A New Existence Theory for Positive Periodic Solutions to Functional Differential Equations with Impulse Effects

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Abstract—The principle of this paper is to deal with a new existence theory for positive periodic solutions to a kind of nonautonomous functional differential equations with impulse actions at fixed moments. Easily verifiable sufficient criteria are established. The approach is based on the fixed-point theorem in cones. The paper extends some previous results and obtains some new results. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Some evolution processes are distinguished by the circumstance that at certain instants their evolution is subjected to a sudden change, for example, in population biology, the diffusion of chemicals, the spread of heat, the radiation of electromagnetic waves, the maintenance of a species through instantaneous stocking and harvesting, etc. Mathematically, this leads to an impulse dynamical system. Differential equations involving impulse effects occurs in many applications:
physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. Therefore, the study of this class of dynamical systems has gained prominence and it is a rapidly growing field. See, for instance the monographs [1–5]. Now, some qualitative properties such as oscillation, asymptotic behavior, stability, and existence of solutions are investigated extensively by many authors [6–12]. However, only a little has been done for the periodicity of nonautonomous impulsive differential equations (see [13]), especially the study based on the fixed-point theorem in cones.

Very recently, the authors [14] investigated the existence of positive periodic solutions for the following generalized nonautonomous functional differential equations,

\[ \dot{y}(t) = -a(t) y(t) + g(t, y(t - \tau(t))) , \]

where \( a(t) \in C(R, (0, \infty)) \), \( \tau(t) \in C(R, R) \), \( g \in C(R \times [0, \infty), [0, \infty)) \), and \( a(t), \tau(t), g(t, y) \) are all \( \omega \)-periodic functions. \( \omega > 0 \) is a constant.

It is well known that the functional differential equation (1.1) includes many mathematical ecological equations.

For example, the general functional differential equations [15],

\[ \dot{y}(t) = -a(t) y(t) + b(t) f(t, y(t - \tau(t))) , \]

the hematopoiesis model [15–17],

\[ \dot{y}(t) = -a(t) y(t) + b(t) e^{-\beta(t) y(t-\tau(t))} , \]

more general the model of blood cell production [15,16,18,19],

\[ \dot{y}(t) = -a(t) y(t) + b(t) \frac{1}{1 + y(t - \tau(t))^{n}}, \quad n > 0, \]

and the more general Nicholson’s blowflies model [15,16,20–22],

\[ \dot{y}(t) = -a(t) y(t) + b(t) y(t - \tau(t)) e^{-\beta(t) y(t-\tau(t))} , \]

To the knowledge of the authors, there are very few works on the existence of positive periodic solutions for equation (1.1), even for (1.2)–(1.6), except [14,15]. The main results of [14] are as follows.

**Theorem 1.1.** (See [14]) Equation (1.1) has at least one \( \omega \)-periodic positive solution, provided one of the following conditions holds.

(i) \[ \liminf_{u \to 0} \min_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u} > 1 \quad \text{and} \quad \limsup_{u \to \infty} \max_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u} < 1; \]

(ii) \[ \limsup_{u \to 0} \max_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u} < 1 \quad \text{and} \quad \liminf_{u \to \infty} \min_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u} > 1. \]

However, the study for the periodicity of the functional differential equation with impulse effects is still in an initial stage of its development, and is far from systematic study. As far as we known, no work has been done for the existence of periodic solutions for the functional differential equation (1.1) with impulse effects.
In this paper, we devote ourselves to exploring the periodicity of the following system,

$$\dot{y}(t) = -a(t)y(t) + g(t, y(t - \tau(t))), \quad t \neq t_j, \quad j \in \mathbb{Z},$$  

$$y(t^+_j) = y(t^-_j) + I_j(y(t_j)),$$  

(1.7)

where \(y(t^+_j)\) and \(y(t^-_j)\) represent the right and the left limit of \(y(t_j)\), respectively, in this paper, it is assumed that \(y\) is left continuous at \(t_j\).

In system (1.7), the following is assumed.

\((H_1)\) \(a(t) \in C(R, [0, \infty))\), \(\tau(t) \in C(R, R)\), \(g \in C(R \times [0, \infty), [0, \infty))\), \(I_j \in C([0, \infty), [0, \infty))\), and \(a(t), \tau(t), g(t, y)\) are all \(\omega\)-periodic functions. \(\omega > 0\) is a constant.

\((H_2)\) There exists a positive integer \(p\) such that \(t_{j+p} = t_j + \omega, \ I_{j+p} = I_j, \ j \in \mathbb{Z}\). Without loss of generality, we also assume that

\([0, \omega) \cap \{t_j : j \in \mathbb{Z}\} = \{t_1, t_2, \ldots, t_p\}\).

In this paper, we present a new existence theory by applying a well-known fixed-point theorem in cones (see Theorem 1.2). The key steps is to find the Green's function of (1.7) and a function \(\psi\) such that the appropriate operator \(\Phi\) satisfies the condition \(y \neq \Phi y + \lambda \psi\) in the cited fixed-point theorem. It seems to be difficult to utilize the norm-type expansion and compression theorem to prove our main results (see [24] for details).

To conclude this section, we state a fixed-point theorem in cones which will be needed in this paper.

**Theorem 1.2.** (See [23, 24].) Let \(X\) be a Banach space and \(K\) is a cone in \(X\). Assume \(\Omega_1, \Omega_2\) are open subsets of \(X\) with \(0 \in \Omega_1, \ \Omega_1 \subset \Omega_2\). Let

$$\Phi : K \cap (\Omega_2 \setminus \Omega_1) \to K$$

be a continuous and completely continuous operator such that

(i) \(\|\Phi x\| \leq \|x\|\) for \(x \in K \cap \partial \Omega_1\),

(ii) there exists \(\psi \in K \setminus \{0\}\) such that \(x \neq \Phi x + \lambda \psi\) for \(x \in K \cap \partial \Omega_2\) and \(\lambda > 0\).

Then, \(\Phi\) has a fixed point in \(K \cap (\Omega_2 \setminus \Omega_1)\).

**Remark 1.1.** In Lemma 1.1, if (i) and (ii) are replaced by

(i) \(\|\Phi x\| \leq \|x\|\) for \(x \in K \cap \partial \Omega_2\), and

(ii) there exists \(\psi \in K \setminus \{0\}\) such that \(x \neq \Phi x + \lambda \psi\) for \(x \in K \cap \partial \Omega_1\) and \(\lambda > 0\), then \(\Phi\) has a fixed point in \(K \cap (\Omega_2 \setminus \Omega_1)\).

**2. Existence of Positive Periodic Solutions**

We now consider "the linear problem",

$$\dot{y}(t) = -a(t)y(t) + \sigma(t), \quad t \neq t_j, \quad j \in \mathbb{Z},$$  

$$y(t^+_j) = y(t^-_j) + I_j(y(t_j)),$$  

(2.1)

where \(\sigma(t) \in C(R, [0, \infty))\) is an \(\omega\)-periodic function and the rest parameters satisfy Hypothesis \((H_1)\) and \((H_2)\).

Note that (2.1) is not really a linear problem since the impulse functions are not necessarily linear. However, if \(I_j\) are linear \((j = 1, 2, \ldots, p)\), then (2.1) is a linear impulsive problem.

The next result is fundamental in our discussion.
LEMMA 2.1. $y(t)$ is an $\omega$-periodic solution of equation (2.1) is equivalent to $y(t)$ is an $\omega$-periodic solution of the following integral equation,

$$y(t) = \int_t^{t+\omega} G(t, s) \sigma(s) \, ds + \sum_{j:t_j \in [t, t+\omega)} G(t, t_j) I_j^*(y(t_j)), \quad (2.2)$$

where

$$G(t, s) = \frac{e^{\int_t^s a(\xi) \, d\xi}}{e^{\int_{t_0}^s a(\xi) \, d\xi} - 1}. \quad (2.3)$$

PROOF. Suppose $y(t)$ is an $\omega$-periodic solution of equation (2.1). Let

$$y(t) = e^{-\int_0^t a(\xi) \, d\epsilon} u(t).$$

Thus, $u(t)$ satisfies the following impulse problem,

$$\begin{cases}
\dot{u}(t) = \sigma^*(t), & t \neq t_j, \quad j \in Z, \\
u(t^+_j) = u(t^-_j) + I_j^*(u(t_j)), \\
u(t + \omega) = e^{\int_0^{t+\omega} a(\xi) \, d\epsilon} u(t),
\end{cases}$$

where

$$\sigma^*(t) = e^{\int_0^t a(\xi) \, d\epsilon} \sigma(t)$$

and

$$I_j^*(u(t_j)) = e^{\int_0^{t_j} a(\xi) \, d\epsilon} I_j(e^{-\int_0^{t_j} a(\xi) \, d\epsilon} u(t_j)), \quad \text{for } j \in Z.$$

If $t \in (t_j, t_{j+1}]$, $j \in Z$, we have that

$$u(t) = u(t_j^+) + \int_{t_j}^t \sigma^*(s) \, ds.$$

On the other hand,

$$u(t^-_j) = u(t^+_j) + \int_{t_j}^{t_j} \sigma^*(s) \, ds.$$

Thus, if $t \in (t_j, t_{j+1}]$, $j \in Z$, we obtain

$$u(t) = u(t_j^-) + \int_{t_j}^t \sigma^*(s) \, ds + I_j^*(u(t_j))$$

$$= u(t_j^-) + \int_{t_j}^{t_j} \sigma^*(s) \, ds + \int_{t_j}^t \sigma^*(s) \, ds + I_j^*(u(t_j))$$

$$= u(t_j^-) + \int_{t_j}^{t_j} \sigma^*(s) \, ds + I_j^*(u(t_j)).$$

(2.4)

For every $t \in R$, there exists a $j \in Z$ such that $t \in (t_j, t_{j+1}]$, then $t + \omega \in (t_j + \omega, t_{j+1} + \omega] = (t_{j+\omega}, t_{j+\omega+1}]$. So, from (2.4), we obtain

$$u(t + \omega) = u(t_{j+\omega}^+) + \int_{t_{j+\omega}^-}^{t+\omega} \sigma^*(s) \, ds + I_{j+\omega}^*(u(t_{j+\omega})).$$
In consequence,

\[ u(t + \omega) = u(t_{j+1}^+) + \int_{t_{j+1}}^{t+\omega} \sigma^*(s) \, ds + \sum_{k: t_k \in (t_{j+1}, t+\omega)} I_k^* (u(t_k)) \]

\[ = u(t_{j+1}^+) + \int_{t_{j+1}}^{t+\omega} \sigma^*(s) \, ds + \sum_{j: t_j \in [t, t+\omega)} I_j^* (u(t_j)) \]

\[ = u(t) + \int_{t}^{t+\omega} \sigma^*(s) \, ds + \sum_{j: t_j \in [t, t+\omega)} I_j^* (u(t_j)). \]  

(2.5)

Then, from (2.5), we get

\[ u(t + \omega) - u(t) = \int_{t}^{t+\omega} \sigma^*(s) \, ds + \sum_{j: t_j \in [t, t+\omega)} I_j^* (u(t_j)) \]

and

\[ u(t) \left[ e^{\int_{t}^{t+\omega} a(\xi) \, d\xi} - 1 \right] = \int_{t}^{t+\omega} \sigma^*(s) \, ds + \sum_{j: t_j \in [t, t+\omega)} I_j^* (u(t_j)). \]

Thus,

\[ u(t) = \frac{\int_{t}^{t+\omega} \sigma^*(s) \, ds + \sum_{j: t_j \in [t, t+\omega)} I_j^* (u(t_j))}{e^{\int_{t}^{t+\omega} a(\xi) \, d\xi} - 1}. \]

Then, for every \( t \in \mathbb{R} \), we have

\[ y(t) = \frac{e^{-\int_{t}^{t+\omega} a(\xi) \, d\xi}}{e^{\int_{t}^{t+\omega} a(\xi) \, d\xi} - 1} \int_{t}^{t+\omega} e^{\int_{t}^{s} a(\xi) \, d\xi} \sigma(s) \, ds \]

\[ + \frac{e^{-\int_{t}^{t+\omega} a(\xi) \, d\xi}}{e^{\int_{t}^{t+\omega} a(\xi) \, d\xi} - 1} \sum_{j: t_j \in [t, t+\omega)} e^{\int_{t}^{t_j} a(\xi) \, d\xi} I_j(y(t_j)) \]

\[ = \int_{t}^{t+\omega} G(t, s) \sigma(s) \, ds + \sum_{j: t_j \in [t, t+\omega)} G(t, t_j) I_j(y(t_j)). \]

On the other hand, (2.1) is satisfied by substituting the periodic solution \( y(t) \) of equation (2.2).

This completes the proof of Lemma 2.1.

From Lemma 2.1, we can obtain following Lemma 2.2 easily.

**Lemma 2.2.** \( y(t) \) is an \( \omega \)-periodic solution of equation (1.7) is equivalent to \( y(t) \) is an \( \omega \)-periodic solution of the following integral equation,

\[ y(t) = \int_{t}^{t+\omega} G(t, s) g(s, y(s - \tau(s))) \, ds + \sum_{j: t_j \in [t, t+\omega)} G(t, t_j) I_j(y(t_j)), \]  

(2.6)

where \( G(t, s) \) is defined by (2.3) in Lemma 2.1.

Let \( X \) be a real Banach space, and \( K \) a closed, nonempty subset of \( X \). \( K \) is a cone provided:

(i) \( \alpha u + \beta v \in K \), for all \( u, v \in K \) and all \( \alpha, \beta \geq 0 \); and

(ii) \( u, -u \in K \) imply \( u = 0 \).
Define
\[ PC(R) = \{ y : R \to R \mid y_{|_{(t_j, t_{j+1})}} \in C(t_j, t_{j+1}), \exists y(t^-) = y(t_j), y(t^+) = y(t_{j+1}), j \in \mathbb{Z} \}. \]

Consider the Banach space,
\[ X = \{ y(t) : y(t) \in PC(R), y(t + \omega) = y(t) \}, \]
with norm,
\[ \| y \| = \sup_{t \in [0, \omega]} \{|y(t)| : y \in X\}. \]

Define an operator on \( X \) as following,
\[ (\Phi y)(t) = \int_t^{t+\omega} G(t, s) g(s, y(s - \tau(s))) \, ds + \sum_{j : t_j \in [t, t+\omega)} G(t, t_j) I_j(y(t_j)), \]
for \( y \in X \). Clearly, \( \Phi \) is a completely continuous operator on \( X \).

Let
\[ K = \{ y \in X : y(t) \geq 0 \text{ and } y(t) \geq \sigma \| y \| \} \]
where \( 0 < \sigma = A/B < 1 \), and
\[ A := \min \{ G(t, s) : 0 \leq t, s \leq \omega \} = \frac{1}{e^{\int_0^\omega a(\xi) \, d\xi}} - 1 > 0, \]
\[ B := \max \{ G(t, s) : 0 \leq t, s \leq \omega \} = \frac{e^{\int_0^\omega a(\xi) \, d\xi}}{e^{\int_0^\omega a(\xi) \, d\xi}} - 1 > 0. \]  

It is not difficult to verify that \( K \) is a cone in \( X \).

**Lemma 2.3.** Assume that \((H_1)\) and \((H_2)\) holds, then \( \Phi(K) \subset K \).

**Proof.** For any \( y \in K \), we have
\[ \| \Phi y \| \leq B \int_0^\omega g(s, y(s - \tau(s))) \, ds + B \sum_{j=1}^p I_j(y(t_j)) \]
and
\[ (\Phi y)(t) \geq A \int_0^\omega g(s, y(s - \tau(s))) \, ds + A \sum_{j=1}^p I_j(y(t_j)) . \]

So, we have
\[ (\Phi y)(t) \geq \frac{A}{B} \| \Phi y \| = \sigma \| \Phi y \| , \]
i.e., \( \Phi y \in K \). This completes the proof of Lemma 2.3.

For convenience and simplicity in the following discussion, we always use the notations,
\[ g_0 = \liminf_{u \to 0^+} \min_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u}, \quad I_0 = \liminf_{u \to 0^+} \sum_{j=1}^p \frac{I_j(u)}{u}, \]
\[ g^\infty = \limsup_{u \to \infty} \max_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u}, \quad I^\infty = \limsup_{u \to \infty} \sum_{j=1}^p \frac{I_j(u)}{u}, \]
\[ g^0 = \limsup_{u \to 0^+} \max_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u}, \quad I^0 = \limsup_{u \to 0^+} \sum_{j=1}^p \frac{I_j(u)}{u}, \]
\[ g_\infty = \liminf_{u \to \infty} \min_{t \in [0, \omega]} \frac{g(t, u)}{a(t)u}, \quad I_\infty = \liminf_{u \to \infty} \sum_{j=1}^p \frac{I_j(u)}{u}. \]

The main result of the present paper is as follows.
THEOREM 2.1. Assume that \((H_1)\) and \((H_2)\) holds. Then, equation \((1.7)\) has at least one \(\omega\)-periodic positive solution, provided one of the following conditions holds,

(i) \(g_0 + A0 > 1\) and \(g^\infty + B1^\infty < 1\),
(ii) \(g^0 + B1^0 < 1\) and \(g_\infty + A1_\infty > 1\).

COROLLARY 2.1. Assume that \((H_1)\) and \((H_2)\) holds. Then, equation \((1.7)\) has at least one \(\omega\)-periodic positive solution, provided one of the following conditions holds.

(i) \(g_0 = \infty\) or \(I_0 = \infty\) and \(g^\infty = 0\), \(I^\infty = 0\); (sublinear)
(ii) \(g^0 = 0\), \(I^0 = 0\) and \(g_\infty = \infty\) or \(I_\infty = \infty\) (superlinear).

REMARK 2.1. Theorem 2.1 extends Theorems 1.1 in [14].

REMARK 2.2. Noting that if \(g(t, u) = a(t)u\) and \(I_j = 0\), then the existence of positive \(\omega\)-periodic solutions for linear problem,

\[
\dot{y}(t) = -a(t)y(t) + a(t)y(t - \tau(t)),
\]

cannot be guaranteed. So, the conditions in Theorem 2.1 are optimal in some sense.

PROOF OF THEOREM 2.1. First of all, we point out that to find an \(\omega\)-periodic solution of equation \((1.7)\) is equivalent to find a fixed point of the operator \(\Phi\). Utilizing Theorem 1.2, we prove the conclusion under Case (i) or Case (ii).

Case (i). Since

\[g_0 + A0 > 1,\]

according the definition of \(g_0\) and \(I_0\), we know that for

\[0 < \epsilon \leq \frac{(g_0 + A0) - 1}{A + 1},\]

there exists a constant \(r > 0\) such that

\[g(t, u) \geq (g_0 - \epsilon)a(t)u, \quad A \sum_{j=1}^{p} I_j(u) \geq A(I_0 - \epsilon)u, \quad \text{whenever } 0 \leq u \leq r.\]

Thus, if \(y \in K\) with \(\|y\| = r\), then \(y(t) \geq \sigma r\). Let \(\psi \equiv 1\) for \(t \in R\) and we prove that

\[y \neq \Phi y + \lambda \psi, \quad \text{for } y \in K \cap \partial \Omega_1 \text{ and } \lambda > 0,\]

where

\[\Omega_1 = \{u \in X : \|u\| < r\}.\]

If not, there exists \(y_0 \in K \cap \partial \Omega_1\) and \(\lambda_0 > 0\) such that

\[y_0 = \Phi y_0 + \lambda_0 \psi.\]

Let \(\mu = \min_{t \in [0,\omega]} y_0(t)\). Then, for \(t \in R\), we have

\[y_0(t) = (\Phi y_0)(t) + \lambda_0\]

\[= \int_{t}^{t+\omega} G(t, s)g(s, y_0(s - \tau(s)))ds + \sum_{j:t_j \in [t, t+\omega]} G(t, t_j)I_j(y_0(t_j)) + \lambda_0\]
\[ \int_{t}^{t+\omega} (g_0 - \epsilon) G(t, s) a(s) y_0(s - \tau(s)) ds + A \sum_{j=1}^{p} I_j(y_0(t_j)) + \lambda_0 \]

\[ \geq (g_0 - \epsilon) \mu \int_{t}^{t+\omega} G(t, s) a(s) ds + A(I_0 - \epsilon) \mu + \lambda_0 \]

\[ = [(g_0 + A I_0) - (A + 1) \epsilon] \mu + \lambda_0 \]

\[ \geq \mu + \lambda_0, \]

and this implies \( \mu \geq \mu + \lambda_0 \), a contradiction.

On the other hand, since

\[ g^\infty + B I^\infty < 1, \]

according the definition of \( g^\infty \) and \( I^\infty \), we know that for

\[ 0 < \epsilon \leq \frac{1 - (g^\infty + B I^\infty)}{B + 1}, \]

there exists a constant \( r_1 > r \) such that

\[ g(t, u) \leq (g^\infty + \epsilon) a(t) u, \quad B \sum_{j=1}^{p} I_j(u) \leq B(I^\infty + \epsilon) u, \quad \text{for } u \geq r_1, \quad j = 1, 2, \ldots, p. \]

Let \( R = r_1 / \sigma \), so we have

\[ u(t) \geq \sigma \|u\| = \sigma R = r_1, \quad \text{for } u \in K \cap \partial \Omega_2, \quad (2.9) \]

where

\[ \Omega_2 = \{u \in X : \|u\| < R\}. \]

Then, for \( y \in K \) with \( \|y\| = R \), we have

\[ (\Phi y)(t) = \int_{t}^{t+\omega} G(t, s) g(s, y(s - \tau(s))) ds \]

\[ + \sum_{j:j \in (t, t+\omega)} G(t, t_j) I_j(y(t_j)) \]

\[ \leq \int_{t}^{t+\omega} (g^\infty + \epsilon) G(t, s) a(s) y(s - \tau(s)) ds + B \sum_{j=1}^{p} I_j(y(t_j)) \]

\[ \leq (g^\infty + \epsilon) \int_{t}^{t+\omega} G(t, s) a(s) ds \|y\| + B(I^\infty + \epsilon) \|y\| \]

\[ = [(g^\infty + B I^\infty) + (B + 1) \epsilon] \|y\| \]

\[ \leq \|y\|. \]

This implies that

\[ \|\Phi y\| \leq \|y\|, \]

for \( y \in K \cap \partial \Omega_2 \).

Therefore, by Theorem 1.2, it follows that \( \Phi \) has a fixed point \( y \in K \cap (\Omega_2 \setminus \Omega_1) \).

Furthermore, \( r \leq \|y\| \leq R \) and \( y(t) \geq \sigma r > 0 \), which means that \( y(t) \) is an \( \omega \)-periodic positive solution of (1.7).
Case (ii). Since 

\[ g_0 + BI_0 < 1, \]

according the definition of \( g_0 \) and \( I_0 \), we know that for 

\[ 0 < \epsilon \leq \frac{1 - (g_0 + BI_0)}{B + 1}, \]

there exist a constant \( r > 0 \) such that

\[ g(t, u) \leq (g_0 + \epsilon)a(t)u, \quad B \sum_{j=1}^{p} I_j(u) \leq B(I_0 + \epsilon)u, \quad \text{whenever } 0 \leq u \leq r, \quad j = 1, 2, \ldots, p. \]

Thus, if \( y \in K \) with \( \|y\| = r \), then \( \sigma r \leq y(t) \leq r \).

Then, for \( y \in K \) with \( \|y\| = r \), we have

\[
(\Phi y)(t) = \int_{t}^{t+\omega} G(t, s)g(s, y(s - \tau(s)))ds 
+ \sum_{j:t_j \in [t, t+\omega]} G(t, t_j)I_j(y(t_j)) 
\leq \int_{t}^{t+\omega} (g_0 + \epsilon)G(t, s)a(s)y(s - \tau(s))ds + B \sum_{j=1}^{p} I_j(y(t_j)) 
\leq (g_0 + \epsilon)\int_{t}^{t+\omega} G(t, s)a(s)ds\|y\| + B(I_0 + \epsilon)\|y\| 
= [(g_0 + BI_0) + (B + 1)\epsilon]\|y\| 
\leq \|y\|.
\]

This implies that 

\[ \|\Phi y\| \leq \|y\| \]

for \( y \in K \cap \partial \Omega_1 \), where

\[ \Omega_1 = \{u \in X : \|u\| < r\}. \]

On the other hand, since

\[ g_\infty + AI_\infty > 1, \]

according the definition of \( g_0 \) and \( I_0 \), we know that for

\[ 0 < \epsilon \leq \frac{(g_\infty + AI_\infty) - 1}{A + 1}, \]

there exist a constant \( r_1 > r \) such that

\[ g(t, u) \geq (g_\infty - \epsilon)a(t)u, \quad A \sum_{j=1}^{p} I_j(u) \geq A(I_\infty - \epsilon)u, \quad \text{for } u \geq r_1. \]

Let \( R = r_1/\sigma \), so we have,

\[ u(t) \geq \sigma \|u\| = \sigma R = r_1, \quad \text{for } u \in K \cap \partial \Omega_2, \quad (2.10) \]
where \[ \Omega_2 = \{ u \in X : \|u\| < R \} . \]

Let \( \psi \equiv 1 \) for \( t \in R \) and we prove that
\[
y \neq \Phi y + \lambda \psi, \quad \text{for} \quad u \in K \cap \partial \Omega_2 \quad \text{and} \quad \lambda > 0,
\]
If not, there exists \( y_0 \in K \cap \partial \Omega_2 \) and \( \lambda_0 > 0 \) such that
\[
y_0 = \Phi y_0 + \lambda_0 \psi.
\]
Let \( \mu = \min_{t \in R} y_0(t) \). Then, for \( t \in R \), we have
\[
y_0(t) = (\Phi y_0)(t) + \lambda_0
\]
\[
= \int_{t}^{t+\omega} G(t, s) g(s, y_0(s - r(s))) \, ds
\]
\[
+ \sum_{j : t_j \in [t, t+\omega)} G(t, t_j) I_j(y_0(t_j)) + \lambda_0
\]
\[
\geq \int_{t}^{t+\omega} (g_\infty - \epsilon) G(t, s) a(s) y_0(s - r(s)) \, ds
\]
\[
+ A \sum_{j=1}^{p} I_j(y_0(t_j)) + \lambda_0
\]
\[
\geq (g_\infty - \epsilon) \mu \int_{t}^{t+\omega} G(t, s) a(s) \, ds + A (I_\infty - \epsilon) \mu + \lambda_0
\]
\[
= [(g_\infty + AI_\infty) + (A + 1) \epsilon] \mu
\]
\[
\geq \mu + \lambda_0,
\]
and this implies \( \mu \geq \mu + \lambda_0 \), a contradiction.

Therefore, by Theorem 1.2, it follows that \( \Phi \) has a fixed point \( y \in K \cap (\Omega_2 \setminus \Omega_1) \). Furthermore, \( r \leq \|y\| \leq R \) and \( y(t) \geq \sigma r > 0 \), which means that \( y(t) \) is an \( \omega \)-periodic positive solution of (1.7).

This completes the proof Theorem 2.1.

3. EXAMPLES

In this section, we apply the main result obtained in previous section to study some examples which have some biological background.

For short, we assume the following.

\( (H_3) \) \( I_\infty = 0. \)

Consider the system,
\[
y(t) = -a(t) y(t) + b(t) e^{-\beta(t)} y(t - \tau(t)), \quad t \neq t_j, \quad j \in Z,
\]
\[
y(t_j^+) = y(t_j^-) + I_j(y(t_j)).
\]

It follows from Corollary 2.1, we have the following corollaries.

**Corollary 3.1.** Assume that \( (H_2), (H_3) \), and
\( (H_4) \) \( a(t), b(t) \in C(R, (0, \infty)), \beta(t) \in C(R, (0, \infty)), \tau(t) \in C(R, R), I_j \in C([0, \infty), [0, \infty)), j \in Z, a(t), b(t), \tau(t), \) and \( \beta(t) \) are all \( \omega \)-periodic functions. \( \omega > 0 \) is a constant.
Then, equation (3.1) has at least one \( \omega \)-periodic positive solutions.

Consider the system,

\[
\dot{y}(t) = -a(t)y(t) + b(t) \frac{1}{1 + y(t - \tau(t))^n}, \quad n > 0, \quad t \neq t_j, \quad j \in \mathbb{Z},
\]

\[
y(t_j^+) = y(t_j^-) + I_j(y(t_j)).
\]

COROLLARY 3.2. Assume that \((H_2), (H_3), \) and 
\((H_5) \quad a(t), b(t) \in C(R, (0, \infty)), \tau(t) \in C(R, R), I_j \in C([0, \infty), [0, \infty)), j \in \mathbb{Z}, \) and \(a(t), b(t), \tau(t), \) are all \( \omega \)-periodic functions. \( \omega > 0 \) is a constant. Then, equation (3.2) has at least one \( \omega \)-periodic positive solution.

It follows from Theorem 2.1, we have the following corollaries.

Consider the system,

\[
\dot{y}(t) = -a(t)y(t) + b(t) \frac{y(t - \tau(t))}{1 + y(t - \tau(t))^n}, \quad n > 0, \quad t \neq t_j, \quad j \in \mathbb{Z},
\]

\[
y(t_j^+) = y(t_j^-) + I_j(y(t_j)).
\]

COROLLARY 3.3. Assume that \((H_2), (H_3), (H_5), \) and 
\((H_6) \quad b(t) > a(t) \) for \( t \in [0, \omega]. \)

Then, equation (3.3) has at least one \( \omega \)-periodic positive solution.

Consider the system,

\[
\dot{y}(t) = -a(t)y(t) + b(t)y(t - \tau(t)) e^{-\beta(t)} y(t - \tau(t)), \quad t \neq t_j, \quad j \in \mathbb{Z},
\]

\[
y(t_j^+) = y(t_j^-) + I_j(y(t_j)).
\]

COROLLARY 3.4. Assume that \((H_2), (H_3), (H_4), \) and \((H_6) \) hold, then equation (3.4) has at least one \( \omega \)-periodic positive solution.

Corollary 3.1 and Corollary 3.2 can be checked easily. For Corollary 3.3 and Corollary 3.4, since

\[
g_0 = \min_{t \in [0, \omega]} \frac{b(t)}{a(t)} > 1 \quad \text{and} \quad g^\infty = 0 < 1,
\]

then the result follows from Theorem 2.1.

REFERENCES