Classification of Computable Approximations by Divergence Boundings

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Computable and Computably Approximable Reals

A real number $x$ is computably approximable if $x = \lim x_s$ for a computable sequence $(x_s)$ of rational numbers. (CA)

$x$ is computable if $(x_s)$ converges to $x$ effectively in one of the following senses. (EC)

- $(\forall n)(|x - x_n| \leq 2^{-n})$
- $(\forall n)(\forall s \geq n)(|x_s - x_n| \leq 2^{-n})$
- $(\forall n)(\forall s, t \geq n)(|x_s - x_t| \leq 2^{-n})$
- $(\forall n)(\forall s \geq e(n))(|x - x_s| \leq 2^{-n})$ for computable function $e$
- $(\forall n)(\forall s, t \geq e(n)) \left( |x - x_s| \leq \frac{1}{d(n)} \right)$ where $e, d$ are computable and $d$ is unbounded.

($e$ is modulus function and $d$ is the distance function)

There are exceptions for non-computable real numbers. How to measure the non-computability?
The First Measurement

- A sequence \((x_s)\) converges \(h\)-bounded effectively if there are at most \(h(n)\) non-overlapping index-pairs \((s, t)\) such that \(|x_s - x_t| > 2^{-n}\) for all \(n\).

- A real number \(x\) is \(h\)-bounded computable if there is a computable sequence \((x_s)\) of rationals which converges \(h\)-bounded effectively to \(x\). \((h\text{-BC})\)

- A real number \(x\) is \(C\)-bounded computable if it is \(h\)-bc for some \(h \in C\). \((C\text{-BC})\)
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• A real number \(x\) is \(x\) \(C\)-bounded computable if it is \(h\)-bc for some \(h \in C\). \((C\text{-BC})\)

Proposition 1.

1. \(x\) is rational \(\iff\) \(x\) is \(h\)-bc and \(\liminf f(n) < \infty\).

2. \(h\) is unbounded, monotone and computable \(\implies\) \(EC \subset h\text{-BC}\)

3. \((\exists c)(\forall n)(|f(n) - g(n)| \leq c) \implies f\text{-BC} = g\text{-BC}\)
The First Measurement

Theorem 1.

1. (Hierarchy) For any computable functions $f$ and $g$ we have

$$(\forall c)(\exists n)(|f(n) - g(n)| > c) \implies f-\text{BC} \neq g-\text{BC}.$$
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$$(\forall c)(\exists n)(|f(n) - g(n)| > c) \implies f\text{-}BC \neq g\text{-}BC.$$  

2. The $C\text{-}BC$ is a field, if $C$ satisfies

$$(\forall f, g \in C)(\forall c)(\exists h \in C)(\forall n)(f(c + n) + g(c + n) \leq h(n)).$$
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3. Let $WC$ is the class of all weakly computable real numbers, then $WC \subsetneq o(2^n)\text{-BC}$. 
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3. Let $\WC$ is the class of all weakly computable real numbers, then $\WC \subset o(2^n)-\BC$.

4. If $f, g$ are increasing computable functions such that

   $$(\exists \gamma > 1)(\forall c \in \mathbb{N})(\forall n)(f(\gamma n) + n + c < g(n))$$

   then there is a $g$-bc real which is not Turing equivalent to any $f$-bc real number.
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Remark: The classification is coarse. No Ershov-style hierarchy.
The Second Measurement

A real number $x$ is $h$-Cauchy computable if there is a computable sequence $(x_s)$ converging to $x$ such that there are at most $h(n)$ non-overlapping index-pairs $(s, t)$ satisfy

$$s, t \geq n \& 2^{-n} \geq |x_s - x_t| < 2^{-n+1}.$$  

$h$-cEC denotes the class of all $h$-Cauchy computable real numbers.
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Theorem 2.

1. $EC = 0$-cEC $\subsetneq 1$-cEC $\subsetneq 2$-cEC $\subsetneq \cdots \subsetneq \omega$-cEC $\subsetneq \omega$-BC.
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**Theorem 2.**

1. $EC = 0$-$cEC \subsetneq 1$-$cEC \subsetneq 2$-$cEC \subsetneq \cdots \subsetneq \ast$-$cEC \subsetneq \omega$-$cEC = \omega$-$BC$.

2. $\ast$-$cEC$ and $SC$ are incomparable, and $\ast$-$cEC \subsetneq WC$. 
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3. (Hierarchy) $f, g$ computable $\& (\exists \infty n)(f(n) \neq g(n)) \Rightarrow f$-$cEC \neq g$-$cEC$
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4. There are $x, y \in 1$-cEC such that $x + y \notin \ast$-cEC.
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2. $\ast\text{-}cEC$ and $SC$ are incomparable, and $\ast\text{-}cEC \subsetneq WC.$

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4. There are $x, y \in 1\text{-}cEC$ such that $x + y \notin \ast\text{-}cEC.$

**Remark:** There is an Ershov-style hierarchy. The classification is too sensitive to arithmetical operations.
The Third Measurement (More General Form)

A real number $x$ is $(f, e, d)$-effectively computable if there is a computable sequence of rationals converging to $x$ such that there are at most $f(n)$ non-overlapping index-pairs $(s, t)$ satisfy

$$s, t \geq e(n) \& \left( |x_s - x_t| > \frac{1}{d(n)} \right).$$

- $f$ — founding function;
- $e$ — modulus function;
- $d$ — distance function

$(f, e, d)$-EC — class of all $(f, e, d)$-effectively computable real numbers.
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The properties we are interested in:

- Closure properties under arithmetical operations and computable functions; and
- possible nice hierarchy properties.
Reduction of \((f, e, d)\)-Effective Computability

The distance function \(d\) and modulus function \(e\) should be computable, monotone and unbounded \((\text{cmu})\).
Reduction of \((f, e, d)\)-Effective Computability

The distance function \(d\) and modulus function \(e\) should be computable, monotone and unbounded (cmu).

**Theorem 3.**

1. A cmu modulus function \(e\) can be reduced to the identity function \(id(n) := n\), i.e.,

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(f, e, d)\text{-EC} = (f, id, d)\text{-EC} := (f, d)\text{-EC}.
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2. A cmu distance function \(d\) can be reduced to the identity function \(id\), too, i.e.,

\[(f, d)\text{-EC} = (f \circ d^{-1}, id)\text{-EC}

where \(d^{-1}(n) := \min\{t \in \mathbb{N} : d(t) \geq n\}\) (upper inverse function of \(d\)).
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3. A cmu distance function \(d\) can also be reduced to the exponential function \(ep(n) \coloneqq 2^n\), i.e.,
   \[
   (f, d)\text{-EC} = (f \circ d^{-2}, ep)\text{-EC}
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   where \(d^{-2}(n) \coloneqq \min\{t \in \mathbb{N} : d(t) \geq 2^n\}\).
$f$-Effectively Computable Real Numbers

A sequence $(x_s)$ converges $f$-effectively if, for all $n$, there are at most $f(n)$ non-overlapping index-pairs $(s, t)$ such that

$$s, t \geq n \& |x_s - x_t| > 2^{-n}.$$ 

A real $x$ is $f$-effectively computable if there is a computable sequence $(x_s)$ which converges $f$-effectively to $x$.

$f$-EC is the class of all $f$-ec real numbers and $C$-EC := $\bigcup_{f \in C} C$-EC.

$\omega$-EC := $C$-EC for the class of computable functions.
**$f$-Effectively Computable Real Numbers**

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**Proposition 2.**

- $0$-EC = EC;
- $\omega$-EC = $\omega$-BC = DBC;
- $f$-EC $\subseteq$ $f$-BC.

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Finite Bounded Effective Computability

A real number $x$ is bounded effectively computable if $x$ if $f$-ec for a constant function $f$.

$f$-EC is denoted by $k$-EC if $f \equiv k$ and $\ast$-EC := $\bigcup_{k \in \mathbb{N}} k$-EC.
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A real number \( x \) is bounded effectively computable if \( x \) if \( f \)-ec for a constant function \( f \).
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**Theorem 4.**

1. \( k\text{-EC} = (k, d)\text{-EC} \) for any cmu distance function \( d \);
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$*$-EC is closed under the arithmetical operations;
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   $\ast$-EC is closed under the arithmetical operations;

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5. The class $\ast$-EC is incomparable with SC ($1$-EC $\not\subseteq$ SC and SC $\not\subseteq$ $\ast$-EC).
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A real number $x$ is bounded effectively computable if $x$ is $f$-ec for a constant function $f$.

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6. $\ast$-EC $\subsetneq WC$. 

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Computably Bounded Effective Computability

\[ \text{DBC} = \omega-\text{EC} := \bigcup \{ f-\text{EC} : f \text{ is computable} \}. \]

Theorem 5.

1. (Hierarchy of the classes:) For any computable functions \( f, g \) we have

\[ (\exists \infty n)(f(n) < g(n)) \implies g-\text{EC} \not\subseteq f-\text{EC}; \]
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**Theorem 5.**

1. **(Hierarchy of the classes:)** For any computable functions $f, g$ we have

   $$\quad (\exists \infty n)(f(n) < g(n)) \Rightarrow g-\text{EC} \nsubseteq f-\text{EC};$$

2. **(Hierarchy of the Turing degrees:)** For any computable functions $f, g$ we have

   $$\quad (\exists \gamma > 1)(\exists \infty n)(f(\gamma n) < g(n)) \Rightarrow (\exists x \in g-\text{EC})(\forall y \in f-\text{EC})(x \neq_T y);$$
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$$\left(\exists \gamma > 1 \right) \left(\exists^{\infty} n \right) \left( f(\gamma n) < g(n) \right) \implies \left(\exists x \in g\text{-EC} \right) \left(\forall y \in f\text{-EC} \right) \left( x \not\equiv_T y \right);$$

3. $SC \subsetneq WC \subsetneq o(2^n)\text{-EC};$
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3. \( \text{SC} \not\subseteq \text{WC} \not\subseteq o(2^n) \text{-EC}; \)

4. \( \text{SC} \not\subseteq o_e(2^n) \text{-EC}, \) where \( o_e(2^n) := \{ f \in o(2^n) : f \text{ is computable} \}. \)
Bounding by Function Classes

$C'\text{-EC} = \ast\text{-EC}$ is a field for the class $C'$ of constant functions.

**Theorem 6.**

If $C'$ is a function class which contains all constant functions and is closed under the addition and composition, then the class $C'\text{-EC}$ is a field.
Bounding by Function Classes

\( \mathcal{C}' \cdot \text{EC} = \ast \cdot \text{EC} \) is a field for the class \( \mathcal{C}' \) of constant functions.

**Theorem 6.**

If \( \mathcal{C}' \) is a function class which contains all constant functions and is closed under the addition and composition, then the class \( \mathcal{C}' \cdot \text{EC} \) is a field.

For any function \( f \), let \( \theta(f) := \{ g : (\exists a, b, c)(\forall n)(g(n) \leq a f(b + n) + c) \} \).

**Corollary.** Let \( f, g \) be monotone functions.

1. The class \( \theta(f) \cdot \text{EC} \) is a field;

2. \( f \in o(g) \implies \theta(f) \cdot \text{EC} \subsetneq \theta(g) \cdot \text{EC} \).
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Thank you