Euler Discretization of Second-Order Terminal Sliding Mode Control Systems

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Abstract—In this paper, discretization behaviors of second-order terminal sliding mode control (TSMC) systems under Euler method are studied. Upper bounds for the system steady states are established. Furthermore, some period properties are also investigated. Simulations are present to verify the theoretical analysis.

I. INTRODUCTION

Sliding mode control (SMC) [1], [2], [3] has been extensively studied and applied in practice due to its attractive features such as simplicity and invariance as well as robustness to matched uncertainties (that is, the uncertainties satisfy a matching condition) [1], [4]. The characteristic feature of SMC systems is that a sliding mode occurs on a prescribed manifold, where switching control is employed to maintain the state on that manifold. Conventional manifolds are usually linear hyperplanes which guarantee the asymptotic stability. To achieve the finite time convergence, a new technique called terminal sliding mode control (TSMC) has been developed in [11], [12]. In TSMC systems, a nonlinear switching manifold is applied so that system states reach the equilibrium in finite time.

Study on discretization effects on SMC systems has been undertaken recently due to the need for digital control of dynamical systems in process manufacturing industry. A primary reason for this study is that there are some intrinsic dynamic properties within the discretized SMC systems which do not appear in their continuous-time counterpart systems. Periodic behaviors are a common phenomenon in the discretized SMC systems. Despite some research work done on discretized SMC systems, the impact of discretization on the dynamical behaviors of TSMC systems is still not well understood. In [9], a class of first-order discrete-time system with power rule has been explored. In this paper, discretization behaviors of second-order TSMC systems under Euler method are investigated. Periodic orbits are observed and analyzed. Finally, the analysis is verified by simulation studies.

II. SECOND-ORDER SYSTEM MODELS AFTER DISCRETIZATION

Consider a 2nd-order system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a_1 x_1 - a_2 x_2 + u
\end{align*}
\]

with the TSM control, \( u = u_{eq} - K \text{sgn}(s) \), where

\[
\begin{align*}
s &= \dot{x}_1 + \beta x_1^{q/p} \\
&= x_2 + \beta x_1^{q/p}
\end{align*}
\]

and

\[
u_{eq} = a_1 x_1 + a_2 x_2 - \beta q/p x_1^{(q-p)/p} x_2
\]

where \( q \) and \( p \) are odd positive integers satisfying \( \frac{1}{2} < q/p < 1 \) or equivalently, \( 2q - p > 0 \) as addressed in [11]. Applying the above TSM control law, after Euler discretization, the discretized system is shown below,

\[
\begin{align*}
x_1(k+1) &= x_1(k) + T \Delta x_2(k) \\
x_2(k+1) &= x_2(k) + T(-\beta q/p x_1^{(q-p)/p}(k)x_2(k)) - K \text{sgn}(s(k))
\end{align*}
\]

where \( T \) is the sampling period.

If we rewrite (1) and (2) as

\[
x(k+1) = A x(k) - KT b \text{sgn}(s(k))
\]

where \( b = [0\ 1]^T \), then

\[
A = \begin{bmatrix} 1 & T \\
0 & 1 - \beta T q/p x_1^{(q-p)/p}(k) \end{bmatrix}
\]

III. STABILITY AND BOUNDEDNESS ANALYSIS

Let’s take a look at

\[
\Delta s(k) = s(k+1) - s(k)
\]

\[
x(k+1) = x_1(k+1) + \beta x_1^{q/p}(k+1)
\]

\[
x_1(k+1) = x_1(k) + T(-\beta q/p x_1^{(q-p)/p}(k)x_2(k)) - K \text{sgn}(s(k)) + \beta x_1^{q/p}(k+1)
\]

Because \( x_2(k) = s(k) - \beta x_1^{q/p}(k) \),

\[
s(k+1) = s(k) - \beta x_1^{q/p}(k) - \beta T q/p x_1^{(q-p)/p}(k)x_2(k) - K T sgn(s(k)) + \beta x_1^{q/p}(k+1)
\]

Because \( x_2(k) = x_1(k+1) - x_1(k) \),

\[
s(k+1) = s(k) - \beta T q/p x_1^{(q-p)/p}(k)x_1(k+1) - x_1(k) + \beta x_1^{q/p}(k+1) - K T sgn(s(k))
\]
Then,
\[
\triangle s(k) = -\beta q/px_1^{(q-p)/p}(k)(x_1(k+1) - x_1(k)) \\
+\beta x_1^{q/p}(k)(x_1(k+1) - x_1^{q/p}(k)) - KT \text{sgn}(s(k)) \\
= -\beta q/px_1^{q/p}(k)(x_1(k+1) - x_1^{q/p}(k)) + \beta x_1^{q/p}(k) \\
((x_1(k+1) - x_1(k))^{q/p} - 1) - KT \text{sgn}(s(k))
\]
Let \(\frac{x_1(k+1)}{x_1(k)} = \gamma\), then
\[
\triangle s(k) = -\beta q/px_1^{q/p}(k)(\gamma - 1) + \beta x_1^{q/p}(k) \\
(\gamma^{q/p} - 1) - KT \text{sgn}(s(k))
\]
Because \(\gamma = \frac{x_1(k+1)}{x_1(k)} = \frac{x_1(k) + T x_2(k)}{x_1(k)} = \frac{x_1(k) + T O(T)}{x_1(k)} = 1 + O(T),\)
\[
|1 - \gamma + p/q(\gamma^{q/p} - 1)| = O(T) + p/q((1 + O(T))^{q/p} - 1) = O(T) + p/q(1 + q/pO(T) + O(T^2) - 1) = O(T)
\]
So, one has
\[
|\beta q/px_1^{q/p}(k)(\gamma - 1) + \beta x_1^{q/p}(k)(\gamma^{q/p} - 1)| = O(T)
\]
To get the bound of states, we introduce the following lemma which appears in [10].

**Lemma 1:** For the scalar dynamics
\[
z(k+1) = z(k) + g(k) - \varepsilon \text{sgn}(z(k)) \tag{5}
\]
if \(|g(k)| < \gamma, \gamma > 0\) and \(\gamma < \varepsilon\), then the state \(z\) converges asymptotically to the range confined by
\[
|z| \leq \varepsilon + \gamma < 2\varepsilon \tag{6}
\]
From the above lemma, one can conclude if \(|1 - \beta q/px_1^{(q-p)/p}(k)(\gamma - 1) + \beta x_1^{q/p}(k)(\gamma^{q/p} - 1)| < KT,\) then \(|s(k)| < 2KT\). The following analysis is based on this result.

**Theorem 1:** If there exists a \(d, 0 < d < 1\) such that \(|1 - \beta Tq/px_1^{(q-p)/p}(k)| < d\), then the states of system (1) and (2) are bounded by the following inequalities.
\[
|x_1(\infty)| \leq \frac{1}{1 - \beta \cdot KT(2 + 1/(1 - d))} \tag{7}
\]
\[
|x_2(\infty)| \leq KT/(1 - d) \tag{8}
\]
**Proof:** Now we are ready to give the bound for system states. Let’s begin with \(x_2(k)\), from (2), one has
\[
x_2(k+1) = (1 - \beta T q/px_1^{(q-p)/p}(k))x_2(k) - KT \text{sgn}(s(k)) \tag{9}
\]
Assume that \(|1 - \beta T q/px_1^{(q-p)/p}(k)| < d < 1\), then the bound of \(x_2(k+1)\) is
\[
|x_2(k+1)| \leq d|k(x_2(0)| + KT \sum_{i=0}^{k} d^i
= d|k(x_2(0)| + KT(1 - d^{k+1})/(1 - d)
\]
As \(k \to \infty,\)
\[
|x_2(\infty)| \leq KT/(1 - d)
\]
Because \(s(k) = x_2(k) + \beta x_1^{q/p}(k),\)
\[
x_1^{q/p}(k) = 1/\beta(s(k) - x_2(k))
\]
One can derive the bound of state \(x_1\) as
\[
|x_1(\infty)| \leq [1/\beta \cdot KT(2 + 1/(1 - d))]^{p/q}
\]
The phase portrait of a typical trajectory is drawn in Fig. 1 where the system trajectory first converges to the nonlinear switching line and then approaches the origin while switching between the zone of \(s > 0\) and \(s < 0\), respectively. The periodic behaviors will be investigated in the next section.

To get a better understanding of how the system trajectory moves towards the nonlinear switching line from somewhere in the phase plane, another phase portrait of the system is displayed in Fig. 2.

In Fig. 2, the phase plane is divided into six regions. In Regions I, II and VI, \(s > 0\) whereas in Regions III, IV and V, \(s < 0\). In the following, we will show that wherever the system state starts, the system trajectory eventually approaches and crosses the switching line. Due to the symmetry of the six regions, only the phase portrait of system trajectory starting from Regions I, II and VI are analyzed.

First, let’s first assume that state starts from Region I where \(s(k) > 0\). If the state starts from Region I, because \(|1 - \beta T q/px_1^{(q-p)/p}(k)| < 1, x_2(k)\) will move down to Region II. Meanwhile, \(x_1(k)\) increases because \(x_2(k) > 0\) in Region I. One can see if \(x(k)\) reaches \(x_2 = 0, x_1(k+1) = x_1(k)\). When states gets into Region II starting from \(x_2 = 0\), assuming that \(x_2(k) = 0^-,\) because \(s(k) > 0\), from (2), one can see \(x_2(k+1) = -KT < 0\). Because when \(x_2(k)\) first gets into Region II, \(|x_2(k)|\) is small, \(x_2(k)\) decreases for a while and \(x_1(k)\) decreases due to the fact that \(x_2(k) < 0\). Finally the system state will approach and cross the switching line \(s = 0\).
If the system state starts from Region VI, there are two possibilities, one is, after a while, the state enters Region I because $x_1(k)$ increases due to $x_2(k) < 0$ in Region VI. If the state enters Region I, it is the same as the scenario we have analyzed above. The other possibility is the system state is always within Region VI where $s(k) > 0$, $x_1(k) < 0$ and $x_2(k) > 0$. One can see $x_1(k)$ will be increasing due to $x_2(k) > 0$. Now to show that the system state will approach and cross $s = 0$, we need to prove that $x_2(k)$ is decreasing. Because $|1 - \beta T q/p x_1^{(q-p)/p}(0)| < 1$, one can see $x_2(k)$ does decrease.

From the above analysis, one can understand that the system state will approach and cross the switching line $s = 0$ from wherever the state starts.

In fact, if the system trajectory starts from within the bound, then after a specific number of steps, the system state will approach and cross the switching line. In the following, we will explain it.

Now we assume that $|1 - \beta T q/p x_1^{(q-p)/p}(0)| > 1$, because $\beta T q/p x_1^{(q-p)/p}(0)$ is always positive, one can see the condition $|1 - \beta T q/p x_1^{(q-p)/p}(0)| > 1$ implies $1 - \beta T q/p x_1^{(q-p)/p}(0) < -1$. This condition leads to $|x_1(0)| > (p x_1^{(q-p)/p})^{p/q-p}$. Without loss of generality, we assume that the system trajectory starts from Region I where $x_1(0) > 0$ and $x_2(0) > 0$. In the next step, from (1), because $x_1(k+1) = x_1(k) + T x_2(k)$ and $x_2(0) > 0$, one can see the value of $x_1(1)$ will be increasing. Now let’s have a look at $x_2(1)$, because $1 - \beta T q/p x_1^{(q-p)/p}(0) < -1$ and $\text{sgn}(s(0)) > 0$, we have $x_2(1) < 0$ and $|x_2(1)| > x_2(0)$. So in the next step, $x(1)$ moves straight down from Region I to Region II or Region III. If $x(1)$ still has not been attracted to the switching line, we look at the next step $x(2)$. If $x(1)$ is in Region II, from (1), because $x_1(1) > 0$, one can see $x_2(2)$ will move towards the switching line until it is attracted to it.

If $x(1)$ is in Region III, which means $x(1)$ misses the switching line. In the next step, same as in the first scenario, $x(2)$ will move towards left because the value of $x_1(2)$ is decreasing. At the same time, $x(2)$ will increase because $1 - \beta T q/p x_1^{(q-p)/p}(0) < -1$ and $\text{sgn}(s(1)) = -1$. After $x(2)$ goes into Region IV, there are two possibilities. One is $x(2)$ is outside the bound, as we analyzed above, the system trajectory will be attracted to the switching line. The other possibility is that the state $x(2)$ is within the bound. In this case, as we just analyzed above, one can see that in the next step, $x(3)$ will jump to Region V or Region VI. If it jumps into Region V, it will go towards the switching line, if it jumps into Region VI, it is just as we analyzed above, in accordance with the scenario where $x(0)$ starts from Region III. Note that because the value of bound is small, with the normal setting of parameters, the possibility of $x(3)$ staying inside the bound is extremely small.

Singularity is one drawback of TSMC systems. A solution to avoid this singularity is derived in [13], which requires that the system states start from a prescribed sector of state space, $\Omega = \{x : x_1 > 0\} \cap \{x : s > 0\}$. The system trajectory starting from $\Omega$ will result in switching manifolds $s = 0$, $x_1 = 0$ being reached sequentially and hence the singularity problem will not occur.

In the discretized system, from (9), one can see

$$x_2(k+1) = x_2(k) - \beta T q/p x_1^{(q-p)/p}(k) s(k) + \beta^2 T q/p x_1^{(q-p)/p}(k) - KT \text{sgn}(s(k))$$

which shows that the singularity may still occur in the term $\beta T q/p x_1^{(q-p)/p}(k) s(k)$. If $x_1(k) = 0$ whereas $s(k) \neq 0$, then $x_2(k+1)$ is infinity. In fact, this term also appears in $u_{eq}$. The selection of the initial state is now very crucial. In the following, the issue of singularity will be addressed.

To better illustrate the singularity, let’s have a look at the following example where $T = 0.05$, $q = 3$, $p = 5$ and $K = \beta = 1$. Now if we let $x(0) = (0.1, -2)^T$, then one can see $x_1(1) = 0$ and $x_2(1) = -1.7993$. In the next step, singularity occurs which sees $x_1(2) = -0.09$ and $x_2(2) = \infty$. Note that in this case $(\frac{2}{\beta T q/p})^{p/q-p} = 2.7557 \times 10^{-5}$. System trajectory is shown in Fig. 4.
Now let’s change the initial state to \( x(0) = (0.1, -2.001)^T \). Then, the next two steps are \( x_1(1) = -5 \times 10^{-5}, x_2(1) = -1.8002 \) and \( x_1(2) = -0.0901, x_2(2) = 1.0868 \). System trajectory is shown in Fig. 5.

Let the initial state be \( x(0) = (0.1, -2.001)^T \). Then, the next two steps are \( x_1(1) = -5 \times 10^{-6}, x_2(1) = -1.7994 \) and \( x_1(2) = -0.09, x_2(2) = 5.3735 \). System trajectory is shown in Fig. 6.

Note that the system trajectories in Fig. 5 and Fig. 6 depict the analysis shown above. Now one can see that starting from \( \Omega \) can also avoid the singularity in the discretized TSMC system. Furthermore, from the above phase portrait analysis, to get a satisfactory phase portrait without an abrupt leap, we should let the system state start as \( x_0 > (\frac{2}{p+p+q})^{q-p} \) and \( s(0) > 0 \) in practical applications.

### IV. Periodic behaviors observations and simulation studies

Periodic behaviors have been found in second-order discretized TSMC systems. In early research work [5], [6], [7], [8], [10], orbits with multiple periods have been found and analyzed. In the discretized TSMC system, there are a few properties similar to some found in the discretized conventional SMC systems. Assume that its symbolic sequence \( s = s_0, s_1, s_2, \ldots \) is with period-\( L \) where \( s_i = \text{sgn}(s(i)) \). Then we can show that the following equation holds.

\[
\sum_{i=0}^{L-1} s_i = 0 
\quad (10)
\]

Equation (10) implies the period of the symbolic sequence is always even. Another fact which has been observed in the periodic orbits is the number of symbolic sequence with the same sign is no greater than 2. That means the symbolic sequence is only the combination of \(+--\) and \(++--\).

However, a major difference between periodic behaviors of these two discretized systems is that the final states in periodic orbits of discretized TSMC systems are determinant and not relying on initial states and where the system trajectory starts. Recall the early work done on discretized SMC systems, the final states in periodic orbits always depend on the initial states and there is no rule to obey. It is expected that the final states of periodic orbits with any periods could be determined by some algebraic equations. Another interesting periodic behavior is that in simulations, only period-2 and period-4 orbits appear. Authors reckon these phenomena could be explained by the fact that continuous TSMC systems achieve finite time convergence. To explain these phenomena in mathematics is the current research topic authors are working on.

In the following, several simulation case studies are presented to support our analysis.

First let’s assume \( \beta = K = 1, \quad q = 3, \quad p = 5, \quad T = 0.1 \), the system trajectory starting from \((1,1)\) is shown in Fig.
where it converges to a period-4 orbit. Starting from a different initial state, \( x(0) = (-0.01, 0.01)^T \), results in the same period-4 orbit with the final four states which are 

\[
(-0.0041, -0.1251), (-0.0084, 0.0423), (-0.0041, 0.1251), (0.0084, -0.0423)
\]

in Fig. 8.

Fig. 7. System trajectory with the initial state \( x(0) = (1, 1) \) and \( T = 0.1, q = 3, p = 5 \).

Fig. 8. Period-4 orbit with initial state \( x(0) = (-0.01, 0.01) \) and \( T = 0.1, q = 3, p = 5 \).

A period-2 orbit can appear with a different initial state. An orbit with period-2 is found in Fig. 9 with \( x(0) = (0.01, 0.01)^T \). The final two states are 

\[
(-0.0035, 0.0702), (0.0035, -0.0702)
\]

Note that the calculated bounds of states from Theorem 1 are

\[
|x_1(\infty)| < 0.2782, \quad |x_2(\infty)| < 0.2641
\]

which are a bit conservative comparing with real values of the final states.

Now, let’s change the values of \( q \) and \( p \) as \( q = 5 \) and \( p = 7 \) and make \( T = 0.05 \). Without any surprises, The period-2 and period-4 orbits are also coexisting. The simulation results are shown in Fig. 10 and Fig. 11.

The final two states in the period-2 orbit are 

\[
(0.0007, -0.0291), (-0.0007, 0.0291)
\]

and the final four states in the period-4 orbit are

\[
(-0.0016, 0.0077), (-0.0012, 0.0560), (0.0016, -0.0077), (0.0012, -0.0560)
\]

The above simulation results verify the analysis which concludes that the period-2 and period-4 orbits are coexisting. The period of the orbit depends on the initial condition. A series of simulations have been conducted to compare the period regions (the region where the system trajectory ending up with a periodic orbit starts) with different \( q \) and \( p \). We found that if \( p - q = 2 \), then increasing the values of \( q \) and \( p \) leads to disappearance of period-4 orbits. Due to the limit of space, the simulation results are not presented. This phenomenon is due to the fact that when \( q \) and \( p \) are getting larger, the system is behaving more like a Euler discretized linear SMC system which has been found to have only period-2 orbits [8].
V. CONCLUSION

In this paper, the discretization behaviors of second-order TSMC systems have been studied with simulation verifications. As it has already been expected, periodic orbits appear in the steady state of the system trajectory. The insightful understanding of these dynamics will help develop preventive measures for ill-behaviors due to discretization in digital TSMC systems design.

REFERENCES