CHAOS SYNCHRONIZATION FROM AN INVARIANT MANIFOLD APPROACH

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Abstract. In this paper, a chaos synchronization method via an invariant manifold approach is proposed. The essence of the method is that the error dynamics between the transmitter and receiver are pushed and forced to stay in the pre-selected invariant manifolds in which the error will converge to zero asymptotically. The well known input-output linearization method is used to discuss the design. Simulation results on the Lorenz system are presented to shown the effectiveness of the method.

Keywords. Chaos synchronization, invariant manifold, feedback, linearization

1 Introduction

Chaotic systems have been studied and are known to exhibit complex dynamical behaviors in the past two decades. The interest in chaotic systems lies mostly in their complex, unpredictable behavior, extreme sensitivity to both initial conditions and parameter variations, and their rich broad-spectrum and noise-like signals. Many theories of chaos control and chaos synchronization have been developed.

Although the first paper on synchronization motions of chaos systems was published in 1983 [4], it is Pecora and Carroll’s work [10] that has provoked intensive research on chaotic synchronization and has opened a way to envisage engineering applications of chaos [1]. In their work, chaos synchronization was studied in the situation where a state variable (or variables) of a chaotic system is used as an input to drive a subsystem that is a replica of part of the original system (called drive-response, or transmitter-receiver, or master-slave synchronization formalisms). They find that the replica subsystem sometimes synchronizes to the chaotic evolution of the original system. The condition for this synchronization to occur depends on the original chaotic system and the choice of the part of the original system that is used...
as the replica subsystem. The choice of the part of the original system used as the replica subsystem is up to the Lyapunov exponents of receiver (called conditional Lyapunov exponents, because the receiver state is driven by the state of the transmitter). A necessary condition for synchronization is that the conditional Lyapunov exponents are negative. Usually, no analytical form of conditional Lyapunov exponents is available, the numerical solution is needed but it is approximate, so in general this approach doesn’t guarantee an effective synchronization. Evidence was found recently that two mutually coupled chaotic systems that have negative conditional Lyapunov exponents can actually desynchronize [12]. Another synchronization scheme is the unidirectionally coupled method [9] that utilizes a form of feedback control with constant feedback coefficients to make the state of the chaotic system synchronize with the transmitter. The unidirectionally coupled method is easy to implement and does not require any computer analysis of the system behavior in practical applications. More importantly, it is always theoretically possible to have a stable synchronization for all kinds of chaotic systems through coupled coefficients and coupled state adjustments. The main task for a designer is to decide the gains of feedback according to the error of transmitter state and receiver states. This problem is essentially about how to choose the linear feedback coefficients to ensure that the error dynamics between the transmitter and receiver systems approach zero. This is closely associated with the linear observer theory, which is particularly suitable for the unidirectionally coupled systems [7].

So far many papers have been published to address chaos synchronization with mainly the linear observer method. There are two ways to use the linear observer in the unidirectional coupled case. One way is that the signal transmitted consists of a nonlinear part and part of state of the original chaos system, so that no nonlinear part appears in the error dynamics [5]. The other way is that, although the output signal of the transmitter system is a linear combination of states, its nonlinear part is required to satisfy the Lipschitz condition, and high gain is needed in this case [6]. In [7], the Lie algebra was used to choose proper output of the transmitter for synchronization. Synchronization of the so-called Lur’e system was discussed in [8]. The extended Kalman filter (EKF) is another example to determine the linear feedback gains [2], [3]. The EKF algorithm adjusts the feedback coefficients adaptively according to the structure of the receiver and driving signal. It eliminates the difficulty of choosing the feedback coefficients in coupled synchronization methods. However the EKF algorithm uses linear approximation of a chaotic system.

Essentially, from the control theory perspective, the design of synchronization is about how to design the output of the transmitter and the receiver to make sure the error dynamics between them approaches zero asymptotically. Based on the invariant manifold (set) theory [11], in this paper, we develop a new kind of design using the invariant manifold approach. The original idea was proposed in [13]. Instead of choosing the coefficients of feedback, the
idea is to choose invariant manifold(s) in which the state of receiver will synchronize with the state of the transmitter, and the aim of control is to push the state of the transmitter to reach and stay in the invariant manifold(s). Once this is done, the dynamics in the invariant manifold(s) will ensure the synchronization. The input-output linearization method is used in the design. This paper is organized as follows: Section 2 introduces the invariant manifold approach to chaotic synchronization. Section 3 is the simulation. The conclusion is given in Section 4.

2 Chaos Synchronization: An Invariant Manifold Approach

The chaos system can be described as follows:

\[ \frac{dx}{dt} = f(x), \]  

(1)

where \( x \in \mathbb{R}^n \) is the state vector and \( f \in \mathbb{R}^n \) is a smooth nonlinear vector function. In this paper, we consider unidirectional synchronization, so equation (1) is called the transmitter system. The receiver system has the general form as follows:

\[ \frac{d\hat{x}}{dt} = f(\hat{x}) + g(\hat{x})\gamma. \]  

(2)

where \( \hat{x} \) is the state vector of the receiver system, \( f(\hat{x}) \) is the same one as in equation (1) and \( g(\hat{x}) \) is a smooth function to be defined. \( \gamma \in \mathbb{R}^m \) is the control. Subtract (2) from (1) and let \( \hat{e}_x = x - \hat{x} \), we have the error dynamics as follows:

\[ \frac{d\hat{e}_x}{dt} = f(x) - f(\hat{x}) - g(\hat{x})\gamma. \]  

(3)

The task of chaos synchronization is to force the two coupled systems, i.e. the transmitter system and the receiver system, to be synchronized by designing a control \( \gamma \) which is attached to the receiver system (2), such that \( \hat{e}_x \to 0 \) when \( t \to \infty \).

An invariant manifold means that there exists a neighborhood around the manifold such that in the neighborhood any state will be eventually attracted to the manifold and stay in it thereafter. For a control system, If the invariant manifold is chosen carefully such that the state of system started from the manifold is asymptotically stable, then the designing of control system is just to choose a control in order to push the state controlled into the invariant manifold. Once states controlled reach the invariant manifold, the system states will stay in them forever and be asymptotically stabilized. In the context of chaos synchronization, if the error \( \hat{e}_x \) under the control \( \gamma \) converges
to zero within the invariant manifold, i.e. the outputs of the receiver and the transmitter systems will match, then the chaos synchronization is realized.

Similar to the input-output linearization approach for the output control of nonlinear systems, if the internal dynamics (or called zero dynamics) is asymptotically stable, even the output control can control only partial states of the nonlinear dynamics, the entire nonlinear system is still asymptotically stable [11]. Due to the freedom of attaching control signals to the receiver system, carefully selected manifold maybe become the stable invariant manifolds so when the system state under the control arrives at the invariant manifolds, it will stay in them forever and be driven to the equilibrium by the dynamics on the invariant manifold. Naturally, we can extend this idea to chaos synchronization. Instead of controlling the entire states of $\hat{e}_x$, we attempt to control only some of the states of $\hat{e}_x$ so that once these states are controlled to reach zero, the rest of the states will be automatically driven to zero by the internal force within the chaotic system. This characteristic can be used for developing a simple and effective design of invariant manifold(s).

To formulate it mathematically, without loss of generality, let us assume there exists a one dimensional manifold $s$ such that once $s = 0$ is reached asymptotically by an appropriate control $\gamma$, then the error state vector $e_x$ will be driven by the sub-dynamics of (3) subject to $s = 0$ to asymptotically converge to zero. This assumption applies to the case where $s = 0$ is chosen in such way that $s$ is a linear function of $e_{x,i}$, yet $e_{x,i}$ are some components of error vector $e_x$ and the error dynamics under the constraint $s = 0$ is asymptotically stable. To make $s = 0$ an invariant manifold, we take the Lyapunov function $V = (1/2)s^2$, and design the control to satisfy the condition

$$\dot{V} = s\dot{s} < 0$$  \hspace{1cm} (4)

According to the Global Invariant Set Theorem [11], all the solutions will globally asymptotically converge to the largest invariant set in the set of all the points where $\dot{V} = 0$. In our particular setting, the largest invariant set confined by $s = 0$ is the invariant manifold which we want to achieve. Once $s = 0$ is reached, under our assumption, the error dynamics (3) under the specially selected manifold $s = 0$ is asymptotically stable. Hence the chaos synchronization is realized.

The question is that does the assumption of stability of the internal dynamics (or zero dynamics) hold for chaotic system? The answer is a yes. Many chaotic systems can be classified as dissipative systems, i.e. the trajectories of chaotic systems do not diverge. A typical example is the Lorenz system whose trajectories are bounded. This suggests that there exists a contracting force within chaotic dynamics, which is the key factor maintain the boundedness of trajectories. Therefore, for dissipative chaotic systems, there must exist as least one-dimensional manifolds (maybe more than one-dimensional manifolds) within the chaotic dynamics, which can be used as the invariant manifold(s). Taking advantage of these manifolds (rather eliminating them) would give rise to a control strategy which deliver a desired
control performance using as little control force as possible.

The method to select the invariant manifold is not unique. In this paper, we consider the method to build up the invariant manifold using the input-output linearization approach. Let $y = h(x)$ be the output, where $h$ is a scalar smooth field on $\mathbb{R}^n$. We use the well-known input-output linearization approach to reformulate (1) as

$$\frac{d\mu_i}{dt} = \mu_{i+1}, i = 1, \ldots, r - 1, \quad (5)$$
$$\frac{d\mu_r}{dt} = a(\mu, \phi), \quad (6)$$
$$\frac{d\phi}{dt} = w(\mu, \phi). \quad (7)$$

where $\mu_i = L_i^{-1}h(x)$ for $i = 1, \ldots, r - 1$, $a(\mu, \phi) = L_r^T h(x)$, $r$ being the relative degree of $h(x)$, with the output defined as $y = \mu_1$ and state vectors $\mu = (\mu_1, \ldots, \mu_r)^T$, $\phi = (\phi_1, \ldots, \phi_{n-r})^T$. By means of input-output linearization, the dynamics (1) is decomposed into an external (input-output) part and an internal (unobservable) part. Now the asymptotical stability problem on system (1) is already turned into the same problem as in system (5)–(7). In parallel to the equations (5)–(7), the input-output linearization form for the receiver system is as follows:

$$\frac{d\hat{\mu}_i}{dt} = \hat{\mu}_{i+1}, i = 1, \ldots, r - 1, \quad (8)$$
$$\frac{d\hat{\mu}_r}{dt} = a(\hat{\mu}, \hat{\phi}) + b(\hat{\mu}, \hat{\phi}) \gamma, \quad (9)$$
$$\frac{d\hat{\phi}}{dt} = w(\hat{\mu}, \hat{\phi}). \quad (10)$$

where $\hat{\mu}_i = L_i^{-1}h(\hat{x})$ for $i = 1, \ldots, r - 1$, $\hat{a}(\hat{\mu}, \hat{\phi}) = L_r^T h(\hat{x})$, $\hat{b}(\hat{\mu}, \hat{\phi}) = L_g L_r^{-1} h(\hat{x})$, $r$ being the relative degree of $h(\hat{x})$, with the output defined as $\hat{y} = \hat{\mu}_1 = h(\hat{x})$ and state vectors $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_r)^T$, $\hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_{n-r})^T$. Letting $e_i = \mu_i - \hat{\mu}_i, i = 1, \ldots, r, e_\phi = \phi - \hat{\phi}$ subtracting (8, 9, 10) from (5, 6, 7), then the error dynamics is as follows:

$$\frac{de_i}{dt} = e_{i+1}, i = 1, \ldots, r - 1, \quad (11)$$
$$\frac{de_r}{dt} = a(\mu, \phi) - a(\hat{\mu}, \hat{\phi}) - b(\hat{\mu}, \hat{\phi}) \gamma, \quad (12)$$
$$\frac{de_\phi}{dt} = w(\mu, \phi) - w(\hat{\mu}, \hat{\phi}). \quad (13)$$

According to the linearization theory, the asymptotically stable problem of
error dynamics in (3) has been changed into the same problem as in the error dynamics (11)–(13), therefore the chaos synchronization problem appears between (1) and (2) has been turned into the same problem between (5)–(7) and (8)–(10). So, the chaos synchronization task is to make sure the error dynamics (11)–(13) is asymptotically stabilized.

What we are going to do next is to build up an invariant manifold for all the observable error states in (11)–(13) to guarantee that the observable error states are asymptotically stabilized. Setting

$$s = e_r + c_{r-1} e_{r-1} + ... + c_1 e_1.$$ \hspace{1cm} (14)

where $c_i > 0, i = 1, ..., r - 1$ and $1, c_{r-1}, c_{r-2}, ..., c_1$ are coefficients of a Hurwitz polynomial. So, $\dot{s} = \dot{e}_r + c_{r-1} \dot{e}_{r-1} + ... + c_1 \dot{e}_1 = \dot{e}_r + c_{r-1} e_r + c_{r-2} e_{r-1} + ... + c_1 e_2$. If letting

$$v = \dot{e}_r = a(\mu, \phi) - a(\hat{\mu}, \hat{\phi}) - b(\hat{\mu}, \hat{\phi}) \gamma = - \sum_{i=2}^{r} c_{i-1} e_i - k(\sum_{j=1}^{r-1} c_j e_j + e_r), \hspace{1cm} (15)$$

where $k > 0$, then there is

$$ss = -k(\sum_{j=1}^{r-1} c_j e_j + e_r)(\sum_{j=1}^{r-1} c_j e_j + e_r) = -s^2 \leq 0. \hspace{1cm} (16)$$

According to the Invariant Set Theorem [11], the largest invariant set confined by $s = 0$ is the invariant manifold which we want the observable error dynamics to achieve. Once $s = 0$ is reached, all the observable error states in (11)–(13) under the specially selected manifold $s = 0$ is asymptotically stable. We also assume that the specially selected manifold satisfies the error zero dynamics in (11)–(13) is asymptotically stable once the observable error dynamics is asymptotically stabilized. Hence the chaos synchronization is realized. From the (15), the control for chaos synchronization is given as follows:

$$\gamma = b(\hat{\mu}, \hat{\phi})^{-1}(a(\mu, \phi) - a(\hat{\mu}, \hat{\phi}) + \sum_{i=2}^{r} c_{i-1} e_i + k(\sum_{j=1}^{r-1} c_j e_j + e_r)). \hspace{1cm} (17)$$

In the following, we shall demonstrate the effectiveness of the design by using the Lorenz system.
3 Simulations

The Lorenz system controlled is described as follows:

\[
\begin{align*}
\frac{dx_1}{dt} &= c(x_2 - x_1), \\
\frac{dx_2}{dt} &= rx_1 - x_2 - x_1x_3, \\
\frac{dx_3}{dt} &= x_1x_2 - bx_3.
\end{align*}
\]

where \( f = (c(x_2 - x_1), rx_1 - x_2 - x_1x_3, x_1x_2 - bx_3)^T \). Using \( c = 10, r = 28 \) and \( b = 8/3 \), the system shows the well-known butterfly chaos (see figure 1).

Just like what already claimed in Section 2, we consider the unidirectional synchronization in which the transmitter system can be depicted by equation (18). When choosing output as \( y = h(x) = x_1 \), we have all the relations needed in the linearization as follows:

\[
\begin{align*}
L^0_h &= h(x) = x_1, \\
L^1_h &= \frac{\partial h}{\partial x} f = c(x_2 - x_1), \\
L^2_h &= \frac{\partial c(x_2 - x_1)}{\partial x} f = -c^2(x_2 - x_1) + c(rx_1 - x_2 - x_1x_3).
\end{align*}
\]

The relative degree is 2 for output \( h = x_1 \). Now we try to build up a non-singular transform, let \( \mu_1 = h = x_1, \mu_2 = c(x_2 - x_1) \). We check \( \frac{\partial \mu_1}{\partial x} g = 0, \)

![Figure 1: Lorenz chaos in 3D space.](image)
for that let \( g = (0, 1, 0)^T \). Without loss of generality, let \( \phi_1 = x_3 \), we get the transformation as

\[
\mu = (\mu_1, \mu_2, \phi_1)^T = (x_1, c(x_2 - x_1), x_3)^T. \tag{20}
\]

since \( \frac{\partial \mu}{\partial x} = \begin{pmatrix} 1 & 0 & 0 \\ -c & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \) is non-singular matrix, so the transformation is non-singular. Therefore we have

\[
x = (x_1, x_2, x_3)^T = (\mu_1, \frac{1}{c} \mu_2 + \mu_1, \phi_1)^T. \tag{21}
\]

The reformulated transmitter system ia as follows:

\[
\begin{align*}
\frac{d\mu_1}{dt} & = \mu_2, \\
\frac{d\mu_2}{dt} & = a(\mu, \phi), \\
\frac{d\phi_1}{dt} & = \frac{\mu_2^2 + \mu_1 \mu_2}{c} - b\phi_1.
\end{align*} \tag{22}
\]

where \( a(\mu, \phi) = L^2_2h = -c^2(x_2 - x_1) + c(rx_1 - x_2 - x_1 x_3) = c(r - 1)\mu_1 - (c + 1)\mu_2 - c\mu_1 \phi_1 \).

Figure 2: Time responses of Lorenz system.
Now we are able to give the receiver system in two different forms as follows:

\[
\begin{align*}
\frac{d\hat{x}_1}{dt} &= c(\hat{x}_2 - \hat{x}_1), \\
\frac{d\hat{x}_2}{dt} &= r\hat{x}_1 - \hat{x}_2 - \hat{x}_1\hat{x}_3 + \gamma, \\
\frac{d\hat{x}_3}{dt} &= \hat{x}_1\hat{x}_2 - b\hat{x}_3.
\end{align*}
\]  

(23)

and

\[
\begin{align*}
\frac{d\hat{\mu}_1}{dt} &= \hat{\mu}_2, \\
\frac{d\hat{\mu}_2}{dt} &= a(\hat{\mu}, \hat{\phi}) + b(\hat{\mu}, \hat{\phi})\gamma, \\
\frac{d\hat{\phi}_1}{dt} &= \hat{\mu}_1^2 + \frac{\hat{\mu}_1\hat{\mu}_2}{c} - b\hat{\phi}_1.
\end{align*}
\]

(24)

where \((\hat{x}_1, \hat{x}_2, \hat{x}_3)\) and \((\hat{\mu}_1, \hat{\mu}_2, \hat{\phi}_1)\) are the states of the receiver system. \(a(\hat{\mu}, \hat{\phi}) = -c^2(\hat{x}_2 - \hat{x}_1) + c(r\hat{x}_1 - \hat{x}_2 - \hat{x}_1\hat{x}_3) = c(r - 1)\hat{\mu}_1 - (c + 1)\hat{\mu}_2 - c\hat{\mu}_1\hat{\phi}_1, b(\hat{\mu}, \hat{\phi}) = L_g L_f^h = \frac{\partial g}{\partial x} = (-c, c, 0)(0, 1, 0) = c > 0.\) The error dynamics can be obtained from the difference between equations (22) and (24) as follows:

\[
\begin{align*}
\frac{de_1}{dt} &= e_2, \\
\frac{de_2}{dt} &= a(\mu, \phi) - a(\hat{\mu}, \hat{\phi}) - b(\hat{\mu}, \hat{\phi})\gamma, \\
\frac{de_\phi}{dt} &= \mu_1^2 - \hat{\mu}_1^2 + \frac{\hat{\mu}_1\hat{\mu}_2}{c} - \frac{\hat{\mu}_1\mu_2}{c} - b(\phi_1 - \hat{\phi}_1).
\end{align*}
\]

(25)

where \(e_1 = \mu_1 - \hat{\mu}_1, e_2 = \mu_2 - \hat{\mu}_2, e_\phi = \phi_1 - \hat{\phi}_1.\) So, if the observable error dynamics \(e_1, e_2\) are asymptotically stable, then \(e_1 \to 0, e_2 \to 0, t \to \infty.\) Therefore \(\hat{\mu}_1 \to \mu_1, \hat{\mu}_2 \to \mu_2, t \to \infty,\) and of course, \(\hat{\mu}_1^2 \to \mu_1^2\) and \(\hat{\mu}_1\hat{\mu}_2 \to \mu_1\mu_2.\) So the last equation in (25) has the form \(\frac{de_\phi}{dt} = -b e_\phi,\) therefore \(e_\phi \to 0, t \to \infty.\) The zero dynamics is asymptotically stabilized. From (17), the control \(\gamma\) is chosen as:

\[
\gamma = b^{-1}(\hat{\mu}, \hat{\phi})(a(\mu, \phi) - a(\hat{\mu}, \hat{\phi}) + c_1 e_2 + k(c_1 e_1 + e_2))
\]

\[
= \left(\frac{kc_1}{c} - c_1 - k + c + r\right)(x_1 - \hat{x}_1) + (c_1 + k - c - 1)(x_2 - \hat{x}_2) - (x_1 x_3 - \hat{x}_1 \hat{x}_3).
\]

In simulations, \(k = 10, c_1 = 10.\) Figure 2 is the time response of Lorenz system, where \(e_1 = x_1 - \hat{x}_1, e_2 = x_2 - \hat{x}_2, e_3 = x_3 - \hat{x}_3.\) One can see the zero dynamic \(e_3\) gradually converges to zero after the observable dynamics approaches to zero. Figure 3 is the control.
4 Conclusion

Different from the conventional error feedback approach to design chaos synchronization, in which the feedback gain sometime is hard to obtain, this paper presented a new design method for chaos synchronization by the invariant manifold approach. Although the discussion above is for the single input single output systems, the idea can be extended to the multi input multi output systems.

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6 References


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