A new adaptive diffusion equation for image noise removal and feature preservation

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Abstract

Anisotropic diffusion can remove noise to some extent in image processing. However the contradiction between diffusion and preservation still exists. In this paper, a new nonlinear diffusion model for image noise removal and feature preservation is presented. This model treats inhomogeneity region and image feature adaptively by discontinuity measure and local gradient information. A well balance between diffusion and preservation is also made in this new diffusion method. Experiments results show that the proposed method has high performance compared to other literature methods and is an ideal edge-preserving filtering method. In addition, we use block-based noise estimation to estimate deviation in diffusion equation.

1. Introduction

Denoising techniques have been profoundly studied in computer vision fields for many years. Assuming an image is contaminated by additive noise, which can be represented by

\[ u(i,j) = u_0(i,j) + n(i,j) \]

(1)

Where \( u_0(i,j) \) represents the original image, \( u(i,j) \) denotes the observed noisy image, and \( n(i,j) \) signifies the signal-independent noise. Noise is recognized as high-frequency signals and can, therefore, be removed by the process of low-pass filtering or smoothing, unfortunately at the expense of some high-frequency information (i.e. edges). Wavelet-based methods, statistical methods, and diffusion filter methods have successfully been used to remove noise from digital images. [1][2][3]

The use of partial differential equations (PDEs) in image denoising has got wide use over the past years. PDEs belong to one of the most important parts of mathematical analysis, are closely related to the physical world. One of the main interests in using PDEs is that the theory is well established and its main benefits of usage of PDEs is that it takes the image smoothing to a continuous domain. We are engrossed by diffusion coming from PDEs. The simplest PDE model is convolution of image \( u(x, y) \) with a Gaussian function \( G = C \sigma^2 \exp(-((x^2 + y^2)/2\sigma^2)) \), where \( C \) and \( \sigma \) are constants, \( (x, y) \in \mathbb{R}^2 \). However, this model diffuses the whole image equally in all directions regardless of image space feature. Therefore, to some extent, this diffusion process blurs image edge. To overcome the shortcoming of Gaussian filter, many researchers have introduced useful model. [4][5] Further studies are still necessary in this aspect. The goal in this paper is to develop the idea of smoothing image without destroying edges and boundaries. We attain this goal by combining the idea of well-balanced diffusion proposed by Celia [6] with discontinuity measure put forward by Ke Chen [7].

This paper is organized in the following way. In Section II, we introduce the method we use for noise removal. Details are given to show how we obtain and solve the associated nonlinear PDEs coming from the adaptive and balanced diffusion method. Finite difference approximations and some implementation details are explained in Section III. Numerical results are also given in Section III. In the numerical experiments, we compare our method with some related algorithms in the literature. Finally, we draw a conclusion in section IV.

2. The adaptive and balanced Model

In this section we describe the proposed diffusion equation model.

An important contribution for edge or boundaries preservation was made by Perona and Malik.[8][9] They proposed the following diffusion equation:

\[ u_t = \Delta (\alpha \| \nabla u \|) \quad \text{in} \quad \Omega \in \mathbb{R}^2 \]

(2)
\[ u(x, y, 0) = u_0(x, y) \]

Where \( g(s) \) is a nonincreasing, smooth, monotonic function, such that \( 0 \leq g(1-\log(r/c)) \leq 1 \). This model diffuses region highly where \( \nabla \) is small, and therefore image edge where \( \nabla \) is large will be well preserved. However, this equation has no ability to classify edge and noise where the gradient \( \nabla \) is also large and the function \( g(s) \) is close to zero at every isolated noise point. So noise will remain when we apply this model to smooth image.

Alvarez, Lions, and Morel [10] modified the “mean curvature flow” equation with the idea of P-M model and proposed the following model,

\[ \frac{\partial u}{\partial t} = \nabla \cdot (g(\nabla u) \nabla u) \quad (3) \]

Although the ALM model has superiority over the previous model, it still has no ability in preserving image corner and edge. Nordstron [11] introduced a forcing term to improve P-M equation resulting in the following model,

\[ \frac{\partial u}{\partial t} = \nabla \cdot (g(\nabla u) \nabla u) - u + u \quad (4) \]

The right term in the above equation has the property of stabilizing \( u \) close to the original image \( u_0 \). However, this model does not eliminate noise well. Hence, Chia [6] proposed a modified parabolic equation (Clia model) for image smoothing.

\[ \frac{\partial u}{\partial t} = \nabla \cdot (g(\nabla u) \nabla u) - u + u \quad (5) \]

Where \( g(\nabla u) \) is image to be processed, \( u(x, y, t) \) is its smoothed one in the scale “\( t \)”, and \( g(\nabla u) \) is the local estimate of \( \nabla u \) used for image denoising. In general, we set parameter \( \lambda \) equal to Gaussian filter deviation \( \sigma \). In equation (5), \( \nabla \cdot (g(\nabla u) \nabla u) \) is a balanced term that has property of forcing the diffused image to be original one avoiding severe diffusion.

As we know, the function \( g(\nabla u) \) utilizes image spatial gradient \( \nabla \) to determine diffusion coefficient. However, this local discontinuities measure \( \nabla \) is acutely sensitive to any local intensity change and is not always robust. It is a well-known fact that image features and noise both can generate discontinuity. So the problem how to classify noise and image feature arises. Fortunately, contextual information, for example, the attribute of neighboring pixels, can be used to detect noise point in noisy surroundings and then can make difference between noisy point and feature point. [14] Inspired by the idea of Punam K [7][13][14][15], we define a measure which has the above property as inhomogeneity \( H(x, y) \).

In the following we will further explain how to get this measure \( H(x, y) \) in details. For a pixel in an image \( (x, y) \), we define its neighborhood, as

\[ N_u(r) = \{ (i, j) | x-r \leq i \leq x+r, \quad y-r \leq j \leq y+r \} \quad (6) \]

where \( r \geq 0 \) and \( r \in Z \) is a parameter that determines the size of its neighborhood. To facilitate presentation, we define the boundary region of a neighborhood of pixel \( (x, y) \) of size \( r \) by

\[ B_u(r) = \{ (i, j) | (i, j) \in N_u(r) - N_u(\lceil r \rceil) \} \quad (7) \]

As defined in [14], the uniformity criterion of calculating the similarity between pixel \( (x, y) \) and the ensemble of pixels in the boundary region of its neighborhood of size.

\[ \psi_{u}(x, y) = \frac{\sum_{(i, j) \in B_u(r)} \exp\left(-\frac{1}{2}(x-y)^{2}+(i-j)^{2}\right)}{2\pi r} \quad (8) \]

Where \( |B_u| \) is the cardinality of \( B_u(r) \) and \( \sigma \) is a parameter reflecting the statistics of local intensity difference of an image. According to [13][14] we first make differences, \( x-y \), between the pixel \( (x, y) \) and its the nearest neighborhood. Then, we sort the local intensity differences and choose the lower 90 percent values among them. After that, we estimate the mean and deviation of the remaining differences. We assume that the remaining local intensity difference has the property of Gaussian distribution, so we get parameter \( \sigma \).

\[ \sigma = \mu + 3\sigma \quad (9) \]

We get optimal local scales \( r \) when \( \psi(x, y) \) is maximum of point \( (x, y) \). In order to determine affinity between two adjacent pixels where one pixel is a member of the nearest neighborhood of the other mutually, we define two digital regions centered at pixel \( (x, y) \) and the other at pixel \( (i, j) \).

\[ N_u(x, y) = \{ (v, w) | v-r_n \leq x \leq v+r_n, \quad w-r_n \leq y \leq w+r_n \} \quad (10) \]

\[ N_u(i, j) = \{ (v, w) | v-r_n \leq i \leq v+r_n, \quad w-r_n \leq j \leq w+r_n \} \quad (11) \]

Where \( r_n = \min(r_u, r_i) \)

\[ d_{u} (x, y) = \sum_{(i, j) \in B_u(r)} \exp\left(-\frac{1}{2}(x-y)^{2}+(i-j)^{2}\right) \quad (12) \]

\[ d_{i} (i, j) = \sum_{(i, j) \in B_u(r)} \exp\left(-\frac{1}{2}(x-y)^{2}+(i-j)^{2}\right) \quad (13) \]

Here,

\[ d_{u} (x, y) = \begin{cases} (x-y)^2 + (i-j)^2 & \text{if } u(x, y) - u(i, j) > 0 \\ 0 & \text{else} \end{cases} \quad (14) \]

\[ d_{i} (i, j) = \begin{cases} (x-y)^2 + (i-j)^2 & \text{if } u(i, j) - u(x, y) > 0 \\ 0 & \text{else} \end{cases} \]

Then, we get the disconnectedness between two adjacent pixels \( (x, y) \) and \( (i, j) \), and define as follows,

\[ \psi_{u}(x, y) = \frac{\sum_{(i, j) \in B_u(r)} \exp\left(-\frac{1}{2}(x-y)^{2}+(i-j)^{2}\right)}{2\pi r} \quad (15) \]

Finally, we utilize contextual discontinuities measure to modify Chia [6] model and get modified model as follows:

\[ \sum_{i=0}^{i=\sigma} \sum_{j=0}^{j=\sigma} \exp\left(-\frac{1}{2}(i-j)^{2}\right) \quad (16) \]
where function \( g(x) = \frac{1}{1 + kx^2} \), \( g(x) = \exp(-\frac{x^2}{2\sigma^2}) \) denotes the image discontinuity measure of pixel \((x, y)\) inhomogeneity as,

\[
H(i, j) = \frac{\sum_{x \in N_1(i, j), y \in N_2(i, j)} H(i, j)}{|N_1(i, j)|}
\]

Unlike previous methods, equation (16) combines local and contextual information of pixel \((i, j)\) to control diffusion magnitude. We make a comparison between local and contextual measures in figure 1. The first column is an original medical image, added Gaussian noise one and added pepper-and-salt noise one respectively. The middle column and the last one contain gradient images and discontinuity measure images correspond to the first column. This figure shows that the ability of discontinuity measure to distinguish noise and feature is much better than gradient. This measure is able to get well performance even in the heavy pepper-and-salt noise environment. This experiment demonstrates that the proposed model makes good difference between edge and noisy point. Our proposed model diffuses image adaptively according to combination of gradient and discontinuity information.

**Figure 1. The comparision of gradient and discontinuity measure**

In our model, inspired by the method proposed by Dong-Hyuk Shin [16], we utilize block-based deviation noise estimation algorithm, which is based on both block-based and filtering-based approaches. It selectively smooth little image blocks by a block-based approach and filters selected smooth blocks using a filtering-based approach. During iteration, we adopt this method to calculate the deviation of the whole image \( u_t \).

\[
\partial_t u_t - \nabla \cdot (g_u \nabla u_t) = g_u \nabla \cdot (\lambda \nabla u_t)
\]

3. Implementation and numerical results

As discussed above, we get the final adaptive and balanced diffusion model

\[
\frac{\partial u_t}{\partial t} = \nabla \cdot (g_u \nabla u_t) - \lambda \nabla \cdot (\nabla g_u \nabla u_t)
\]

Using Newmann’s boundary conditions we get \( u^{n+1} = u^n + \Delta t D(u^n) \), \( u^n = u_o \) here,

\[
D(u) = g_u(E(u) - g_u(0))
\]

and the diffusion term

\[
E(u) = \frac{1}{2} \int_{0}^{\infty} E(u, s) ds
\]

In the following, we discuss the results of our experiments with the proposed model. Our first experiment (figure 2) is walk man image that contains many small features. We compare our method with P-M model [8], ALM model [10] and C model [6]. Figure 2 shows that P-M model can make good results in noise removal but fail in edge preservation. ALM model has good ability of keeping feature but no ability in smoothing and C model blurs image to some extent. However, our model gets good result in both aspects.

Table 1 gives result image (walk man) SNR (signal to noise ratio) of our method and other methods SNR.

Except for SNR test discussed above, the root mean square error (RMSE) is applied to evaluate the results quantitatively during diffusion and RMSE is defined as

\[
\text{RMSE}(\hat{x}(i, j)) = \sqrt{\text{E}[(\hat{x}(i, j) - x(i, j))^2]}
\]

where \( x \) is the original image and \( \hat{x} \) is the de-noised image. \( \text{E}(\cdot) \) denotes the expectation operator. Figure 3 demonstrates that our model (dash line) has the smallest RMSE among four models and the error is stable after 50 iterations.

**Figure 2. Walkman image.**

Top left - original, top right - noisy image, middle left - result of P-M model, middle right -
result of ALM model, bottom left- C model, bottom right- result of our model.

Table 1. SNR test (walkman image)

<table>
<thead>
<tr>
<th>Model</th>
<th>SNR (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-M model</td>
<td>69.12</td>
</tr>
<tr>
<td>ALM model</td>
<td>105.76</td>
</tr>
<tr>
<td>C model</td>
<td>108.54</td>
</tr>
<tr>
<td>Our model</td>
<td>140.21</td>
</tr>
</tbody>
</table>

Figure 3. RMSE results of our model and C, ALM, P-M model

Figure 2, 3 and table 1 show that our method is better than three models and has high performance.

4. Conclusion

Image denoising plays in an important role in image processing. Traditional anisotropic diffusion method is sensitive to local noise points and small feature and must make a tradeoff between noise removal and feature preservation. In this paper we proposed a new diffusion image denoising equation which combines local and contextual information of image pixel to control diffusion coefficient and can get a well balance between result image and original one. It utilizes inhomogeneity measure and gradient information to decide diffusion magnitude. Subjective and objective tests show that this model has high performance than traditional image denoising method. Fast block-based noise estimate method is also used in the diffusion method.

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Reference