Domain of attraction of hysteresis-series based chaotic attractors

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Abstract

This paper discusses the domain of attraction and its sensitivity for a class of chaotic attractors generated by using second-order linear systems with hysteresis-series. It is found that the domain of attraction of the chaotic attractors is determined by an unstable limit cycle. The chaotic dynamical behaviors are demonstrated by using the Poincaré map. The sensitivity of the domain of attraction with respect to the system parameters is studied and some simulation results are presented.

1. Introduction

Chaotic oscillators may have great potential for engineering applications such as secure communication, fluid mixing, vibration analysis and biological systems, etc. Generating chaos for engineering applications has been studied extensively. Among them, generating chaos with linear unstable second-order systems and hysteresis was reported. Newcomb and his colleagues developed some hysteresis chaos based on the study of fibrillation of the heart [1983, 1984]. Saito and his colleagues have obtained a number of results on chaos generation, including generation of one-scroll [1985] and double-scroll [1995, 1996, 1998] chaos with hysteresis switched linear second-order autonomous systems, including some sufficient conditions for chaos generation and bifurcation analysis [Satio, 1995]; a hysteresis chaos generator and its OTA implementation [Nakagawa & Satio, 1996], and the chaos generation region [Satio, 1998]. Storace et al. have obtained some results in this direction as well, including a simple realization of hysteresis chaos generator, where the hysteresis is not the limit case of a rectangular cycle [1998], studies of the dynamics [1999], and bifurcation analysis [2001]. N-scroll chaotic oscillators generated by linear unstable second-order
systems with a double-hysteresis block was proposed and implemented by Han et al. [2003]. Recently, Moreno and his colleagues adopted the transformation point method in the analysis of piecewise-linear oscillators with hysteresis [2003]. It was found that an unstable periodic orbit obtained from a corresponding limit cycle is the closure of the domain of attraction in the second-order systems case.

This paper studies the stability margin of a kind of chaotic oscillators, which is composed of an unstable linear second-order system and a hysteresis-series. The paper is organized as follows: Section 2 introduces the hysteresis-series switched system. The domain of attraction of chaotic attractors is proposed in Section 3. Section 4 demonstrates the dynamical behaviors of the system. Conclusions are drawn in Section 5.

2. A Hysteresis Switched System

Considering the block diagram shown in Fig.1, which represents a linear second-order system with a feedback control of a hysteresis-series.

![Fig. 1. Second-order system with a feedback of hysteresis-series.](image)

This system can be described by

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + 2\alpha y + H(x, n)
\end{align*}
\]  

(1)

where \(x\) and \(y\) are state variables, \(\alpha\) is a positive constant, \(H(x, n)\) is a hysteresis-series, as shown in Fig.2 and described by

\[
H(x, n) = \sum_{i=1}^{n} h_i(x)
\]  

(2)

and

\[
h_i(x) = \begin{cases} 
1 & \text{for } x > i-1 \\
0 & \text{for } x < i
\end{cases}
\]  

(3)

The equilibrium points of system (1) are located on the \(x\)-axis, which are given by

\[O_x = [0, 1, \ldots, n-1, n]\]
The eigenvalues of each subsystem (corresponding to each hysteresis subfunction), \( \lambda = \alpha \pm j\beta \), are the same. The solution of system (1) can be expressed as

\[
\begin{align*}
X(t) &= e^{\alpha t} \left[ X(0) \cos(\beta t) + \frac{1}{\beta} (Y(0) - \alpha X(0)) \sin(\beta t) \right] \\
Y(t) &= e^{\alpha t} \left[ Y(0) \cos(\beta t) + \frac{\alpha}{\beta} (Y(0) - \alpha X(0) - \frac{\beta^2}{\alpha} X(0)) \sin(\beta t) \right]
\end{align*}
\]

where \( \beta = \sqrt{1 - \alpha^2} \), \( (X, Y)^T = (x-m, y)^T \), if \( x \in S_m \) \((0 \leq m \leq n)\), and \( X(0), Y(0) \) are the initial conditions.

With suitable parameters, system (1) can create \((n+1)\)-scroll chaotic attractors (Han, et al. 2003).

3. Domain of Attraction of Chaotic Attractors

The hysteresis-series switched system (1) can exhibit dynamical behaviors similar to the second-order systems with relay type of hysteresis, such as limit cycles and chaotic attractors [Morene, et al. 2003]. Chaotic attractors exist if there exists an attractor with exponential divergence and boundedness of the trajectories inside the domain of attraction of the attractor. The exponentially divergent trajectories of system (1) inside the domain can be implemented with a pair of system conjugate complex eigenvalues with positive real parts. The trajectories that remain bounded inside the domain can be implemented by suitable switching of the hysteresis-series (2). The closed trajectory \( L \) shown in Fig.3 is called a limit cycle.

According to [Morene, et al. 2003], there is an unstable limit cycle \( L \), which determines the domain of attraction of the chaotic attractor. Therefore, with suitable parameters \( \alpha \) and \( n \), a trajectory of system (1) starting inside \( L \) may be just wandering within it, so that chaotic behaviors occur. The trajectory of the chaotic attractor travels in all the \( n+1 \) sub-planes \( S_m \) \((m = 0, 1, \ldots, n)\), which can be considered as being connected by subtrajectories of subsystems of (1).

The limit cycle shown in Fig.3 corresponds to a 3-scroll chaotic attractor of system (1), in which the trajectory covers all the three sub-planes, denoted by \( S_0, S_1 \) and \( S_2 \), respectively. It
can be observed that \((X, Y)^T=(x, y)^T\), if \(x \in S_0\); \((X, Y)^T=(x-1, y)^T\), if \(x \in S_1\); and \((X, Y)^T=(x-2, y)^T\), if \(x \in S_2\). The solution of the limit cycle can be obtained by combining pieces of solutions of Eq.(4) on each sub-plane. Note that \(c=n-b\), where \(b\) and \(c\) are the negative and positive \(x\)-coordinates of the limit cycle when \(y=0\), as shown in Fig.3(a), where \(n\) is the number of the sub-hysteresis functions of the hysteresis-series.

Notice that the unstable limit cycle \(L\) bounds the domain of attraction of the chaotic attractors. For example, when \(n=2\) and 5, the 3- and 6-scroll chaotic attractors generated by system (1) and the trajectories of \(L\) are shown in Fig.4, where \(C\) stands for the area covered by the chaotic trajectory.

One can notice from Fig.4 that both the horizontal and vertical coordinates of the limit cycle \(L\) increase with the increase of the scroll numbers. The initial value for Fig.4 (b) is
(2.0528, 2.02), which incurs an unbounded trajectory if \( n = 2 \), and the trajectory is bounded when \( n = 5 \), which is outside the region covered by the chaotic trajectory (but within the domain of attraction).

![Diagram](image)

**Fig.4. Limit-cycle and the chaotic trajectories \( \alpha = 0.0625 \). (a) 3-scroll; (b) 6-scroll.**

4. **Dynamics of the Hysteresis-series Switched System**

In this section, the chaotic dynamics occurring in the hysteresis-series switched systems are studied.

4.1 **Poincaré map of the hysteresis-series switched system**
In the following, we use the Poincaré map approach to analyze the occurrence of chaos. The positive \( x \)-axis is chosen as the Poincaré section. When \( n=2 \), the Poincaré map with three different parameters of system (1) are shown in Fig.5.

From Fig.5, one can see the following:

1) Point \( L \) in Fig.5 corresponds to the unstable limit cycle that bounds the domain of attraction. With the increase of \( \alpha \), the horizontal coordinate \( L_0 \) of the limit cycle decreases.
Fig. 5. Poincaré map of system (1) \( n=2 \). (a) \( \alpha=0.0625 \); (b) \( \alpha=0.0896 \); (c) \( \alpha=0.1 \).

2) If the maximum vertical coordinate of the Poincaré map is smaller than \( L_1 \) (the vertical coordinate of point \( L \)), as shown in Fig. 5 (a) when \( \alpha = 0.0625 \), then all trajectories will stay within the domain of attraction if initial conditions are within it.

3) Fig. 5 (b) shows a limiting case of occurrence of chaos when \( \alpha = \alpha_{\text{max}} = 0.0896 \). The maximum vertical coordinate of the Poincaré map equals \( L_1 \). In other words, the outer border of the region covered by the chaotic trajectory is the limit cycle.

4) Fig. 5 (c) shows that the maximum vertical coordinate of the Poincaré map is larger than \( L_1 \), which means that some trajectories may move out of the domain of attraction even if the initial conditions are within it.

Fig. 6. Poincaré map of system (1) when \( n=4, \alpha=0.1 \).
With the same $\alpha=0.1$ as that in Fig.5 (c), which creates the unbounded trajectory when $n=2$, chaos appears if the number of the hysteresis functions is increased to 4. The Poincaré map of the 5-scroll chaotic attractor is shown in Fig.6, where the maximum vertical coordinate is smaller than $L1$.

### 4.2 Stability and sensitivity analyses

The stability of system (1) depends on the system parameter $\alpha$ and the number of the hysteresis functions, $n$. From the Poincaré map shown in Fig.5 (c) and Fig.6, one can see that when parameter $\alpha$ is fixed, with the increase of $n$, the unbounded trajectory may become bounded, or the unstable trajectory may become stable but chaotic. Fig.7 shows the maximum value of $\alpha$ for the occurrence of chaos versus $n$.

![Fig.7. The maximum value $\alpha$ of system (1) versus $n$.](image)

Fig.7 shows that, with the increase of $n$, the maximum value of parameter $\alpha$ for the occurrence of chaos in system (1) increases. The limit cycle determines the domain of attraction, which also depends on $\alpha$ and $n$. Fig.8 shows the horizontal coordinate of the limit cycle in the Poincaré map, $L0$, versus $n$, while $\alpha=\alpha_{\text{max}}$. Fig.8 indicates that the maximum $x$-coordinate of the limit cycle, when $y=0$, increases linearly with respect to $n$.

![Fig.8. $L0$ versus $n$, while $\alpha=\alpha_{\text{max}}$.](image)
The limit cycle trajectory is sensitive to parameter $\alpha$. The relationship of $L_0$ versus $\alpha$, when $n=4$, is shown in Fig. 9, where the maximum $x$-coordinate (when $y=0$) of the limit cycle, $L_0$, decreases with the increase of the system parameter $\alpha$, when $n$ is fixed.

\[
\text{Fig. 9. } L_0 \text{ versus parameter } \alpha \text{ (n=4).}
\]

5. Conclusion

In this paper, the domain of attraction of hysteresis-series switched systems has been studied. It has been found that an unstable limit cycle determines the boundary of the domain of attraction of the chaotic attractors. The size of the domain of attraction is determined by the parameter $\alpha$ and the number of hysteresis functions in the hysteresis-series. This paper is the first one that studies the domain of attractions of the chaotic attractors generated by using second-order unstable linear systems with hysteresis-series. Future work will be conducted for multi-directional hysteresis-series-based chaos.

References


