Portfolio Selection in the Enlarged Markovian Regime-switching Market

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We introduce a set of \( j^{th} \) Markovian geometric jump assets to complete the Markovian regime switching market.

We consider the portfolio optimization problem in the enlarged Markovian regime-switching market.

We establish the relationship between the optimal portfolio problems in the enlarged market and the original market.
2 Background of Completing Market

Karatzas et al. [6] introduced a way to complete the incomplete market by adding the fictitious stocks for the original incomplete market. By using the results of utility maximization problem in the complete market and the relationship between the complete market and the original incomplete market, Karatzas et al. [6] solved the utility maximization problem in the original incomplete market.

Corcuera et al. [3] completed the geometric Lévy market by adding the power-jump assets.

Corcuera et al. [4] considered the optimal investment problems in the completed geometric Lévy market with the new power-jump assets.
★ Niu [9] studied the optimal investment problems in a simple Lévy market\(^1\) which was completed by adding the power-jump assets introduced by Corcuera et al. [3].

★ Guo [5] introduced the change-of-state contract to complete the Markovian regime-switching market.

★ Zhang et al. [10] introduced the \(j^{th}\) Markovian jump assets to complete Markovian regime-switching market. One potential problem of the \(j^{th}\) Markovian jump assets is that it can take negative values with positive probability.

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\(^1\)In the simple Lévy market, the log stock price is driven by a Brownian motion plus a Poisson process.
Continuous time Markov chain:

- Canonical state space: a finite set of unit vectors

\[ E := \{ e_1, e_2, \ldots, e_N \} \]

where \( e_i \in \mathbb{R}^N \) and the \( j^{th} \) component of \( e_i \) is the Kronecker delta \( \delta_{ij} \).

- Generator of Markov chain \( \{ X(t) \} \): \( \Lambda := [\lambda_{ij}]_{i,j=1,2,\ldots,N} \)

- The Markov chain is assumed to be irreducible.
★ Interest rate, appreciation rate and volatility:

\[ r(t) := \langle r, X(t) \rangle, \quad r := (r_1, \ldots, r_N)' \]
\[ \mu_0(t) := \langle \mu_0, X(t) \rangle, \quad \mu_0 := (\mu_0^1, \ldots, \mu_0^N)' \]
\[ \sigma_0(t) := \langle \sigma_0, X(t) \rangle, \quad \sigma_0 := (\sigma_0^1, \ldots, \sigma_0^N)' \]

where \( \langle \cdot \rangle \) is the scalar product in \( \mathbb{R}^N \) and \( \{X(t)\} \) is a continuous time Markov chain as introduced before.

★ Markovian regime-switching market

\[ dB(t) = r(t)B(t)dt \]
\[ dS_0(t) = S_0(t) \left[ \mu_0(t)dt + \sigma_0(t)dW(t) \right] \]
Marked point process generated by Markov chain:

- Let \( \{T_n | n = 1, 2, \ldots, N\} \) be the jump epochs of the chain \( \mathbf{X} \) and \( \mathbf{X}_n := \mathbf{X}(T_n) \). Then \( \{(T_n, \mathbf{X}_n)\} \) is a marked point process generated by the Markov chain and we can define the random measure

\[
\Phi(t, j) := \Phi([0, t] \times \mathbf{e}_j) = \sum_{n \geq 1} 1_{(T_n \leq t, \mathbf{X}_n = \mathbf{e}_j)} ,
\]

(1)

- From Last and Brandt [7], the unique predictable projection of \( \Phi(t, j) \) is given by:

\[
\phi(t, j) := \int_0^t \lambda_j(s) ds ,
\]

(2)

where

\[
\lambda_j(s) := \sum_{i \neq j} 1_{\{\mathbf{X}(s-) = \mathbf{e}_i\}} \lambda_{ij} .
\]

(3)
For any $j = 1, 2, \ldots, N$,

$$
\tilde{\Phi}(t, j) := \Phi(t, j) - \phi(t, j)
$$

are basic $(F^X, \mathcal{P})$-martingales for the marked point process associated with the Markov chain $X$, where $F^X := \{F^X(t) | t \in T\}$.

$\tilde{\Phi}(t, j)$ is called the $j^{th}$ Markov jump martingale.
Now for each \( j = 1, 2, \ldots, N \), we introduce a set of \( j^{th} \) Markovian geometric jump assets as follows

\[
dS_j(t) = S_j(t-) \left( \mu_j(t-) dt + \sigma_j(t-) \tilde{\Phi}(dt, j) \right), j = 1, \ldots, N, \quad (5)
\]

where

\[
\mu_j(t-) := \langle \mu_j, X(t-) \rangle, \quad \sigma_j(t-) := \langle \sigma_j, X(t-) \rangle,
\]

with

\[
\mu_j := (\mu_j^1, \mu_j^2, \cdots, \mu_j^N)^t \in \mathbb{R}^N, \quad \sigma_j := (\sigma_j^1, \sigma_j^2, \cdots, \sigma_j^N)^t \in \mathbb{R}^N.
\]
We shall show that the following market enlarged with the $j^{th}$ Markovian geometric jump assets

$$
\begin{align*}
 dB(t) &= r(t)B(t)dt, \\
 dS_0(t) &= S_0(t) \left( \mu_0(t)dt + \sigma_0(t)dW_0(t) \right), \\
 dS_j(t) &= S_j(t-) \left( \mu_j(t-)dt + \sigma_j(t-)d\tilde{\Phi}_j(t) \right), \quad j = 1, 2, \cdots, N.
\end{align*}
$$

is arbitrage free and complete.
Suppose $Q$ and $P$ are equivalent and let

$$L(t) := E \left[ \frac{dQ}{dP} \bigg| \mathcal{G}(t) \right] ,$$

where $\mathcal{G}(t) := \sigma\{W_0(s), X(s) | s \in [0, t]\}$

**Theorem 1.** There exist $G$-predictable processes

$$\psi_j := \{\psi_j(t) | t \in T\}, \ j = 0, 1, \cdots, N$$

such that

$$L(t) = 1 + \int_0^t L(s-)\psi_0(s)dW_0(s) + \sum_{j=1}^N \int_0^t L(s-)\psi_j(s)\tilde{\Phi}(ds, j) . \quad (6)$$
By the generalize Girsanov’s theorem for jump-diffusion process, we have under $\mathcal{Q}$ the processes

$$W_0^Q(t) := W_0(t) - \int_0^t \psi_0(s)ds, \quad t \in \mathcal{T},$$

is a standard Brownian motion and for $j = 1, 2, \cdots, N$,

$$\bar{\Phi}^Q(t, j) := \Phi(t, j) - \int_0^t (1 + \psi_j(s))\phi(ds, j), \quad t \in \mathcal{T},$$

are ($\mathcal{Q}, \mathcal{G}(t)$)-martingales.
The price processes under $Q$

$$dB(t) = r(t)B(t)dt,$$

$$dS_0(t) = S_0(t)\left( [\mu_0(t) + \sigma_0(t)\psi_0(t)]dt + \sigma_0(t)dW_0^Q(t) \right),$$

$$dS_j(t) = S_j(t-1)\left( [\mu_j(t-1) + \sigma_j(t-1)\psi_j(t)\lambda_j(t)]dt + \sigma_j(t-1)\bar{\Phi}^Q(dt,j) \right), j = 1, 2, \ldots, N.$$

**Martingale conditions**

$$\mu_0(t) + \sigma_0(t)\psi_0(t) = r(t), \quad (7)$$

$$\mu_j(t-1) + \sigma_j(t-1)\lambda_j(t)\psi_j(t) = r(t-1). \quad (8)$$

The martingale conditions can be obtained from Proposition 10.1.8 of Musiela and Rutkowsk [8].
From martingale conditions, we can determine $\psi_0(t)$ and $\psi_j(t)$ by

$$
\psi_0(t) = \frac{r(t) - \mu_0(t)}{\sigma_0(t)}, \quad \psi_j(t) = \frac{r(t-) - \mu_j(t-)}{\sigma_j(t-)\lambda_j(t)}, \quad j = 1, \ldots, N.
$$

Note that

$$
\lambda_j(t) = 0 \text{ iff } X(t-) = e_j.
$$

Thus we can only determine $\psi_j(t)$ when $X(t-) \neq e_j$, however this is sufficient for us to determine the equivalent martingale measure $Q$. Indeed, when $X(t-) = e_j$, $\tilde{\Phi}(dt, j) = 0$, and, therefore $\psi_j(t)$ has no impact on the value of the right side of equation (6) when $X(t-) = e_j$. 

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Now if we set

\[ \psi_0 := \left( \frac{r_1 - \mu_1^0}{\sigma_0^1}, \frac{r_2 - \mu_2^0}{\sigma_0^2}, \ldots, \frac{r_N - \mu_N^0}{\sigma_0^N} \right), \]

\[ \psi_j := \left( \frac{r_1 - \mu_j^1}{\sigma_j^1 \lambda_{1j}}, \ldots, \frac{r_{j-1} - \mu_j^{j-1}}{\sigma_j^{j-1} \lambda_{j-1,j}}, 0, \frac{r_{j+1} - \mu_j^{j+1}}{\sigma_j^{j+1} \lambda_{j+1,j}}, \ldots, \frac{r_N - \mu_j^N}{\sigma_j^N \lambda_{N,j}} \right), \]

\[ j = 1, 2, \ldots, N, \]

we can write \( \psi_j(t), j = 0, 1, \ldots, N, \) as follows:

\[ \psi_0(t) = \langle \psi_0, X(t) \rangle, \quad \psi_j(t) = \langle \psi_j, X(t- \cdot) \rangle, \quad (10) \]

for each \( j = 1, 2, \ldots, N. \)
Theorem 2. Assume that \( \mu_j^j = r_j \), for all \( j = 1, 2, \ldots, N \), and that \( L(t) \) is given by (6) with \( \psi_j(t), j = 0, 1, \ldots, N \), given by (10). Define a new measure \( Q \) equivalent to \( P \) on \( \mathcal{G}(T) \) by

\[
\frac{dQ}{dP}
\bigg|_{\mathcal{G}(T)} := L(T) .
\] (11)

Then under \( Q \) the price processes of the securities in the enlarged market admit the following representations:

\[
\begin{align*}
    dB(t) &= r(t)B(t)dt \\
    dS_0(t) &= S_0(t) \left( r(t)dt + \sigma_0(t)dW^Q_0(t) \right) \\
    dS_j(t) &= S_j(t- \left( r(t-)dt + \sigma_j(t-)\tilde{\Phi}^Q(dt,j) \right) , \quad j = 1, 2, \ldots, N ,
\end{align*}
\]

and, therefore the discounted price processes of the securities in the enlarged market are \( (\mathcal{G}, Q) \)-martingales, and, the enlarged market is arbitrage-free. Moreover, the market is complete.
The Finance Market:

\[ dB(t) = r(t)B(t)dt , \]
\[ dS_0(t) = S_0(t) \left( \mu_0(t-) dt + \sigma_0(t)dW_0(t) \right) , \]
\[ dS_j(t) = S_j(t-) \left( \mu_j(t-) dt + \sigma_j(t-)d\tilde{\Phi}_j(t) \right) , \quad j = 1, 2, \cdots, N . \]
★★ \( \tilde{\pi}_j(t) \): the fraction of wealth invested in \( S_j(t) \) at time \( t \), for each \( j = 0, 1, 2, \ldots, N \).

★★ The corresponding wealth process, denoted as \( R^{\tilde{\pi}} := \{ R^{\tilde{\pi}}(t) | t \in \mathcal{T} \} \), is governed by:

\[
\frac{dR^{\tilde{\pi}}(t)}{R^{\tilde{\pi}}(t-)} = \left[ r(t) + \sum_{j=0}^{N} \tilde{\pi}_j(t)(\mu_j(t-) - r(t)) \right] dt \\
+ \tilde{\pi}_0(t)\sigma_0(t)dW_0(t) + \sum_{j=1}^{N} \tilde{\pi}_j(t)\sigma_j(t-)d\bar{\Phi}_j(t) .
\] (12)
Objective:

\[ V(t, z, e_i) = \sup_{\tilde{\pi} \in \mathcal{A}} V^{\tilde{\pi}}(t, z, e_i) = \sup_{\tilde{\pi} \in \mathcal{A}} E_{t, z, i}[U(R^{\tilde{\pi}}(T))] . \]

where \( E_{t, z, i}[\cdot] \) is the conditional probability given \( R^{\tilde{\pi}}(t) = z \) and \( X(t) = e_i \) under \( \mathcal{P} \).
Logarithmic Utility: \( U(z) = \log(z) \)

**Theorem 3.** (i) For each \((t, z) \in \mathcal{T} \times \mathbb{R}^+\) and \(i = 1, 2, \ldots, N\),

\[
V(t, z, e_i) = \log(z) + h(t, e_i) ,
\]

where

\[
h(t, e_i) = E_{t,i}\left\{ \int_t^T \left( r(s) + \frac{(\mu_0(s) - r(s))^2}{2\sigma_0(s)^2} \right. \right.
\]

\[
- \sum_{j=1}^N 1\{x_{(s^-)} \neq e_j\} \lambda_j(s) \left[ \log \left( 1 - \frac{\mu_j(s^-) - r(s)}{\sigma_j(s^-)\lambda_j(s)} \right) - \frac{\mu_j(s^-) - r(s)}{\sigma_j(s^-)\lambda_j(s)} \right] \left. \right\} ds \right\} ,
\]

and \( E_{t,i}[\cdot] \) is the conditional expectation given \( X(t) = e_i \) under \( \mathcal{P} \).
(ii) Let \( \tilde{\pi}^*(s) := (\tilde{\pi}_0^*(s), \tilde{\pi}_1^*(s), \cdots, \tilde{\pi}_N^*(s))' \) and \( \tilde{\pi}_j^*(s) \) is defined by setting

\[
\tilde{\pi}_0^*(s) := \frac{\mu_0(s-)-r(s)}{\sigma_0(s)^2},
\]

\[
\tilde{\pi}_j^*(s) := 1\{x(s-) \neq e_j\} \frac{\mu_j(s-)-r(s)}{\sigma_j(s-)^2 \lambda_j(s) - \sigma_j(s-)(\mu_j(s-)-r(s))},
\]

\( j = 1, 2, \cdots, N, \quad s \in [t, T]. \)

Then \( \tilde{\pi}^* := \{\tilde{\pi}^*(s)|s \in [t, T]\} \) is the optimal portfolio strategy to the portfolio selection problem with logarithmic utility.
Proof. By applying a generalized version of Itô’s differentiation rule for jump-diffusion processes, we can obtain the following explicit expression of the wealth process (12):

\[
R^{\tilde{\pi}}(T) = R^{\tilde{\pi}}(t) \exp \left\{ \int_t^T \left[ r(s) + \sum_{j=0}^N \tilde{\pi}_j(s)(\mu_j(s) - r(s)) - \frac{\tilde{\pi}_0(s)^2\sigma_0(s)^2}{2} \right] ds \\
+ \sum_{j=1}^N \int_t^T \left[ \log(1 + \tilde{\pi}_j(s)\sigma_j(s-)) - \tilde{\pi}_j(s)\sigma_j(s-) \right] \lambda_j(s) ds \\
+ \int_t^T \tilde{\pi}_0(s)\sigma_0(s)dW_0(s) + \sum_{j=1}^N \int_t^T \log(1 + \tilde{\pi}_j(s)\sigma_j(s-))d\Phi_j(s) \right\}. (14)
\]
Therefore,

\[
V^{\tilde{\pi}}(t, z, \mathbf{e}_i) = E_{t,z,i}[U(R^{\tilde{\pi}}(T))] = \log(z) + h^{\tilde{\pi}}(t, \mathbf{e}_i),
\]  

(15)

where

\[
h^{\tilde{\pi}}(t, \mathbf{e}_i) := E^{t,i} \left[ \int_t^T \left( r(s) + \sum_{j=0}^{N} \tilde{\pi}_j(s)(\mu_j(s) - r(s)) - \frac{\pi_0(s)^2\sigma_0(s)^2}{2} \right) ds \\
+ \sum_{j=1}^{N} \int_t^T \left( \log(1 + \tilde{\pi}_j(s)\sigma_j(s) - \tilde{\pi}_j(s)\sigma_j(s)) - \tilde{\pi}_j(s)\sigma_j(s) \right) \lambda_j(s) ds \right].
\]  

(16)
So, to determine the optimal portfolio strategy, it suffices to maximize for each \( s \in [t, T] \), the following Hamiltonian:

\[
H(\tilde{\pi}_0(s), \tilde{\pi}_1(s), \cdots, \tilde{\pi}_N(s)) \\
:= r(s) + \sum_{j=0}^{N} \tilde{\pi}_j(s)(\mu_j(s) - r(s)) - \frac{\tilde{\pi}_0(s)^2 \sigma_0(s)^2}{2} \\
+ \sum_{j=1}^{N} \left( \log(1 + \tilde{\pi}_j(s)\sigma_j(s)) - \tilde{\pi}_j(s)\sigma_j(s) \right) \lambda_j(s) .
\]

(17)
Theorem 4. Suppose $h(t, e_i)$ is as in Theorem 3. Then $h(t, e_i), i = 1, 2, \ldots, N$, satisfy the following system of coupled linear differential equations:

$$
\frac{dh}{dt}(t, e_i) + r_i + \frac{(\mu_i^0 - r_i)^2}{2(\sigma_0^i)^2} - \sum_{j \neq i, j=1}^{N} \lambda_{ij} \left[ \log \left( 1 - \frac{\mu_j^i - r_i}{\sigma_j^i \lambda_{ij}} \right) + \frac{\mu_j^i - r_i}{\sigma_j^i \lambda_{ij}} \right] + \sum_{j=1}^{N} \lambda_{ij} h(t, e_j) = 0 ,
$$

with boundary conditions

$$
h(T, e_i) = 0, \quad i = 1, 2, \ldots, N.
$$
Remark 5. Let $G := (G_1, \cdots, G_N)'$ with $G_i$ given by

$$G_i = r_i + \frac{(\mu^i_0 - r_i)^2}{2(\sigma^i_0)^2} - \sum_{j \neq i, j = 1}^N \lambda_{ij} \left[ \log(1 - \frac{\mu^i_j - r_i}{\sigma^i_j \lambda_{ij}}) + \frac{\mu^i_j - r_i}{\sigma^i_j \lambda_{ij}} \right],$$

\[i = 1, 2, \ldots, N,\] \hspace{1cm} (18)

Then from Bronson [2, Chapter 8.4] and Proposition A4.1 of Asmussen [1], $h(t) := (h(t, e_1), \cdots, h(t, e_N))'$ has the following closed form

$$h(t) = (T - t) \exp(-\Lambda t) e \otimes \nu G + D[\exp(\Lambda(T - t)) - I]G$$

where $D = (\Lambda - e \otimes \nu)^{-1}$, $e = (1, 1, \cdots, 1)'$, $\nu$ is the stationary distribution of the Markov chain, and $I$ is the identity matrix.
Power Utility: \( U(z) = z^\alpha \)

HJB Equation:

\[
V_t(t, z, e_i) + \sup_{\tilde{\pi}} \left\{ \left[ r_i + \sum_{j=0}^{N} \tilde{\pi}_j (\mu^i_j - r_i) \right] zV_z(t, z, e_i) \right. \\
+ \sum_{j=1}^{N} \left[ V(t, z(1 + \tilde{\pi}_j \sigma^i_j), e_i) - V(t, z, e_i) - V_z(t, z, i) z\tilde{\pi}_j \sigma^i_j \right] \lambda_{ij} \\
+ \frac{1}{2} \tilde{\pi}_0^2 (\sigma^i_0)^2 z^2 V_{zz}(t, z, e_i) \right\} + \sum_{j=1}^{n} \lambda_{ij} V(t, z, e_j) = 0 ,
\]

Boundary conditions:

\[
V(T, z, e_i) = z^\alpha , \quad i = 1, 2, \ldots , N .
\]
Theorem 6. Let $g(t) := (g(t, e_1), g(t, e_2), \ldots, g(t, e_N))'$ denote a “classical” solution to the following system of coupled linear differential equations with terminal conditions $g(T, e_i) = 1$:

$$\frac{dg}{dt}(t, e_i) = -\left[ \alpha r_i + \frac{\alpha}{2(1 - \alpha)} \frac{(\mu_0^i - r_i)^2}{(\sigma_0^i)^2} \right.$$

$$+ \sum_{j=1, j \neq i}^{N} \lambda_{ij} (1 - \alpha) \left(1 - \frac{\mu_j^i - r_i}{\lambda_{ij} \sigma_j^i}\right) \frac{\alpha}{\alpha - 1}$$

$$+ \sum_{j=1, j \neq i}^{N} \lambda_{ij} \left(\alpha \left(1 - \frac{\mu_j^i - r_i}{\lambda_{ij} \sigma_j^i}\right) - 1\right) \right] g(t, e_i) - \sum_{j=1}^{N} \lambda_{ij} g(t, e_j).$$

Then

★ $v(t, z, e_i) := z^\alpha g(t, e_i)$ is a solution to the system of HJB equations (19) with boundary conditions (20);
The Feynman-Kac formula yields the following stochastic representation for \( g(t, e_i) \):

\[
g(t, e_i) = E_{t,i} \left\{ \exp \left\{ \int_t^T \alpha r(s) + \frac{\alpha}{2(1 - \alpha)} \left( \frac{\mu_0(s-) - r(s)}{\sigma_0(s)} \right)^2 \right\} \right. \\
+ \sum_{j=1}^N 1_{\{X(s-) \neq e_j\}} \lambda_j(s) \left( \alpha \left(1 - \frac{\mu_j(s-) - r(s)}{\lambda_j(s)\sigma_j(s-)} \right) - 1 \right) \\
\left. + \sum_{j=1}^N 1_{\{X(s-) \neq e_j\}} (1 - \alpha) \lambda_j(s) \left( 1 - \frac{\mu_j(s-) - r(s)}{\lambda_j(s)\sigma_j(s-)} \right)^{\frac{\alpha}{\alpha - 1}} \right\} \right. \\
\left. ds \right\},
\]

where \( E_{t,i}[\cdot] \) is the conditional expectation given \( X(t) = e_i \) under \( \mathcal{P} \).
Proof. To solve the HJB equation (19), we try to fit a solution of the form

$$v(t, z, e_i) = z^\alpha g(t, e_i)$$

Substitute above trial equation into HJB equation (19), we finally obtained that $g(t, e_i)$ need to be satisfied equation (21).

For the proof of the Feynman-Kac representation of $g(t, e_i)$, it is standard.
Remark 7. Let \( \mathbf{H} = \text{diag}(H_1, \cdots, H_N) \) with \( H_i \) given by

\[
H_i = \alpha r_i + \frac{\alpha}{2(1 - \alpha)} \frac{(\mu_0^i - r_i)^2}{(\sigma_0^i)^2} \]
\[
+ \sum_{j=1, j \neq i}^N \lambda_{ij} (1 - \alpha) \left( 1 - \frac{\mu_j^i - r_i}{\lambda_{ij} \sigma_j^i} \right) \frac{\alpha}{\alpha - 1} \]
\[
+ \sum_{j=1, j \neq i}^N \lambda_{ij} \left( \alpha (1 - \frac{\mu_j^i - r_i}{\lambda_{ij} \sigma_j^i}) - 1 \right) , \]
\[
i = 1, 2, \cdots, N ,
\]

Then also from Bronson [2, Chapter 8.4], we have

\[
g(t) = \exp\left[(\mathbf{H} + \mathbf{\Lambda})(T - t)\right] \mathbf{e} ,
\]

where \( g(t) = (g(t, e_1), \cdots, g(t, e_N))' \).
Theorem 8. Suppose \( v(t, z, i) \) is as given in Theorem 6. Then

1. \( V(t, z, e_i) = v(t, z, e_i) \) for all \((t, z, e_i) \in [0, T] \times \mathbb{R}^+ \times E;\)

2. Let \( \tilde{\pi}^*(s) := (\tilde{\pi}_0^*(s), \tilde{\pi}_1^*(s), \cdots, \tilde{\pi}_N^*(s))' \) and \( \tilde{\pi}_j^*(s) \) is defined by setting

\[
\tilde{\pi}_0^*(s) := \frac{\mu_0(s-) - r(s)}{(1 - \gamma)\sigma_0(s)^2},
\]

\[
\tilde{\pi}_j^*(s) := 1\{x(s-) \neq e_j\} \left(1 - \frac{\mu_j(s-) - r(s)}{\lambda_j(s)\sigma_j(s-)}\right)^{\frac{1}{\gamma-1}} - 1, \quad j = 1, 2, \cdots, N,
\]

for all \( s \in [t, T]. \) Then \( \tilde{\pi}^* := \{\tilde{\pi}^*(s) | s \in [t, T]\} \) is the optimal portfolio strategy for the portfolio selection problem with power utility.
In this section, we shall regard the appreciation rates $\mu_j(t-), \ j = 1, \cdots , N$ of the $j^{th}$ Geometric Markovian jump securities as “free” parameters and give the relationships between the optimization problem in the enlarged market and in the original market in both logarithmic and power utility cases.

In what follows, we shall denote by $V_{\mu}$ the value function in the enlarged market enlarged market to emphasize its dependence on the appreciation rates $\mu(t-):= (\mu_1(t-), \cdots , \mu_N(t-))'$ of the $N$ Geometric Markovian jump securities in the enlarged market.
The optimization problem in the original market:

$\tilde{\pi}_0(t)$: the fraction of the wealth invested in $S_0(t)$ in the original market.

$R^{\tilde{\pi}_0} := \{ R^{\tilde{\pi}_0}(t) | t \in T \}$: the corresponding wealth process

$$\frac{dR^{\tilde{\pi}_0}(t)}{R^{\tilde{\pi}_0}(t)} = \left[ r(t) + \tilde{\pi}_0(t)(\mu_0(t) - r(t)) \right] dt + \tilde{\pi}_0(t)\sigma_0(t)dW_0(t) .$$ (21)

$V_0(t, z, e_i)$: the value function in the original market defined by

$$V_0(t, z, e_i) = \sup_{\tilde{\pi}_0 \in A_0} E_{t, z, i}[U(R^{\tilde{\pi}_0}(T))].$$
Theorem 9. Let $V_r(t, z, e_i)$ be the value function in the enlarged market with the appreciation rates of the $N$ Geometric Markovian jump securities equal to the risk-free interest rate. Then:

1. For the logarithmic utility,

$$V_0(t, z, e_i) = V_r(t, z, e_i) = \inf_{\mu} V_{\mu}(t, z, e_i)$$

$$= \log(z) + E_{t,i} \left[ \int_t^T \left( r(s) + \frac{\mu_0(s-)-r(s)^2}{2\sigma_0(s)^2} \right) ds \right] ,$$

and the corresponding optimal portfolio strategies are:

$$\tilde{\pi}_0^*(s) = \frac{\mu_0(s-)-r(s)}{\sigma_0(s)^2} , \quad \tilde{\pi}_j^*(s) = 0 , \quad j = 1, 2, \cdots , N . \quad (22)$$
2. For the power utility,

\[ V_0(t, z, e_i) = V_r(t, z, e_i) = \inf_{\mu} V_{\mu}(t, z, e_i) \]

\[ = z^\alpha E_{t,i} \left\{ \exp \left\{ \int_t^T \left[ \alpha r(s) + \frac{\alpha}{2(1 - \alpha)} \left( \frac{\mu_0(s-) - r(s)}{\sigma_0(s)} \right)^2 \right] ds \right\} \right\}, \]

and the corresponding optimal portfolio strategies are:

\[ \tilde{\pi}_0^*(s) := \frac{\mu_0(s-) - r(s)}{(1 - \alpha)\sigma_0(s)^2}, \quad \tilde{\pi}_j^*(s) := 0, \quad j = 1, 2, \cdots, N. \]
Proof. Note that the set of admissible strategies in the original incomplete market \( \mathcal{A}_0 \) can be regarded as the set of admissible strategies in the enlarged market \( \mathcal{A} \) if the following constraints are imposed

\[
\pi_j \equiv 0, \quad j = 1, 2, \cdots, N.
\]

Consequently, we must have the following inequalities:

\[
V_0(t, z, \mathbf{e}_i) \leq V_{\bm{\mu}}(t, z, \mathbf{e}_i), \quad i = 1, 2, \cdots, N.
\]

Since we assume in this section that the appreciation rates of the “fictitious” shares are “free” parameters,

\[
V_0(t, z, \mathbf{e}_i) \leq \inf_{\bm{\mu}} V_{\bm{\mu}}(t, z, \mathbf{e}_i), \quad i = 1, 2, \cdots, N. \tag{23}
\]
From the expression of $V_\mu$ in the Theorem 3 for logarithmic utility, we can easily obtain

$$V_r(t, z, e_i) = \inf V_\mu(t, z, e_i)$$

and the corresponding optimal strategy is given by

$$\tilde{\pi}_0^*(s) = \frac{\mu_0(s-) - r(s)}{\sigma_0(s)^2},$$

$$\tilde{\pi}_j^*(s) = 0 \quad j = 1, 2, \cdots, N.$$ 

The above optimal portfolio strategy can be said belong to $\mathcal{A}_0$ since

$$\tilde{\pi}_j^*(s) = 0 \quad j = 1, 2, \cdots, N.$$ 

Therefore, we have

$$\inf_{\mu} V_\mu(t, z, e_i) = V_r(t, z, e_i) \leq V_0(t, z, e_i)$$
References


