2\tau)}] \sup_{-\tau \leq t \leq \tau} E|x(t)|^2
\]
with \( \beta_1 = C(2T - 2\tau) \), where \( C(\cdot) \) has been defined in Lemma 3.4. Choosing
\[
\alpha = \sqrt{\frac{Me^{-\lambda(T - 2\tau)}}{\beta_1 A}}
\]
we obtain that
\[
\sup_{T - \tau \leq t \leq 2T - \tau} \mathbb{E}|x(t)|^2 \leq \kappa(\Delta) \sup_{-\tau \leq t \leq \tau} E|x(t)|^2,
\] (3.6)
where
\[
\kappa(\Delta) = \beta_1 \Delta + 2\sqrt{\beta_1 M \Delta e^{-1/2\Delta(T - 2\tau)}} + M e^{-\Delta(T - 2\tau)}.
\]
Using (3.4) we observe that
\[
\kappa(\Delta) \leq Me^{-\Delta(T - 2\tau)} \leq e^{-3/4\Delta T}.
\]
Since \(\kappa(\Delta)\) increases monotonically with \(\Delta\), there exists a \(\Delta^* > 0\) such that \(\kappa(\Delta) \leq e^{-1/2\Delta T}\) for all \(\Delta \leq \Delta^*\). In (3.6) this gives
\[
\sup_{T - \tau \leq t \leq 2T - \tau} \mathbb{E}|x(t)|^2 \leq e^{-1/2\Delta T} \sup_{-\tau \leq t \leq \tau} E|x(t)|^2.
\] (3.7)
Recall that \(T\) is a multiple of \(\tau\) and hence of \(\Delta\). So, by the flow property of the EM solutions, for any integer \(i \geq 0\)
\[
x(t) = x(t; i T, x_{iT}), \quad \forall t \geq i T.
\]
Repeating the argument above for \(x(t; i T, x_{iT})\) in the same way that (3.7) was obtained we may establish
\[
\sup_{(i+1)T - \tau \leq t \leq (i+2)T - \tau} \mathbb{E}|x(t)|^2 \leq e^{-1/2\Delta T} \sup_{iT - \tau \leq t < iT + \tau} E|x(t)|^2.
\] (3.8)
From this we see that
\[
\begin{align*}
\sup_{(i+1)T - \tau \leq t \leq (i+2)T - \tau} \mathbb{E}|x(t)|^2 & \leq e^{-1/2\Delta T} \sup_{iT - \tau \leq t < iT + \tau} E|x(t)|^2 \\
& \leq e^{-1/2\Delta(i+1)T} \sup_{-\tau \leq t \leq T - \tau} E|x(t)|^2.
\end{align*}
\] (3.9)
Now, by (3.2) and (3.3), we see from (3.5) that
\[
\sup_{\tau \leq t \leq T - \tau} \mathbb{E}|x(t)|^2 \leq [(1 + \alpha) \beta_2 A \|\xi\|_E^2 + (1 + \alpha^{-1}) M \|x_t\|_E^2]
\leq [(1 + \alpha) \beta_2 A + (1 + \alpha^{-1}) M] \sup_{-\tau \leq t \leq \tau} E|x(t)|^2
\]
with \(\beta_2 = C(T - \tau)\). Putting \(\alpha = \sqrt{M/\beta_2 A}\) yields
\[
\sup_{\tau \leq t \leq T - \tau} \mathbb{E}|x(t)|^2 \leq [\beta_2 A + 2\sqrt{\beta_2 \Delta M}] \sup_{-\tau \leq t \leq \tau} E|x(t)|^2.
\]
We can, if necessary, let \(\Delta^*\) be even smaller so that \(\beta_2 A + 2\sqrt{\beta_2 \Delta M} + M \leq 2M\) for all \(\Delta \leq \Delta^*\) and hence
\[
\sup_{\tau \leq t \leq T - \tau} \mathbb{E}|x(t)|^2 \leq 2M \sup_{-\tau \leq t \leq \tau} E|x(t)|^2.
\]
Substituting this into (3.9) and bearing in mind that \(M\) must not be less than 1 we obtain that
\[
\begin{align*}
\sup_{(i+1)T - \tau \leq t \leq (i+2)T - \tau} \mathbb{E}|x(t)|^2 & \leq 2M e^{-1/2\Delta(i+1)T} \sup_{-\tau \leq t \leq \tau} E|x(t)|^2 \\
& \leq 2MC_1 \|\xi\|_E^2 e^{-1/2\Delta(i+1)T}, \quad \forall i \geq 0.
\end{align*}
\]
while
\[ \sup_{0 \leq t \leq T-\tau} \mathbb{E}[x(t)]^2 \leq 2MC_1 \xi^2 \|\xi\|^2_E. \]

Hence,
\[ \mathbb{E}[x(t)]^2 \leq 2MC_1 e^{1/2\lambda T} \|\xi\|^2_E e^{-1/2\lambda t}, \quad \forall t \geq 0. \]

That is, the EM method is exponentially stable in mean square with \( \gamma = 1/2\lambda \) and \( H = 2MC_1 e^{1/2\lambda T} \). This completes the proof. \( \square \)

Theorem 3.1 shows that the exponential stability in mean square of the SDDE (2.1) implies the exponential stability in mean square of the EM method for small \( \Delta \). Let us now establish the converse theorem.

**Theorem 3.2.** Let (H1) hold. Assume that for some \( \Delta > 0 \), the EM method is exponentially stable in mean square on the SDDE (2.1), namely
\[ \mathbb{E}[x(t; 0, \xi)]^2 \leq H \|\xi\|^2_E e^{-\gamma t}, \quad \forall t \geq 0 \]
for all \( \xi \in L^2_{\mathbb{F}}(\mathcal{F}_0; \mathbb{R}^n) \). Suppose we can verify
\[ \beta_3 \Delta + 2\sqrt{\beta_3 H} e^{-1/2(T-2\tau)} + He^{-\gamma(T-2\tau)} \leq e^{-1/2\gamma T}, \tag{3.10} \]
where \( T = \tau(9 + \ln(4 \log(H) / \gamma \tau)) \), \( \beta_3 = C(2T - 2\tau) \) and \( C(\cdot) \) was defined in Lemma 3.4. Then the SDDE is exponentially stable in mean square. More precisely,
\[ \mathbb{E}[y(t)]^2 \leq M \mathbb{E}[\xi]^2 e^{-\lambda t}, \quad \forall t \geq 0 \]
with
\[ \lambda = \frac{1}{4} \gamma \quad \text{and} \quad M = C_1 e^{1/2\gamma T} \left( \beta_4 \Delta + 2\sqrt{\beta_4 \Delta H} + H \right), \]
where \( C_1 \) was defined in Theorem 3.1 and \( \beta_4 = C(T - \tau) \).

The proof of this theorem is absolutely based on the following lemma which can be proved in the same way as Lemmas 3.1 and 3.4 were proved.

**Lemma 3.5.** Let (H1) hold. Then
\[ \sup_{0 \leq t \leq T} \mathbb{E}[y(t; 0, \xi)]^2 \leq C_1 \|\xi\|^2_E, \tag{3.11} \]
where \( C_1 \) was defined in Theorem 3.1. Moreover, write \( y(t; 0, \xi) = y(t) \) and define \( x(t) = x(t; \tau, y_\tau) \) which is the EM solution to the SDDE (2.1) with initial data \( y_\tau \) given at \( t = \tau \). Then
\[ \sup_{\tau \leq t \leq \tau + T} \mathbb{E}[x(t) - y(t)]^2 \leq C(T) \|\xi\|^2_E \Delta \quad \text{for} \; \forall T > 0, \tag{3.12} \]
where \( C(T) \) was defined in Lemma 3.4.

**Proof of Theorem 3.2.** The proof is similar to that of Theorem 3.1 so we only give the outline but highlight the different part.

Fix any initial data \( \xi \), write \( y(t; 0, \xi) = y(t) \) and define \( x(t) = x(t; \tau, y_\tau) \). The exponential stability in mean square of the EM shows
\[ \mathbb{E}[x(t)]^2 \leq H \|y_\tau\|^2_E e^{-\gamma(t-\tau)} \quad \text{for} \; \forall t \geq \tau. \tag{3.13} \]
By (3.12) and (3.13) we can show that
\[
\sup_{T - \tau \leq t \leq 2T - \tau} \mathbb{E}[y(t)]^2 \leq [\beta_3 A + 2\sqrt{\beta_3 H} \Delta e^{-1/2;2\gamma(T - 2\tau)} + H e^{-\gamma(T - 2\tau)}] \sup_{-\tau \leq t \leq \tau} \mathbb{E}[y(t)]^2.
\] (3.14)

Using (3.10) we get that
\[
\sup_{T - \tau \leq t \leq 2T - \tau} \mathbb{E}[y(t)]^2 \leq e^{-1/2;2\gamma T} \sup_{-\tau \leq t \leq \tau} \mathbb{E}[y(t)]^2.
\] (3.15)

Repeating this argument we find that
\[
\sup_{(i+1)T - \tau \leq t \leq (i+2)T - \tau} \mathbb{E}[y(t)]^2 \leq e^{-1/2;2\gamma(i+1)T} \sup_{-\tau \leq t \leq T - \tau} \mathbb{E}[x(t)]^2, \quad \forall i \geq 0.
\] (3.16)

On the other hand, by (3.12) and (3.13), we can show that
\[
\sup_{\tau \leq t \leq T - \tau} \mathbb{E}[y(t)]^2 \leq [\beta_4 A + 2\sqrt{\beta_4 \Delta H} + H] \sup_{-\tau \leq t \leq \tau} \mathbb{E}[y(t)]^2,
\]
where \(\beta_4 = C(T - \tau)\). This, together with (3.11), yields
\[
\sup_{-\tau \leq t \leq T - \tau} \mathbb{E}[y(t)]^2 \leq C_1 [\beta_4 A + 2\sqrt{\beta_4 \Delta H} + H] \|\xi\|^2_E.
\] (3.17)

It then follows from (3.16) and (3.17) that
\[
\mathbb{E}[x(t)]^2 \leq C_1 e^{1/2;2T} [\beta_4 A + 2\sqrt{\beta_4 \Delta H} + H] \|\xi\|^2_E e^{-1/2;2T}, \quad \forall t \geq 0
\]
as required. \(\Box\)

Combining Theorems 3.1 and 3.2 we obtain the following sufficient and necessary theorem.

**Theorem 3.3.** Under hypotheses (H1), the SDDE (2.1) is exponentially stable in mean square if and only if for some \(A > 0\), the EM method is exponentially stable in mean square with rate constant \(\gamma\) and growth constant \(H\) satisfying
\[
\beta_3 A + 2\sqrt{\beta_3 H} \Delta e^{-1/2;2\gamma(T - 2\tau)} + H e^{-\gamma(T - 2\tau)} \leq e^{-1/2;2T},
\] (3.18)
where \(T = \tau(9 + \ln[4 \log(H)/\gamma \tau])\), \(\beta_3 = C(2T - 2\tau)\) and \(C(\cdot)\) was defined in Lemma 3.4.

**Proof.** The sufficient part follows from Theorem 3.2 directly. To show the necessary part, assume that the SDDE (2.1) is exponentially stable in mean square. Then, by Theorem 3.1, there exists a \(A^* > 0\) such that for every \(A < A^*\), the EM method is exponentially stable in mean square on the SDDE with rate constant \(\gamma\) and growth constant \(H\), both of which are independent of \(A\). With these \(\gamma\) and \(H\), compute \(T = \tau(9 + \ln[4 \log(H)/\gamma \tau])\) and note
\[
H e^{-\gamma(T - 2\tau)} \leq H \exp(-\frac{3}{4} \gamma T - e^{\frac{1}{4} (8\gamma \tau + 4 \ln \log(H)) \gamma T} < e^{-3/4;2T} < e^{-1/2;T}.
\]
So we can always find a \(A\) sufficiently small for (3.18) to hold and this proves the necessary part. \(\Box\)

We emphasize that Theorem 3.3 is an “if and only if” result, and hence has important practical implications. Suppose, for example, that we need to find out whether a given SDDE is exponentially stable in mean square or not. Given that we fail to find an appropriate Lyapunov function or functional to show the exponential stability, we can carry out careful numerical simulations. We may then confidently infer that the underlying SDDE is exponential stability in mean square or not according to whether the numerical simulations indicate the same property or not.

**4. Generalized result**

In this section we shall replace the global Lipschitz condition with a more general condition. As a standing hypothesis we assume that the coefficients \(f\) and \(g\) of the SDDE (2.1) are sufficiently smooth so that it has the unique solution
y(t; 0, \xi) for any initial data y_0 = \xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n) given at time t = 0. Moreover, to highlight the role of stepsize \Delta, we will write the EM approximate solution as x_{\Delta}(t; 0, \xi).

As pointed out in the previous section, we recall that the proof of Theorem 3.1 uses only properties (3.1) and (3.2) rather than hypothesis (H1) itself while the proof of Theorem 3.2 makes use of only properties (3.11) and (3.12). So Theorem 3.3 will still hold if hypothesis (H1) is replaced by properties (3.1), (3.2), (3.11) and (3.12). This leads to the following definition and then a generalized result.

**Definition 4.1.** The SDDE (2.1) and the corresponding EM method are said to have Property (P1) if, for sufficiently small \Delta (=\tau/N), the following two conditions are satisfied:

1. There is a positive constant C_1 independent of \xi and \Delta such that
   \[
   \sup_{0 \leq t \leq \tau} [\mathbb{E}[y(t; 0, \xi)]^2 \vee \mathbb{E}[x_{\Delta}(t; 0, \xi)]^2] \leq C_1 \| \xi \|_E^2.
   \]
2. For each T > 0, there is a positive constant C = C(T) independent of \xi and \Delta such that
   \[
   \sup_{\tau \leq t \leq \tau + T} \mathbb{E}[x_{\Delta}(t; 0, \xi) - y(t; \tau, x_{\Delta}; \tau)]^2 \leq C \| \xi \|_E^2 \Delta
   \]
   with \( x_{\Delta, \tau} = \{x_{\Delta}(u; 0, \xi) : 0 \leq u \leq \tau \} \) while
   \[
   \sup_{\tau \leq t \leq \tau + T} \mathbb{E}[x_{\Delta}(t; \tau, y_\tau) - y(t; 0, \xi)]^2 \leq C \| \xi \|_E^2 \Delta
   \]
   with \( y_\tau = \{y(u; 0, \xi) : 0 \leq u \leq \tau \} \).

**Theorem 4.1.** Suppose that the SDDE (2.1) and the EM method have Property (P1). Then the SDDE (2.1) is exponentially stable in mean square if and only if for some \Delta > 0, the EM method is exponentially stable in mean square with rate constant \gamma and growth constant H satisfying
   \[
   \beta_3 \Delta + 2\sqrt{\beta_3 H \Delta e^{-1/2\gamma(T-2\tau)}} + HE^{-\gamma(T-2\tau)} \leq e^{-1/2\gamma T},
   \]
where \( T = \tau + \ln[4\log(H)/\gamma] \), \( \beta_3 = C(2T - 2\tau) \) and \( C(\cdot) \) was given by Property (P1).

Let us remark that Property (P1) is a finite-time convergence property while the exponential stability is an asymptotic property. The important feature of Theorem 4.1 is that it transfers the asymptotic problem into a finite-time problem. That is, in order to reveal that the EM method shares exponential mean-square stability with the SDDE, it is enough to prove a theorem like (4.1). The theory above works if y(t; s, \xi) is a solution to a different type of equations e.g. an SDDE with variable time lag; \( \{x_{\Delta}(t; s, \xi)\} \) are obtained by other numerical scheme rather than the EM method or they are solutions to an SDDE with parameter \Delta.

These observations inspire the general treatment of this section.

In this section we shall write \( L^2_{\mathcal{F}}([-\tau; 0]; \mathbb{R}^n) = L^2_{\mathcal{F}} \), for simplicity. Let \( \mathcal{S} \subset (0, \infty) \) be an index set. For example, \( \mathcal{S} = \{\tau/N : N = 1, 2, \ldots, \} \) or \( (0, 1] \). Let \( \mathcal{K} \) be a family of continuous strictly increasing functions \( \kappa : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \kappa(0) = 0 \). Let us now give a number of definitions.
**Definition 5.1.** Let $\Delta > 0$. A stochastic process $\{y(t; s, \xi) : s \in \mathbb{R}_+, s - \tau \leq t < \infty, \xi \in L^2_{\mathcal{F}_t}\}$, which will be written as $\{y(t; s, \xi)\}$ thereafter for simplicity, is said to be an $L^2_{\mathcal{F}_t}$-related $\Delta$-periodic stochastic flow if it satisfies the following three conditions:

1. $\{y(s + u; s, \xi) : -\tau \leq u \leq 0\} = \xi,$
2. $y_t := \{y(t + u; s, \xi) : -\tau \leq u \leq 0\} \in L^2_{\mathcal{F}_t}$ for $\forall t \geq s,$
3. $y(t; s, \xi) = y(t; s + k\Delta, y_{s+k\Delta})$ for $\forall t \geq s + k\Delta$ and $k = 0, 1, 2, \ldots.$

The process is said to be an $L^2_{\mathcal{F}_t}$-related stochastic flow if it is an $L^2_{\mathcal{F}_t}$-related $\Delta$-periodic stochastic flow for any $\Delta > 0$.

**Definition 5.2.** Let $\{y(t; s, \xi)\}$ be an $L^2_{\mathcal{F}_t}$-related stochastic flow and, for each $\Delta \in \mathcal{F}$, let $\{x_\Delta(t; s, \xi)\}$ be an $L^2_{\mathcal{F}_t}$-related $\Delta$-periodic stochastic flow. They are said to have Property (P2) if the following two conditions are satisfied:

1. There is a positive constant $C_1$ independent of $s, \xi$ and $\Delta$ such that
   $$\sup_{0 \leq u \leq \tau} [E|y(s + u; s, \xi)|^2 \vee E|x_\Delta(s + u; s, \xi)|^2] \leq C_1 \|\xi\|^2_{\mathcal{F}_s}.$$
2. There is a function $\kappa \in \mathcal{K}$ and, for each $T > 0$, there is a positive constant $C = C(T)$ independent of $s, \xi$ and $\Delta$, such that
   $$\sup_{\tau \leq u \leq \tau + T} E|x_\Delta(s + u; s, \xi) - y(s + u; s + \tau, x_\Delta(s+\tau))|^2 \leq C \|\xi\|^2_{\mathcal{F}_s} \kappa(\Delta),$$
   and
   $$\sup_{\tau \leq u \leq \tau + T} E|x_\Delta(s + u; s + \tau, y_{s+\tau}) - y(s + u; s, \xi)|^2 \leq C \|\xi\|^2_{\mathcal{F}_s} \kappa(\Delta),$$
   where $x_\Delta(t) := \{x_\Delta(t + u; s, \xi) : -\tau \leq u \leq 0\}$ and $y_t := \{y(t + u; s, \xi) : -\tau \leq u \leq 0\}.$

Let us remark that the exact solution process $\{y(t; s, \xi)\}$ of the SDE (2.1) is an $L^2_{\mathcal{F}_t}$-related stochastic flow while the EM approximate solution process $\{x_\Delta(t; s, \xi)\}$ is an $L^2_{\mathcal{F}_t}$-related $\Delta$-periodic stochastic flow.

**Definition 5.3.** A stochastic process $\{y(t; s, \xi)\}$ is said to be exponentially stable in mean square if there is a pair of positive constants $\lambda$ and $M$ such that
   $$E|y(t; s, \xi)|^2 \leq M \|\xi\|^2_{\mathcal{F}_s} e^{-\lambda(t-s)}$$
   for all $0 \leq s \leq t < \infty$ and $\xi \in L^2_{\mathcal{F}_s}$. We refer to $\lambda$ as the rate constant and $M$ as the growth constant.

The following theorem shows that processes with Property (P2) share exponential stability in mean square.

**Theorem 5.1.** Let $\{y(t; s, \xi)\}$ be an $L^2_{\mathcal{F}_t}$-related stochastic flow while let, for each $\Delta \in \mathcal{F}$, $\{x_\Delta(t; s, \xi)\}$ be an $L^2_{\mathcal{F}_t}$-related $\Delta$-periodic stochastic flow. Suppose that they have Property (P2). Then the process $\{y(t; s, \xi)\}$ is exponentially stable in mean square if and only if for some $\Delta \in \mathcal{F}$, the process $\{x_\Delta(t; s, \xi)\}$ is exponentially stable in mean square with rate constant $\gamma$ and growth constant $H$ satisfying
   $$\beta_3 \kappa(\Delta) + 2\sqrt{\beta_3 H \kappa(\Delta)} e^{-1/2}\gamma \leq e^{-1/2}T, \quad H e^{-\gamma(T-2\tau)} \leq e^{-1/2}T,$$
   where $T = \tau(9 + \ln(4 \log(H) / \gamma T))$, $\beta_3 = C(2T - 2\tau)$ and $C(\cdot)$ was given by Property (P2).

This theorem can be proved in the same way as Theorems 3.1–3.3 were proved but the details are left to the reader. Instead, let us discuss a couple of cases to show the power of the general treatment above.
5.1. Nonautonomous SDDE

Let
\[ f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}. \]

For each \( A \in \mathcal{A} := (0, 1] \), let
\[ f_A : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n \quad \text{and} \quad g_A : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}. \]

We impose the following hypotheses:

(H2) Assume that

- all the functions \( f(x, y, t) \), \( g(x, y, t) \), \( f_A(x, y, t) \) and \( g_A(x, y, t) \) are locally Lipschitz continuous in \( x \) and \( y \);
- \( f(0, 0, t) = f_A(0, 0, t) \equiv 0 \) and \( g(0, 0, t) = g_A(0, 0, t) \equiv 0 \);
- there is a function \( \kappa \in \mathscr{K} \) and a positive constant \( K \) (independent of \( A \)) such that
  \[ \langle x - \bar{x}, f(x, y, t) - f_A(\bar{x}, \bar{y}, t) \rangle \leq \kappa(A)(|x|^2 + |y|^2 + |\bar{x}|^2 + |\bar{y}|^2) + K(|x - \bar{x}|^2 + |y - \bar{y}|^2) \]
  and
  \[ |g(x, y, t) - g_A(\bar{x}, \bar{y}, t)|^2 \leq \kappa(A)(|x|^2 + |y|^2 + |\bar{x}|^2 + |\bar{y}|^2) + K(|x - \bar{x}|^2 + |y - \bar{y}|^2) \]
  for all \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \), \( t \geq 0 \) and \( A \in (0, 1] \).

Note from (H2) that
\[ \langle x, f(x, y, t) \rangle = \langle x, f(x, y, t) - f_A(0, 0, t) \rangle \leq \kappa(A)(|x|^2 + |y|^2) \leq \tilde{K}(|x|^2 + |y|^2), \]
where \( \tilde{K} = K + \kappa(1) \). Similarly,
\[ \langle x, f_A(x, y, t) \rangle \vee |g(x, y, t)|^2 \vee |g_A(x, y, t)|^2 \leq \tilde{K}(|x|^2 + |y|^2). \]

Consider the nonautonomous SDDE
\[ dy(t) = f(y(t), y(t - \tau), t) \, dt + g(y(t), y(t - \tau), t) \, dw(t) \]
on \( t \geq s \) with initial data \( y_s = \zeta \in L^2_{\mathcal{F}_s} \).

**Lemma 5.1.** Under (H2), the SDDE (5.3) has a unique solution \( y(t) \) on \( t \geq s - \tau \). Moreover, for any \( T > 0 \), there is a positive constant \( h(T) \) independent of \( s \) and \( \zeta \) such that
\[ \mathbb{E} \left( \sup_{s \leq t \leq s + T} |y(t)|^2 \right) \leq h(T) \| \zeta \|_{L^2_{\mathcal{F}_s}}^2. \]

**Proof.** Since both \( f \) and \( g \) are locally Lipschitz continuous, Theorem 3.2.2 of Mao [21, p. 95] shows that there is a unique maximal local solution \( y(t) \) on \( t \in [s, \sigma_\infty] \), where \( \sigma_\infty \) is the explosion time. We need to show \( \sigma_\infty = \infty \) a.s. For this purpose, we define, for each integer \( k \geq 1 \), the stopping time
\[ \sigma_k = \inf\{t \geq s : |y(t)| \geq k\}, \]
where we set \( \inf \emptyset = \infty \) as usual. Clearly, \( \sigma_k \) is increasing in \( k \) while \( \sigma_k \leq \sigma_\infty \) a.s. Fix any \( k \) and \( T > 0 \). For any \( t \in [s, s + T] \), we derive by the Itô formula that
\[ |y(t \wedge \sigma_k)|^2 = |y(s)|^2 + \int_s^{t \wedge \sigma_k} \left[ 2\langle y(r), f(y(r), y(r - \tau), r) \rangle + |g(y(r), y(r - \tau), r)|^2 \right] dr + M(t), \]
where
\[ M(t) = 2 \int_s^{t \wedge \sigma_k} (y(r), g(y(r), y(r - \tau), r) \, dB(r)). \]

By (5.1) and (5.2) we have
\[
\begin{align*}
\mathbb{E} \int_s^{t \wedge \sigma_k} [2(y(r), f(y(r), y(r - \tau), r) + g(y(r), y(r - \tau), r))^2] \, dr \\
\leq 3\tilde{K} \mathbb{E} \int_s^{t \wedge \sigma_k} [|y(r)|^2 + |y(r - \tau)|^2] \, dr \\
\leq 3\tau\tilde{K} \|\xi\|_{L^2}^2 + 6\tilde{K} \int_s^t \mathbb{E} \left( \sup_{s \leq u \leq r} |y(u \wedge \sigma_k)|^2 \right) \, dr.
\end{align*}
\]

Moreover, by the Burkholder–Davis–Gundy inequality (cf. Mao [21, p. 7]) and (5.2), we derive that
\[
\begin{align*}
\mathbb{E} \left( \sup_{s \leq u \leq t} M(u) \right) & \leq 4\sqrt{2} \mathbb{E} \left( \int_s^{t \wedge \sigma_k} |y(r)|^2 |g(y(r), y(r - \tau), r)|^2 \, dr \right)^{1/2} \\
& \leq 4\sqrt{2} \mathbb{E} \left( \sup_{s \leq r \leq t} |y(r \wedge \sigma_k)|^2 \right)^{1/2} \int_s^{t \wedge \sigma_k} |g(y(r), y(r - \tau), r)|^2 \, dr \\
& \leq \frac{1}{2} \mathbb{E} \left( \sup_{s \leq r \leq t} |y(r \wedge \sigma_k)|^2 \right) + 16 \int_s^{t \wedge \sigma_k} |g(y(r), y(r - \tau), r)|^2 \, dr \\
& \leq \frac{1}{2} \mathbb{E} \left( \sup_{s \leq r \leq t} |y(r \wedge \sigma_k)|^2 \right) + 16\tau\tilde{K} \|\xi\|_{L^2}^2 + 32\tilde{K} \int_s^t \mathbb{E} \left( \sup_{s \leq u \leq r} |y(u \wedge \sigma_k)|^2 \right) \, dr.
\end{align*}
\]

Making use of these inequalities we obtain from (5.5) that
\[
\begin{align*}
\mathbb{E} \left( \sup_{s \leq r \leq t} |y(r \wedge \sigma_k)|^2 \right) & \leq 2(1 + 19\tau\tilde{K}) \|\xi\|_{L^2}^2 + 76\tilde{K} \int_s^t \mathbb{E} \left( \sup_{s \leq u \leq r} |y(u \wedge \sigma_k)|^2 \right) \, dr.
\end{align*}
\]

The Gronwall inequality gives
\[
\begin{align*}
\mathbb{E} \left( \sup_{s \leq r \leq s + T} |y(r \wedge \sigma_k)|^2 \right) & \leq 2(1 + 19\tau\tilde{K})e^{76\tilde{K} T} \|\xi\|_{L^2}^2.
\end{align*}
\]

(5.6).

This implies
\[
k^2 \mathbb{P} \{ \sigma_k \leq s + T \} \leq 2(1 + 19\tau\tilde{K})e^{76\tilde{K} T} \|\xi\|_{L^2}^2.
\]

Letting \( k \to \infty \) yields
\[
\lim_{k \to \infty} \mathbb{P} \{ \sigma_k \leq s + T \} = 0.
\]

Since \( T > 0 \) is arbitrary, we must have \( \sigma_k \to \infty \) a.s. and hence \( \sigma_\infty = \infty \) a.s. This proves the existence and uniqueness of the global solution. Finally, assertion (5.4) follows from (5.6) by letting \( k \to \infty \) and setting \( h(T) = 2(1 + 19\tau\tilde{K})e^{76\tilde{K} T} \).

Let us now denote this solution process by \( \{ y(t; s, \xi) \} \). It is obvious that \( \{ y(t; s, \xi) \} \) is an \( L_{\mathfrak{F}_t}^2 \)-related stochastic flow.

Now, for each \( \lambda \in (0, 1) \), consider the SDDE
\[
\begin{align*}
dx(t) = f_\lambda(x(t), x(t - \tau), t) \, dt + g_\lambda(x(t), x(t - \tau), t) \, dw(t)
\end{align*}
\]

(5.7).
on $t \geq s$ with initial data $x_s = \xi \in \mathbb{L}^2_{\mathcal{F}}$. This may be regarded as an approximated equation to the SDDE (5.3). Lemma 5.1 shows that under (H2), this SDDE has a unique solution on $t \geq s - \tau$. We denote the solution process by $\{x_{\mathcal{A}}(t; s, \xi)\}$, which is clearly an $\mathbb{L}^2_{\mathcal{F}}$-related stochastic flow and hence of course an $\mathbb{L}^2_{\mathcal{F}}$-related $\mathcal{A}$-periodic stochastic flow. Moreover, it has the property

$$
\mathbb{E} \left( \sup_{s \leq t \leq s + T} |x_{\mathcal{A}}(t; s, \xi)|^2 \right) \leq h(T) \|\xi\|^2_E \tag{5.8}
$$

with the same $h(T)$ defined as above.

**Lemma 5.2.** Under (H2), the solution processes $\{y(t; s, \xi)\}$ and $\{x_{\mathcal{A}}(t; s, \xi)\}$ of equations (5.3) and (5.7), respectively, have Property (P2).

**Proof.** Condition 1 of Property (P2) follows from (5.4) and (5.8). To show Condition 2 there, let us fix $s, \xi, \mathcal{A}$ and $T > 0$ arbitrarily. Write $x_{\mathcal{A}}(t; s, \xi) = x(t)$ on $t \geq s - \tau$ and $y(t; s + \tau, x_{\mathcal{A}}(s + \tau) = y(t)$ on $t \geq s$. It follows from (5.4) and (5.8) that

$$
\mathbb{E}|x(t)|^2 + \mathbb{E}|y(t)|^2 \leq \tilde{C} \|\xi\|^2_E, \quad s \leq t \leq s + \tau + T, \tag{5.9}
$$

where $\tilde{C} = h(\tau)h(T)$. For $t \in [s + \tau, s + \tau + T]$, the Itô formula implies

$$
\mathbb{E}|x(t) - y(t)|^2 = \mathbb{E}\int_{s+\tau}^{t} \left[ 2(x(r) - y(r), f_{\mathcal{A}}(x(r), x(r - \tau), r) - f(y(r), y(r - \tau), r)) + |g_{\mathcal{A}}(x(r), x(r - \tau), r) - g(y(r), y(r - \tau), r)|^2 \right] dr. \tag{5.10}
$$

By (H2) and (5.9) we then obtain that

$$
\mathbb{E}|x(t) - y(t)|^2 \leq 3 \kappa(\mathcal{A}) \mathbb{E}\int_{s+\tau}^{t} \left[ |x(r)|^2 + |y(r)|^2 + |x(r - \tau)|^2 + |y(r - \tau)|^2 \right] dr
+ 3K \mathbb{E}\int_{s+\tau}^{t} \left[ |x(r) - y(r)|^2 + |x(r - \tau) - y(r - \tau)|^2 \right] dr
\leq 12 \tilde{C} T \kappa(\mathcal{A}) \|\xi\|^2_E + 6K \mathbb{E}\int_{s+\tau}^{t} \left( \sup_{s+\tau \leq u \leq r} |x(u) - y(u)|^2 \right) dr.
$$

Since the right-hand side term of the above is increasing in $t$, we must have

$$
\mathbb{E}\left( \sup_{s+\tau \leq u \leq t} |x(u) - y(u)|^2 \right) \leq 12 \tilde{C} T \kappa(\mathcal{A}) \|\xi\|^2_E + 6K \mathbb{E}\int_{s+\tau}^{t} \left( \sup_{s+\tau \leq u \leq r} |x(u) - y(u)|^2 \right) dr.
$$

The Gronwall inequality shows that

$$
\mathbb{E}\left( \sup_{s+\tau \leq t \leq s+\tau+T} |x(t) - y(t)|^2 \right) \leq 12 \tilde{C} T e^{6KT} \kappa(\mathcal{A}) \|\xi\|^2_E.
$$

Similarly, we can show that

$$
\mathbb{E}\left( \sup_{s+\tau \leq t \leq s+\tau+T} |x_{\mathcal{A}}(t; s + \tau, y_{s+\tau}) - y(t; s, \xi)|^2 \right) \leq 12 \tilde{C} T e^{6KT} \kappa(\mathcal{A}) \|\xi\|^2_E.
$$

In other words, we have proved Condition 2 of Property (P2) so the proof of this lemma is complete. □

We can therefore conclude the following sufficient and necessary result from Theorem 5.1.
Theorem 5.2. Under (H2), the solution process \( \{y(t; s, \zeta)\} \) of the SDDE (5.3) is exponentially stable in mean square if and only if for some sufficiently small \( \delta \in (0, 1] \) the solution process \( \{x_\delta(t; s, \zeta)\} \) of the SDDE (5.7) is exponentially stable in mean square with rate constant \( \gamma \) and growth constant \( H \) satisfying

\[
\beta_3 \kappa(A) + 2\sqrt{\beta_3 \kappa(A)}e^{-1/2\gamma(T-2\tau)} + H e^{-\gamma(T-2\tau)} \leq e^{-1/2\gamma T},
\]

where

\[
T = (9 + \ln(4 \log(H)/\gamma \tau)), \quad \beta_3 = 24(T - \tau)\tilde{C}e^{6K(T-\tau)},
\]

\[
\tilde{C} = 4(1 + 19\tau \tilde{K})^2e^{152\tilde{K}(T-\tau)}.
\]

5.2. SDDEs with variable time lag

Let \( \delta : \mathbb{R}_+ \to [0, \tau] \) be a continuous function such that for some positive constant \( \eta \),

\[
|\delta(u) - \delta(v)| \leq \eta (u - v) \quad \forall 0 \leq v < u < \infty. \tag{5.11}
\]

Let \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) satisfy hypotheses (H1). Consider the SDDE with variable time lag

\[
dy(t) = f(y(t), y(t - \delta(t))) \, dt + g(y(t), y(t - \delta(t))) \, dw(t), \tag{5.12}
\]
on \( t \geq s \) with initial data \( y_s = \zeta \in L^2_{\mathcal{F}_s} \). It is known (cf. Mao [22, Theorem 5.3.2]) that the SDDE has a unique solution denoted here by \( y(t; s, \zeta) \) on \( t \geq s - \tau \).

Lemma 5.3. Under (H1), for any \( T > 0 \),

\[
\mathbb{E}|y(t; s, \zeta)|^2 \leq h_1(T)\|\zeta\|^2 \quad \forall s - \tau \leq t \leq s + T \tag{5.13}
\]

and

\[
\mathbb{E}|y(u; s, \zeta) - y(v; s, \zeta)|^2 \leq h_2(T)\|\zeta\|^2 (u - v) \quad \forall s \leq v < u \leq s + T, \tag{5.14}
\]

where

\[
h_1(T) = 3e^{6T(K_1 + K_2)} \quad \text{and} \quad h_2(T) = 4(T K_1 + K_2)h_1(T).
\]

This lemma can be proved in the same way as Lemmas 3.1–3.3 were proved so the details are omitted. It is therefore easy to observe that the solution process \( \{y(t; s, \zeta)\} \) is an \( L^2_{\mathcal{F}_s} \)-related stochastic flow.

Let us now modify the EM method to cope with the variable time lag. Let \( \mathcal{S} = \{\tau/N : N = 1, 2, \ldots\} \). Given a stepsize \( \delta \in \mathcal{S} \) and initial data \( \zeta \in L^2_{\mathcal{F}_s} \) at time \( s \), set

\[
x(s + k\delta) = \zeta(k\delta), \quad -\tau/\delta \leq k \leq 0;
\]

compute the discrete approximations

\[
x(s + (k + 1)\delta) = x(s + k\delta) + f(x(s + k\delta), x(s + k\delta - \ln[\delta(s + k\delta)/\delta]\delta))\delta
\]
\[
+ g(x(k\delta), x(s + k\delta - \ln[\delta(s + k\delta)/\delta]\delta))\Delta w_k, \quad k \geq 1, \tag{5.15}
\]

where \( \Delta w_k = w(s + (k + 1)\delta) - w(s + k\delta) \); and then form the continuous-time approximations

\[
x(t) = x(s) + \int_s^t f(z_1(r), z_2(r)) \, dr + \int_s^t g(z_1(r), z_2(r)) \, dw(r) \quad \text{on } t \geq s \tag{5.16}
\]

with \( x(s + u) = \zeta(u) \) for \( -\tau \leq u \leq 0 \), where

\[
z_1(t) = \sum_{k=0}^{\infty} x(s + k\delta)1_{[s+k\delta,s+(k+1)\delta)}(t)
\]
To highlight the dependence on $A$, $s$ and $\xi$, we shall write this EM approximate solution process as \{x_A(t; s, \xi)\}.

**Lemma 5.4.** Under (H1), for any $T > 0$,
\[
E|x_A(t; s, \xi)|^2 \leq h_1(T)\|\xi\|^2_E, \quad s - \tau \leq t \leq s + T
\]  
and
\[
E|x_A(u; s, \xi) - x_A(v; s, \xi)|^2 \leq h_2(T)\|\xi\|^2_E(u - v), \quad s \leq v < u \leq s + T,
\]
where $h_1(T)$ and $h_2(T)$ are the same as defined in Lemma 5.3.

Again this lemma can be proved in the same way as Lemmas 3.1–3.3 were proved so the details are omitted. It is also easy to see the following flow property:
\[
x_A(t; s, \xi) = x_A(t; s + kA, x_A(s + kA)) \quad \text{for } \forall t \geq s + kA, \quad k = 0, 1, 2, \ldots.
\]
Hence, \{x_A(t; s, \xi)\} is an $L^2_{\mathcal{F}_t}$-related $A$-periodic stochastic flow.

**Lemma 5.5.** Under (H1), the solution process \{y(t; s, \xi)\} of the SDDE (5.12) and the EM approximate processes \{x_A(t; s, \xi)\} ($A \in \mathcal{A}$) have Property (P2).

**Proof.** In view of Lemmas 5.3 and 5.4, we need only to show Condition 2 of Property (P2). Fix $s, \xi, A$ and $T > 0$ arbitrarily. Write $x_A(t; s, \xi) = x(t)$ and $y(t; s + \tau, x_A(s + \tau)) = y(t)$. By Lemma 5.4,
\[
E|x(u) - x(v)|^2 \leq c\|\xi\|^2_E(u - v), \quad s \leq v < u \leq s + \tau + T,
\]  
where $c = h_2(\tau + T)$. Now, for $t \in [s + \tau, s + \tau + T]$, it follows from the definitions of $z_1(t)$ and $z_2(t)$ as well as (5.19) that
\[
E|x(t) - z_1(t)|^2 \leq c\|\xi\|^2_E A;
\]  
while, for $k = \text{Int}[(t - s)/A]$ (i.e. $k$ is such that $s + kA \leq t < s + (k + 1)A$),
\[
E|x(t - \delta(t)) - z_2(t)|^2 \leq c\|\xi\|^2_E (|t - \delta(t) - s - kA + \text{Int}[\delta(s + kA)/A]|)
\]  
\[
\leq c\|\xi\|^2_E (A + \text{Int}[\delta(s + kA)/A]|A|).
\]
Recalling (5.11) and noting
\[
\delta(s + kA) - A \leq \text{Int}[\delta(s + kA)/A] \leq \delta(s + kA),
\]
we derive that
\[
|\delta(t) - \text{Int}[\delta(s + kA)/A]|A|
\]  
\[
\leq \begin{cases} 
\delta(t) - \delta(s + kA) + A \leq (\eta + 1)A & \text{if } \delta(t) \geq \delta(s + kA), \\
\delta(s + kA) - \delta(t) \leq \eta A & \text{if } \delta(t) \leq \delta(s + kA) - A,
\end{cases}
\]
In other words, we always have
\[
|\delta(t) - \text{Int}[\delta(s + kA)/A]|A| \leq (\eta + 1)A.
\]
Substituting this into (5.21) gives
\[
E|x(t - \delta(t)) - z_2(t)|^2 \leq c(\eta + 2)\|\xi\|^2_E A.
\]
By (H1), it is easy to show from (5.12) and (5.16) that for \( t \in [s + \tau, \ s + \tau + T] \),
\[
\mathbb{E}|x(t) - y(t)|^2 \leq 2(K_1 T + K_2) \int_{s+\tau}^{t} [\mathbb{E}|z_1(r) - y(r)|^2 + \mathbb{E}|z_2(r) - y(r - \delta(r))|^2] \, dr.
\]
But, by (5.20),
\[
\mathbb{E}|z_1(r) - y(r)|^2 \leq 2\mathbb{E}|x(r) - y(r)|^2 + 2\mathbb{E}|z_1(r) - x(r)|^2 \leq 2\mathbb{E}|x(r) - y(r)|^2 + 2c\|\xi\|^2 A,
\]
while, by (5.22),
\[
\mathbb{E}|z_2(r) - y(r - \delta(r))|^2 \leq 2\mathbb{E}|x(r - \delta(r)) - y(r - \delta(r))|^2 + 2\mathbb{E}|z_2(r) - x(r - \delta(r))|^2
\leq 2\mathbb{E}|x(r - \delta(r)) - y(r - \delta(r))|^2 + 2c(\eta + 2)\|\xi\|^2 A.
\]
So
\[
\mathbb{E}|x(t) - y(t)|^2 \leq 4c(K_1 T + K_2)(\eta + 3)\|\xi\|^2 A + 4(K_1 T + K_2) \int_{s+\tau}^{t} [\mathbb{E}|x(r) - y(r)|^2 + \mathbb{E}|x(r - \delta(r)) - y(r - \delta(r))|^2] \, dr.
\]
This implies that
\[
\sup_{s + \tau \leq t \leq t_1} \mathbb{E}|x(t) - y(t)|^2 \leq 4c(K_1 T + K_2)(\eta + 3)\|\xi\|^2 A + 8(K_1 T + K_2) \int_{s+\tau}^{t_1} \sup_{s + \tau \leq u \leq r} \mathbb{E}|x(u) - y(u)|^2 \, dr
\]
for \( s + \tau \leq t_1 \leq s + \tau + T \). So the Gronwall inequality yields
\[
\sup_{s + \tau \leq t \leq s + \tau + T} \mathbb{E}|x(t) - y(t)|^2 \leq C(T)\|\xi\|^2 A,
\]
(5.23)
where \( C(T) = 4(K_1 T + K_2)(\eta + 3)h_2(T + \tau)e^{8T(K_1 T + K_2)} \). Similarly, we can show that
\[
\sup_{s + \tau \leq t \leq s + \tau + T} \mathbb{E}|x_A(t; s + \tau, y_{s+\tau}) - y(t; s, \xi)|^2 \leq C(T)\|\xi\|^2 A.
\]
In other words, we have proved Condition 2 of Property (P2) so the proof of this lemma is complete. \( \square \)

By Theorem 5.1 we can therefore conclude the following sufficient and necessary result.

**Theorem 5.3.** Under (H1), the solution process \( \{y(t; s, \xi)\} \) of the SDDE (5.12) is exponentially stable in mean square if and only if for some sufficiently small \( A \in \mathcal{F} \), the EM approximate solution process \( \{x_A(t; s, \xi)\} \) is exponentially stable in mean square with rate constant \( \gamma \) and growth constant \( H \) satisfying
\[
\beta_3 A + 2\sqrt{\beta_3 HA} e^{-1/2\gamma(T - 2\tau)} + H e^{-\gamma(T - 2\tau)} \leq e^{-1/2\gamma T},
\]
where
\[
T = \tau(9 + \ln[4\log(H)/\gamma\tau]), \quad h_2 = 12(K_1(2T - \tau) + K_2)e^{6(2T - \tau)((2T - \tau)K_1 + K_2)},
\]
\[
\beta_3 = 4(K_1 T + K_2)(\eta + 3)h_2 e^{8T(2K_1(2T - \tau) + K_2)}.
\]

6. Linear SDDEs

One of the most important classes of SDDEs is the linear SDDEs which appear frequently in many branches of science and industry. Applying our theory developed above to the linear SDDEs we can obtain very useful “if and only if” result.
Let $\delta : \mathbb{R}_+ \to [0, \tau]$ be the same function defined in Section 5.2. Let $A, B, F_i, G_i$ $(1 \leq i \leq m)$ be $n \times n$-matrices. Consider the linear SDDE with variable time lag
\[
\frac{dy(t)}{dt} = [Ay(t) + By(t - \delta(t))] dt + \sum_{i=1}^{m} [F_i y(t) + G_i y(t - \delta(t))] dw_i(t) \quad (6.1)
\]
on $t \geq s$ with initial data $y_s = \xi \in L^2_{\mathscr{F},s}([-\tau, 0]; \mathbb{R}^n)$.

Let $\mathcal{S} = \{s/N : N = 1, 2, \ldots \}$. Given a stepsize $\Delta \in \mathcal{S}$ the EM approximate solution defined by (5.15) and (5.16) is now computed as follows: Set
\[
x(s + k\Delta) = \xi(k\Delta), \quad -\tau/\Delta \leq k \leq 0;
\]
compute the discrete approximations
\[
x(s + (k + 1)\Delta) = x(s + k\Delta) + [Ax(s + k\Delta) + Bx(s + k\Delta - \ln(\delta(s + k\Delta)/\Delta)\Delta) + \sum_{i=1}^{m} [F_i x(k\Delta) + G_i x(s + k\Delta - \ln(\delta(s + k\Delta)/\Delta)\Delta)] \Delta w_{ik}, \quad (6.2)
\]
for $k \geq 0$, where $\Delta w_{ik} = w_i(s + (k + 1)\Delta) - w_i(s + k\Delta)$; and then form the continuous-time approximations
\[
x(t) = x(s) + \int_s^t [Az_1(r) + Bz_2(r)] dr + \sum_{i=1}^{m} \int_s^t [F_i z_1(r) + G_i z_2(r)] dw_i(r) \quad \text{on} \quad t \geq s \quad (6.3)
\]
with $x(s + u) = \xi(u)$ for $-\tau \leq u \leq 0$, where
\[
z_1(t) = \sum_{k=0}^{\infty} x(s + k\Delta)1_{[s+k\Delta,s+(k+1)\Delta)}(t)
\]
and
\[
z_1(t) = \sum_{k=0}^{\infty} x(s + k\Delta - \ln(\delta(s + k\Delta)/\Delta)\Delta)1_{[s+k\Delta,s+(k+1)\Delta)}(t).
\]

To close this paper we conclude by Theorem 5.3 the following useful “if and only if” result for the linear SDDE (6.1).

**Corollary 6.1.** The SDDE (6.1) is exponentially stable in mean square if and only if for some sufficiently small $\Delta \in \mathcal{S}$, the EM approximate solution is exponentially stable in mean square with rate constant $\gamma$ and growth constant $H$ satisfying
\[
\beta_3 \Delta + 2\sqrt{\beta_3 H A e^{-1/2}(T - 2\tau) + H e^{-\gamma(T - 2\tau)}} \leq e^{-1/2(T - 2\tau)},
\]
where
\[
K_1 = 2(|A|^2 \vee |B|^2), \quad K_2 = 2 \left( \sum_{i=1}^{m} |F_i|^2 \vee \sum_{i=1}^{m} |G_i|^2 \right),
\]
\[
T = \tau(9 + \ln(4 \log(H)/\gamma \tau)), \quad h_2 = 12(K_1(2T - \tau) + K_2) e^{6(2T - \tau)((2T - \tau)K_1 + K_2)},
\]
\[
\beta_3 = 4(K_1 T + K_2)(\eta + 3) h_2 e^{8T(2K_1(T - \tau) + K_2)}.
\]

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References