GLOBAL ATTRACTORS OF REACTION-DIFFUSION SYSTEMS AND THEIR HOMOGENIZATION*

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Abstract In this paper, we study the existence of the global attractor $A^{\varepsilon}$ of reaction-diffusion equation

$$\partial_t u^\varepsilon(x, t) = A^\varepsilon u^\varepsilon(x, t) - f(x, \varepsilon^{-1}x, u^\varepsilon(x, t)),$$

and the homogenized attractor $A^0$ of the corresponding homogenized equation, then give explicit estimates for the distance between the attractor $A^{\varepsilon}$ and the homogenized attractor $A^0$.

Key Words Homogenization; global attractor; reaction-diffusion systems; almost-periodic function; Diophantine conditions.

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1. Introduction and Main Results

We consider the reaction-diffusion system

$$\left\{ \begin{array}{l}
\partial_t u^\varepsilon(x, t) = A^\varepsilon u^\varepsilon(x, t) - f(x, \varepsilon^{-1}x, u^\varepsilon(x, t)), \\
u^\varepsilon(x, t)|_{\partial\Omega} = 0, \quad u^\varepsilon(x, t)|_{t=0} = u_0,
\end{array} \right. \quad (x, t) \in \Omega \times \mathbb{R}^+,$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^3$ and $0 < \varepsilon \leq \varepsilon_0 < 1$. Here $u^\varepsilon = u^\varepsilon(x, t) = (u^\varepsilon_1, \ldots, u^\varepsilon_k)$ is an unknown vector-valued function. The second order elliptic differential operators $A^\varepsilon$ have the form as follows:

$$A^\varepsilon u := \text{diag}(A^\varepsilon_1 u^1, \ldots, A^\varepsilon_k u^k),$$

(1.2)

with

$$A^\varepsilon_i u^l = \sum_{i,j=1}^3 \partial_{x_i} (a^l_{ij}(\varepsilon^{-1}x) \partial_{x_j} u^l(x)),$$

(1.3)

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where the functions $a^l_{ij}(y)$, $l = 1, \cdots, k$, $y \in \mathbb{R}^3$, are assumed to be symmetric, smooth and $Y$-periodic with respect to $y \in \mathbb{R}^3$, where $Y \subset \mathbb{R}^3$ is a fixed cube. The uniform ellipticity condition
\begin{equation}
\sum_{i,j=1}^{3} a^l_{ij}(y) \xi_i \xi_j \geq \nu |\xi|^2, \quad \forall y, \xi \in \mathbb{R}^3,
\end{equation}
is also assumed (with an appropriate $\nu > 0$) to be valid for operators $A^l_{\epsilon}$. We impose that $f(x, y, u)$ is almost-periodic ([1]) with respect to $y \in \mathbb{R}^3$ and satisfies the conditions as follows:
\begin{align}
f & \in C^1(\mathbb{R}^k, \mathbb{R}^k), \quad \partial_z f(x, y, z) \zeta \zeta \geq -C_2 \zeta \zeta, \quad \forall \zeta \in \mathbb{R}^k, \\
|f(x, y, u)| & \leq C(1 + |u|^p), \quad \forall (x, y) \in \Omega \times \mathbb{R}^3, \\
\sum_{l=1}^{k} |f^l u|^p_l p_l & \geq C \sum_{l=1}^{k} |u|^p_{l+2} - C_1, \quad \forall u \in \mathbb{R}^k,
\end{align}
where $p \geq 1, p_i \geq 2(p - 1), i = 1, \cdots, k$. It is assumed also that the initial data $u_0 \in (L^2(\Omega))^k$.

Efendiev and Zelik (see [2]) studied the problem (1.1) when $f(x, y, u)$ is independent of $y$. Fiedler and Vishik (see [3]) studied the case when the $A_{\epsilon} u$ in (1.1) is replaced by $a \Delta u$. In fact, one can obtain the existence of solutions and attractors for (1.1) with $f(x, y, u)$ depending on $y$ by the standard method as those in [4]. However, when estimate the distance between the attractors for (1.1) and the attractors of the homogenized equation, the arguments in [2] or [3] don’t work. We have to overcome these difficulties by combining the ideas in [3], [2] and analyzing carefully the properties of periodic and almost-periodic functions.

In order to simplify our expression, we denote $H = (L^2(\Omega))^k$, $V = (W_0^{1,2}(\Omega))^k$, $F = (L^\infty(\Omega))^k$, $\| \cdot \|_{(W_0^{1, p}(\Omega))^k} = \| \cdot \|_{L^p}$.

**Theorem 1.1** If the assumptions (1.2) – (1.7) hold, and the initial data $u_0 \in H$, then for any $T > 0$, $\epsilon > 0$, the problem (1.1) possesses a unique solution $u^\epsilon(x, t) \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$, $u^\epsilon \in C(R^+; H)$. The mapping $S_{\epsilon}^t: u_0 \rightarrow u^\epsilon(x, t)$ defines a continuous semigroup $S_{\epsilon}^t: H \rightarrow H$. If, furthermore, $u_0 \in V$, then $u^\epsilon(x, t) \in L^\infty([0, T]; V) \cap L^2([0, T]; W^{2,2}(\Omega)), u^\epsilon \in C(R^+; V)$.

**Theorem 1.2** If the assumptions (1.2) – (1.7) hold, and $u_0 \in H$, then for every $\epsilon > 0$, the semigroup $S_{\epsilon}^t$ generated by the equation (1.1) possesses a global compact attractor $A^\epsilon$ in $H$.

Theorem 1.1 can be proved by the Faedo-Galerkin method with the help of R. Temam [4], and the details of the proof are omitted. Similar arguments as in [4] for the problem (1.1) yield the a priori estimates needed about $u^\epsilon(x, t)$ in $H$ and $V$, and we omit the details. Then Theorem 1.2, whose proof is also omitted, can be easily proved by the standard arguments [4, Theorem 1.1.1].

By the standard homogenization theory, one can obtain the homogenized problem (2.11), for which one can prove the similar results to Theorems 1.1 and 1.2. In order to
estimate the $L^2$-distance between the attractors for (1.1) and the attractors of the homogenized equation (2.11), we can obtain the a priori estimates required, whose proofs are also omitted, by the similar arguments as those in [2] under the better initial data condition (see Section 2). Under some additional assumptions (mainly the so-called Diophantine conditions (2.21)), we have

**Theorem 1.3** Let the assumptions of Theorem 1.2, (2.1), (2.2) and the assumptions of Proposition 2.2 (see Section 2) hold. Let $u_0 \in F \cap V$ and let $u^\varepsilon(x,t)$ be the solution, defined in Theorem 1.1, of the problem (1.1), $u^0(x,t) \in L^\infty([0,T];H) \cap L^2([0,T];V)$ be the solution of the problem (2.11), then $\forall t > 0$, we have

$$\|u^\varepsilon(x,t) - u^0(x,t)\|_H \leq C\varepsilon^2 e^{\beta t},$$

where the constant $C > 0$ depends only on $\|u_0\|_{F \cap V}$ and $\beta > 0$ is a constant independent of $u^\varepsilon$ and $u^0$.

**Theorem 1.4** Let the assumptions of Theorem 1.3 and (2.39) hold. Let $A^\varepsilon$ be the global attractor of the equation (1.1) and $A^0$ be the global attractor of the homogenized equation (2.11), and define the fractional convergence rate $k = \frac{2\rho}{3\rho + 3\beta}$, then there exists a constant $C > 0$ such that

$$d(A^\varepsilon, A^0) := \text{dist}_H(A^\varepsilon, A^0) \leq C\varepsilon^k, \quad 0 < \varepsilon \leq \varepsilon_0.$$

2. The Homogenization and the Estimates of Errors

First, we study the homogenization of the problem (1.1). In addition to the assumptions (1.2)–(1.7), we assume the initial data $u_0 \in F \cap V$ and the $f(x,y,z)$ satisfies the conditions as follows:

$$f^l(x,y,z) = \sum_{j=1}^{q} b^l_j(x,y) f^l_j(z), \quad |b^l_j(x,y)| \leq C, \quad (2.1)$$

where $f^l(x,y,z), l = 1, \ldots, k$, are the components of $f(x,y,z)$. Let

$$\sum_{l=1}^{k} |\partial_z f^l(x,y,z)| \leq C_1(|z|^4 + 1). \quad (2.2)$$

Recall that $w \in AP(\mathbb{R}^3)$ (the set of almost-periodic functions) possesses the mean value which can be calculated by:

$$\langle w \rangle = \langle w \rangle_x := \lim_{T \to \infty} \frac{1}{2T^3} \int_{[-T,T]^3} w(x) dx, \quad (2.3)$$

and the Fourier expansion as follows (see [5])

$$w(x) = \sum_{\hat{w}(\xi) \neq 0} \hat{w}(\xi) e^{i(x,\xi)}, \quad (2.4)$$
where the amplitudes $\hat{w}(\xi) \in C$, $\xi \in \mathbb{R}^3$, defined by $\hat{w}(\xi) = \langle w(x)e^{-i(x,\xi)} \rangle$. We denote by Trig($\mathbb{R}^3$) the space of all finite trigonometric polynomials of the form (2.4)

$$\text{Trig}(\mathbb{R}^3) := \left\{ w(x) = \sum_{k=1}^{K} w_k e^{i(x,\xi_k)} : K \in \mathbb{N}, \xi_k \in \mathbb{R}^3, w_k \in C \right\}. \quad (2.5)$$

We state a classical result in the homogenization theory:

**Proposition 2.1** ([6, 7]) Let $g \in W^{-1,2}(\Omega)$ and $v^\varepsilon \in V$ be the solution of the equation $A_\varepsilon v^\varepsilon = g$, where the operator $A_\varepsilon$ is defined by (1.3). Then,

$$\begin{cases}
    v^\varepsilon \rightharpoonup v^0 & \text{weakly in } V, \\
    A_\varepsilon v^\varepsilon \rightharpoonup A_0 v^0 & \text{weakly in } H,
\end{cases} \quad (2.6)$$

where $v^0 \in V$ is a unique solution of the homogenized problem $A_0 v^0 = g$. The operator $A_0$ is defined by the form as follows:

$$A_0^l v^0 = \sum_{i,j=1}^{3} \partial_{x_i} (a_{ij}^0 \partial_{x_j} v^0), \quad A_0 v := \text{diag}(A_0^1 v^0, \ldots, A_0^K v^0), \quad (2.7)$$

and the so-called homogenized coefficients $a_{ij}^0 = \langle a_{ij}^l(y) \rangle + \sum_{m=1}^{3} \langle a_{im}^l(y) \partial_{y_m} N_{m}^l(y) \rangle$ are constants, where the $\mathbf{Y}$-periodic correctors $N_{m}^l(y)$, $m = 1, 2, 3$, $l = 1, \ldots, k$, are the solutions of the auxiliary periodic problem as follows:

$$\sum_{i,j=1}^{3} \partial_{y_i} (a_{ij}^l(y) \partial_{y_j} N_{m}^l(y)) = -\sum_{i=1}^{3} \partial_{y_i} (a_{im}^l(y)), \quad y \in \mathbb{R}^3. \quad (2.8)$$

And the homogenized matrix $A_0$ satisfies the coerciveness condition (1.4).

The following lemma, whose proof is easy and so omitted, will be used in the sequel.

**Lemma 2.1** Let Assumptions (1.6), (2.1) hold and $f(x,y,u^\varepsilon)$ be almost-periodic in $y$, assume $u^\varepsilon \rightharpoonup u^0$ in $H$ ($\varepsilon \to 0$), and denote $f_0(x,u^0) := \langle f(x,y,u^0) \rangle_y$, then we have the result as follows:

$$f(x,\varepsilon^{-1} x, u^\varepsilon) \rightharpoonup f_0(x,u^0) \quad \text{weakly in } H. \quad (2.9)$$

$$f^l(x,u^0) = \sum_{j=1}^{q} b_{lj}^j(x) f_{ji}(u^0). \quad (2.10)$$

Now by the standard homogenization theory we obtain the homogenized problem

$$\begin{cases}
    \partial_t u^0 = A_0 u^0 - f_0(x,u^0), & (x,t) \in \Omega \times \mathbb{R}^+, \\
    u^0|_{\partial \Omega} = 0, & u^0|_{t=0} = u_0.
\end{cases} \quad (2.11)$$

Note that this equation satisfies all assumptions of the equation (1.1), consequently, it admits a unique solution $u^0(x,t) \in L^\infty([0,T]; H) \cap L^2([0,T]; V)$ and (2.11) possesses a global attractor $A_0$ in $H$. 

We now specify additional conditions which enable us to estimate the distance between the solutions \(w'(x, t)\) and \(u^0(x, t)\) in the norm of \(H\). In order to give the distance estimate of \(w'(x, t)\) and \(u^0(x, t)\) in \(H\), we need three propositions (see [2, 3]).

First, we introduce some results about divergence representations. Let \(h(x, y) = h(x_1, \cdots, x_3, y_1, \cdots, y_3)\) be a sufficiently smooth function which is almost-periodic in \(y = (y_1, \cdots, y_3)\), i.e.:

(i) there exists a function \(H(x, w_1, \cdots, w_3) = H(x_1, \cdots, x_3, w_{11}, \cdots, w_{1k_1}, \cdots, w_{31}, \cdots, w_{3k_3})\) which is \(2\pi\)-periodic with respect to each \(w_{ij}\). Here \(w_i = (w_{i1}, \cdots, w_{ik_i}) \in \mathbb{R}^{k_i}, (i = 1, \cdots, 3)\).

(ii) there exists rationally independent frequency \(\alpha_{11}, \cdots, \alpha_{1k_1}, \cdots, \alpha_{3k_3}\) such that

\[
h(x, y) = H(x_1, \cdots, x_3, \alpha_1 y_1, \cdots, \alpha_3 y_3),
\]

where \(\alpha_i = (\alpha_{i1}, \cdots, \alpha_{ik_i})\). Let \(\tilde{H}(x, w) = H(x, w) - H_0(x)\), where

\[
H_0(x) = |T^k|^{-1} \int_{T^k} H(x, w_1, \cdots, w_3)dw_1 \cdots dw_3,
\]

where \(T^k = T^{k_1} \times \cdots \times T^{k_3}\), and \(T^{k_i} = \mathbb{R}^{k_i}/(\mathbb{Z} \cdot 2\pi)^{k_i}\) is the \(k_i\)-dimensional torus. Assume that the Fourier series

\[
H(x, w) = \sum_m H_m(x)e^{im \cdot w}
\]

is convergent. Let

\[
\tilde{h}(x, y) = h(x, y) - H_0(x) = \sum_{m \neq 0} H_m(x) \exp \left( i \sum_{j=1}^3 m_j \alpha_j y_j \right),
\]

where \(m_j = (m_{j1}, \cdots, m_{jk_j}) \in \mathbb{Z}^{k_j}, \alpha_j \in \mathbb{R}^{k_j}\) and \(y_j \in \mathbb{R}\). For any such almost periodic function \(h(x, y)\), we construct a corresponding divergence representation by function \(S_{\sigma}(x, y), \sigma = 1, \cdots, 3\).

\[
\tilde{h}(x, y) = \sum_{\sigma=1}^3 \partial_{y_{\sigma}} S_{\sigma}(x, y).
\]

We shall find \(S_{\sigma}(x, y)\) of the form

\[
S_{\sigma}(x, y) = \sum_{m \in \mathbb{Z}^{k_\sigma}} \eta_{m, \sigma}(x) \exp \left( i \sum_{j=1}^3 m_j \alpha_j y_j \right).
\]

From (2.15) – (2.17) we derive:

\[
\sum_{m \neq 0} H_m(x) \exp \left( i \sum_{j=1}^3 m_j \alpha_j y_j \right) = \tilde{h}(x, y) = \sum_{m \neq 0} \sum_{\sigma=1}^3 m_{\sigma} \cdot \alpha_{\sigma} \eta_m(x) \exp \left( i \sum_{j=1}^3 m_j \alpha_j y_j \right),
\]

(2.18)
So (2.16) will hold if
\[
\sum_{\sigma=1}^{3} m_{\sigma} \cdot \alpha_{\sigma} u_{m}^{n}(x) = -iH_{m}(x),
\]
(2.19)
for all \( m \in \mathbb{Z}^{k} \setminus \{0\} \). Let the following assumptions be satisfied for some positive \( \delta \) and \( \delta' \):
\[
\tilde{b}_{l}^{j} = \tilde{h}_{l}^{j} = \sum_{m \neq 0} H_{lm}^{j}(x) \exp \left( i \sum_{j} m_{j} \alpha_{j} y_{j} \right). \tag{2.20}
\]
\[
|m_{\sigma} \cdot \alpha_{\sigma}| \geq c|m_{\sigma}|^{-(k_{\sigma}-1)+\delta}, \quad \forall m_{\sigma} \in \mathbb{Z} \setminus \{0\}. \tag{2.21}
\]
\[
\left\| H_{lm}^{j}(x) \right\|_{C^{0}(\Omega)} \leq c(1 + |m_{\sigma}|)^{-(k_{\sigma}-1)+\delta}(1 + |m|)^{-(k+k')} \tag{2.22}
\]
\[
\left\| \partial_{x_{\sigma}} H_{lm}^{j}(x) \right\|_{L^{1}(\Omega)} \leq c(1 + |m_{\sigma}|)^{-(k_{\sigma}-1)+\delta}(1 + |m|)^{-(k+k')}. \tag{2.23}
\]
Now we can state the propositions as follows:

**Proposition 2.2**([3]) Let the coefficients \( \tilde{b}_{l}^{j}(x,y) \) of (2.1) satisfy the conditions as follows:

(i) \( \tilde{b}_{l}^{j}(x,y) \) are almost-periodic in \( y, \; j = 1, \ldots, q \);
(ii) the corresponding frequencies \( \alpha_{ij} \) satisfy Diophantine condition (2.21);
(iii) the coefficients \( H_{lm}^{j}(x) \) in the series (2.20) of \( \tilde{b}_{l}^{j}(x,y) = b_{l}^{j}(x,y) - \tilde{b}_{l}^{j}(x) \) satisfy the decay conditions (2.22), (2.23),
then we can represent \( \tilde{b}_{l}^{j}(x,y) \) in the form
\[
\tilde{b}_{l}^{j}(x,y) = \sum_{\sigma=1}^{3} \partial_{x_{\sigma}} S_{l\sigma}^{j}(x,y), \tag{2.24}
\]
which satisfies
\[
\left| S_{l\sigma}^{j}(x,y) \right| \leq C_{0}, \quad \left\| \partial_{x_{\sigma}} S_{l\sigma}^{j}(x,y) \right\|_{L^{1}(\Omega)} \leq C_{0}, \tag{2.25}
\]
here \( \partial_{x_{\sigma}}^{1} \) indicates partial derivatives with respect to the first argument \( x \) of the function \( S_{l\sigma}^{j}(x,y) \).

**Proposition 2.3**([3]) Let the assumptions (1.2)-(1.7), (2.1), (2.2) and Proposition 2.2 hold. Then
\[
\left| (f(x, \varepsilon^{-1}x, u^{\varepsilon}) - f_{0}(x, u^{\varepsilon}), u^{\varepsilon} - u^{0}) \right| \leq \varepsilon C\|u^{\varepsilon} - u^{0}\|_{V}, \tag{2.26}
\]
where the constant \( C > 0 \) depends only on \( \|u_{0}\|_{F\cap V} \).

Denote (see [5]):
\[
u_{1}(t) = u^{0}(t) + \varepsilon \sum_{k=1}^{3} N_{k}(\varepsilon^{-1}x) \partial_{x_{k}} u^{0}(t), \tag{2.27}
\]
where \( N_{k}(\varepsilon^{-1}x), \; k = 1, \; 2, \; 3, \) are the solutions of the problem (2.8). Note that the function \( u_{1}(t) \) doesn’t satisfy the 0-Dirichlet boundary condition. In order to avoid this
difficulty, we introduce a family of cut-off functions \(\tau^\varepsilon(x)\) satisfying two conditions as follows (see [5]): (1) \(\tau^\varepsilon(x)\in C_0^\infty(\Omega)\), \(0 \leq \tau^\varepsilon \leq 1\), \(\tau^\varepsilon(x) \equiv 1\) off the \(\varepsilon\)-neighborhood of the boundary of \(\Omega\); (2) \(\varepsilon|\nabla_x \tau^\varepsilon(x)| \leq C\) in \(\Omega\), where the constant \(C\) is independent of \(\varepsilon\). Thus we take

\[
w^\varepsilon(t) = u^0_\varepsilon(t) - \varepsilon(1 - \tau^\varepsilon(x)) \sum_{k=1}^{3} N_k(\varepsilon^{-1}x) \partial_{x_k} u^0(t)
= u^0(t) + \varepsilon \tau^\varepsilon(x) \sum_{k=1}^{3} N_k(\varepsilon^{-1}x) \partial_{x_k} u^0(t).
\]

Then, obviously, \(w^\varepsilon(t) \in V\). We need the proposition as follows:

**Proposition 2.4** ([2]) Let the assumption (1.4) hold, and let \(w^\varepsilon(t)\), \(A_\varepsilon u^\varepsilon\), \(A_0 u^0\) be defined by (2.28), (1.3), (2.7) respectively, \(u^\varepsilon(t)\), \(u^0(t)\) be the solution of the equation (1.1), (2.11) respectively. Then

\[
(A_\varepsilon u^\varepsilon(t) - A_0 u^0(t), u^\varepsilon(t) - w^\varepsilon(t)) \leq C \varepsilon^\frac{2}{3} \|u^0(t)\|_{2,2}^2,
\]

where the constant \(C > 0\) is independent of \(\varepsilon\).

**Proof of Theorem 1.3** Denote \(v(x,t) = u^\varepsilon(x,t) - u^0(x,t)\). Subtracting (2.11) from (1.1), we get

\[
\partial_t v = A_\varepsilon u^\varepsilon - A_0 u^0 - (f(x,\varepsilon^{-1}x, u^\varepsilon) - f_0(x, u^\varepsilon)) - (f_0(x, u^\varepsilon) - f_0(x, u^0)).
\]

Multiplying both sides of (2.30) by \(v\) and integrating over \(\Omega\), we obtain

\[
(\partial_t v, v) = (A_\varepsilon u^\varepsilon - A_0 u^0, v) - (f(x,\varepsilon^{-1}x, u^\varepsilon) - f_0(x, u^\varepsilon), v) - (f_0(x, u^\varepsilon) - f_0(x, u^0), v).
\]

To prove the theorem, we estimate each term of the right-hand side of (2.31) respectively. Using Proposition 2.4, we derive

\[
\sum_{l=1}^{k} (A_\varepsilon^l u^\varepsilon_l(t) - A_0^l u^0_l(t), v^l(t)) = \sum_{l=1}^{k} (A_\varepsilon^l u^\varepsilon_l(t) - A_0^l u^0_l(t), u^\varepsilon_l(t) - w^\varepsilon_l(t))
+ \sum_{l=1}^{k} (A_\varepsilon^l u^\varepsilon_l(t) - A_0^l u^0_l(t), v^l(t) - u^\varepsilon_l(t) + w^\varepsilon_l(t))
\leq C \varepsilon^\frac{2}{3} \|u^0\|_{2,2}^2 + \|A_\varepsilon u^\varepsilon - A_0 u^0\|_H \cdot \|v - u^\varepsilon + w^\varepsilon\|_H. \tag{2.32}
\]

Note that the definitions (2.27), (2.28) and (2.5) imply the estimate

\[
\|v(t) - u^\varepsilon(t) + w^\varepsilon(t)\|_H \leq C \varepsilon \|u^0(t)\|_V. \tag{2.33}
\]

Similar methods as in [2] for the equation (1.1) and (2.11) yield

\[
\int_T^{T+1} \|A_\varepsilon u^\varepsilon(t)\|_H^2 dt + \int_T^{T+1} \|A_0 u^0(t)\|_H^2 dt + \int_T^{T+1} \|u^0(t)\|_V^2 dt + \int_T^{T+1} \|u^0(t)\|_{2,2}^2 dt
\leq Q(\|u_0\|_{F\cap V}), \tag{2.34}
\]
for the appropriate function \( Q \) independent of \( T \geq 0 \) (here we have implicitly used the elliptic regularity estimate \( \| u_0 \|_{2, 2} \leq C \| A_0 u_0 \|_H \)). Inserting the estimate (2.33) to (2.32) and integrating over \( t \in [0, T] \) then taking the estimate (2.34) into account, we have

\[
\sum_{i=1}^{k} \int_0^T \left( A_i u_i(t) - A_0 u_0(t) \right) \, dt \leq \varepsilon \frac{2}{3} Q(\| u_0 \|_{F \cap V}) T. \tag{2.35}
\]

Applying (2.26) to the second term of the right-hand side of (2.31), integrating over \( t \in [0, T] \), using Minkowski-inequality and (2.34), we obtain

\[
\int_0^T \left| f(x, \varepsilon^{-1} x, u^\varepsilon(t)) - f_0(x, u^\varepsilon(t)), v(t) \right| \, dt \leq \varepsilon Q(\| u_0 \|_{F \cap V}) T. \tag{2.36}
\]

Assumption (2.2) implies

\[
\int_0^T \left| (f_0(x, u^\varepsilon) - f_0(x, u^0), v) \right| \, dt = \int_0^T \left| \left( \int_0^1 f'(s(u^\varepsilon + (1-s)u^0)) ds \cdot v, v \right) \right| \, dt \leq C_2 T \int_0^T \| v \|_H^2 \, dt. \tag{2.37}
\]

Integrating (2.31) over \( t \in [0, T] \) and taking account of (2.35)-(2.37), we get

\[
\| v(T) \|^2_{0, 2} \leq \varepsilon \frac{2}{3} Q(\| u_0 \|_{F \cap V}) T + 2\varepsilon Q(\| u_0 \|_{F \cap V}) T + 2C_2 \int_0^T \| v(T) \|^2_H \, dt. \tag{2.38}
\]

Applying Gronwall’s inequality to (2.38) proves Theorem 1.3.

Now we are ready to derive the error’s estimates for the global attractors \( A^\varepsilon \) and \( A^0 \). To this end, we need some additional information about \( A^0 \) which we in fact require to be exponentially attracting with exponential rate \( \rho > 0 \). We assume there exists a constant \( C = C(\varepsilon_0) \) such that for all \( t \geq 0 \)

\[
d := \text{dist}_H(u^0, A^0) \leq C e^{-\rho t}, \tag{2.39}
\]

holds, uniformly for all \( u_0 \in \bigcup_{0 < \varepsilon \leq \varepsilon_0} A^\varepsilon \), where \( \text{dist}_H \) means the nonsymmetric Hausdorff distance (see [4]), i.e.

\[
\text{dist}_H(A, B) := \sup_{x \in A} \inf_{y \in B} \| x - y \|_H. \tag{2.40}
\]

**Proof of Theorem 1.4** Let

\[
\mathcal{B} := \bigcup_{0 < \varepsilon \leq \varepsilon_0} A^\varepsilon. \tag{2.41}
\]

Pick \( 0 < \varepsilon \leq \varepsilon_0 \) and \( u^\varepsilon \in A^\varepsilon \subset \mathcal{B} \), arbitrarily. For \( t \geq 0 \) chosen below consider \( u_0 \in A^\varepsilon \) such that

\[
S_t^\varepsilon u_0 = u^\varepsilon. \tag{2.42}
\]
Then Theorem 1.3 and (2.39) imply

\[ d(u^\varepsilon, A^0) \leq d(u^\varepsilon, u^0) + d(u^0, A^0) \leq C\varepsilon^{\tilde{T}} e^{\beta t} + C e^{-\rho t}. \]  \hspace{1cm} (2.43)

Choose \( t \geq 0 \), such that \( \varepsilon^{\frac{2}{3}} e^{\beta t} = e^{-\rho t} \), thus \( t = -\frac{\ln \varepsilon}{\beta + \rho} \). Substituting this choice of \( t \) back into (2.43), because of the arbitrariness of \( u^\varepsilon \), we prove Theorem 1.4.

References

[1] Levitan B., Almost-Periodic Functions, Gostekhizdat, Moscow, 1953