On the Connectivity of Graphs Embedded in Surfaces

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In a 1973 paper, Cooke obtained an upper bound on the possible connectivity of a graph embedded in a surface (orientable or nonorientable) of fixed genus. Furthermore, he claimed that for each orientable genus $\gamma > 0$ (respectively, nonorientable genus $\gamma \neq 0$) there is a complete graph of orientable genus $\gamma$ (respectively, nonorientable genus $\gamma$) and having connectivity attaining his bound. It is false that there is a complete graph of genus $\gamma$ (respectively, nonorientable genus $\gamma$) for every $\gamma$ (respectively $\gamma$) and that is the starting point of the present paper. Ringel and Youngs did show that for each $\gamma > 0$ (respectively, $\gamma > 0$, $\gamma \neq 2$) there is a complete graph $K_n$ which embeds in $S_\gamma$ (respectively, $N_\gamma$) such that $n$ is the chromatic number of surface $S_\gamma$ (respectively, the chromatic number of surface $N_\gamma$). One then easily observes that the connectivity of this $K_n$ attains the upper bound found by Cook. This leads us to define two kinds of connectivity bound for each orientable (or nonorientable) surface. We define the maximum connectivity $\kappa_{\text{max}}$ of the orientable surface $S_\gamma$ to be the maximum connectivity of any graph embeddable in the surface and the genus connectivity $\kappa_{\text{gen}}(S_\gamma)$ of the surface to be the maximum connectivity of any graph which genus embeds in the surface. For nonorientable surfaces, the bounds $\kappa_{\text{max}}(N_\gamma)$ and $\kappa_{\text{gen}}(N_\gamma)$ are defined similarly. In this paper we first study the uniqueness of graphs possessing connectivity $\kappa_{\text{max}}(S_\gamma)$ or $\kappa_{\text{gen}}(N_\gamma)$. The remainder of the paper is devoted to the study of the spectrum of values of genera in the intervals $[\gamma(K_n)+1, \gamma(K_{n+1})]$ and $[\gamma(K_n)+1, \gamma(K_{n+1})]$ with respect to their genus and maximum connectivities.

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1. INTRODUCTION

Throughout this paper, $\kappa(G)$ will denote the (vertex-)connectivity of the graph $G$. Similarly, denote by $\gamma(G)$ (respectively, $\bar{\gamma}(G)$) the orientable (resp. nonorientable) genus of $G$. For any real number $x$, denote by $\lceil x \rceil$ the least integer greater than or equal to $x$ and by $\lfloor x \rfloor$ the greatest integer less than or equal to $x$.

The maximum connectivity of a surface (orientable or nonorientable) is defined to be the maximum value of connectivity taken over all graphs embeddable in the surface. We denote this parameter by $\kappa_{\text{max}}(S_{\gamma})$ when $S_{\gamma}$ is orientable and by $\kappa_{\text{max}}(N_{\gamma})$ when $N_{\gamma}$ is nonorientable. It is well known that $\kappa_{\text{max}} = 5$ for both the plane and the projective plane.

In his 1973 paper, Cook [2] proved the following two results.

Theorem 1.1. (a) If $G$ is any nonplanar graph and if $G$ is embedded in the orientable surface $S_{\gamma}$, then

$$\kappa(G) \leq \left\lfloor \frac{5 + \sqrt{1 + 48\gamma}}{2} \right\rfloor,$$

while

(b) If $G$ has nonorientable genus at least 2 and if $G$ is embedded in the nonorientable surface $N_{\gamma}$, then

$$\kappa(G) \leq \left\lfloor \frac{5 + \sqrt{1 + 24\gamma}}{2} \right\rfloor.$$

Note that the bound in part (b) of the above theorem also holds for graphs having nonorientable genus 1.

Cook then claimed that, by results of Ringel and Youngs [14] (resp. Ringel [12]; see also [13]), for each value of $\gamma > 0$ (resp. $\gamma > 0, \gamma \neq 2$), there is a complete graph $K_{\gamma}$ embeddable in $S_{\gamma}$ (resp. $N_{\gamma}$) such that $\kappa(K_{\gamma}) = \kappa_{\text{max}}(S_{\gamma})$ (resp. $\kappa(K_{\gamma}) = \kappa_{\text{max}}(N_{\gamma})$). Therefore, $\kappa_{\text{max}}(S_{\gamma}) = \left\lfloor \frac{5 + \sqrt{1 + 48\gamma}}{2} \right\rfloor$ and $\kappa_{\text{max}}(N_{\gamma}) = \left\lfloor \frac{5 + \sqrt{1 + 24\gamma}}{2} \right\rfloor$. Our first question is: Are these complete graphs the only graphs which attain Cook’s bounds for $\kappa_{\text{max}}(S_{\gamma})$ (resp. for $\kappa_{\text{max}}(N_{\gamma})$)?

It is well known that there are infinitely many 5-connected graphs which are planar and infinitely many 5-connected graphs with nonorientable genus 1. It has also been shown by Negami [8–10] that there are infinitely many 6-connected graphs embeddable in each closed surface with nonpositive Euler characteristic. On the other hand, it also well known (see, for example, [16]) that any 7-connected graph embeddable in a surface of
genus $\gamma$ or $\bar{\gamma}$ has size bounded above by a function of the genus and hence there are at most only a finite number of 7-connected graphs embeddable in any surface, orientable or nonorientable. Thus for any surface with $\gamma \geq 2$ (resp. with $\bar{\gamma} \geq 3$), the number of graphs embeddable in that surface and having connectivity $\kappa$, $7 \leq \kappa \leq \kappa_{\max}$, is finite.

Furthermore, Cook also claimed that the complete graphs attaining his bounds for surfaces $S_\gamma$ and $N_\gamma$ are genus embeddable there. This is not necessarily the case. It is well known, in fact, that as genus increases, there are many surfaces which do not admit a genus embedding of any complete graph.

The maximum connectivity of any graph having orientable (resp. nonorientable) genus $\gamma$ (resp. $\bar{\gamma}$) will be denoted by $\kappa_{\max}(S_\gamma)$ (resp. $\kappa_{\max}(N_\gamma)$) and will be called the genus connectivity of the surface $S_\gamma$ (resp. of surface $N_\gamma$). Clearly, for any orientable surface $S_\gamma$ (resp. nonorientable surface $N_\gamma$) $\kappa_{\max}(S_\gamma) \leq \kappa_{\max}(S_\gamma)$ and $\kappa_{\max}(N_\gamma) \leq \kappa_{\max}(N_\gamma)$. But is there a surface with genus connectivity strictly less than its maximum connectivity? And precisely when does strict inequality hold in these inequalities?

It is obvious that $\kappa_{\max}(S_\gamma)$ and $\kappa_{\max}(N_\gamma)$ are monotone nondecreasing functions of $\gamma$ and $\bar{\gamma}$, respectively. However, it will be shown that, surprisingly, when the genus increases, $\kappa_{\text{gen}}$ may in fact decrease. For example, it will be shown later that $\kappa_{\text{gen}}(S_{35}) = 23$, but $\kappa_{\text{gen}}(S_{36}) < 23$. Why does this counterintuitive behavior of $\kappa_{\text{gen}}(S_\gamma)$ and $\kappa_{\text{gen}}(\bar{\gamma})$ occur? It was these three questions which motivated the present paper.

In the second section of the present paper, the uniqueness of graphs having maximum connectivity $\kappa_{\max}(S_\gamma)$ or $\kappa_{\max}(N_\gamma)$ will be studied. In Section 3 a surface is defined to be in Class A or B according to whether or not the genus and maximum connectivities are equal. Consider the integer interval $[\gamma(K_n) + 1, \gamma(K_{n+1})]$. It is shown that for $n \geq 30$, this interval must contain the genus of at least one surface in Class B. Similarly, for $n \geq 18$, it is shown that there is a nonorientable surface $N_j$ in Class B for some $j \in [\gamma(K_n) + 1, \gamma(K_{n+1})]$.

In Section 4 it is shown that each interval $[\gamma(K_n) + 1, \gamma(K_{n+1})]$ and $[\gamma(K_n) + 1, \gamma(K_{n+1})]$ can be written as the union of two subintervals, the lower subinterval consisting only of genera of Class A surfaces and the upper consisting entirely of genera of Class B surfaces. In Section 5 the location of the “breaking point” between these two subintervals is investigated.

2. ON THE UNIQUENESS OF GRAPHS WITH CONNECTIVITY $\kappa_{\max}(S_\gamma)$ OR $\kappa_{\max}(N_\gamma)$

It is well known, by Euler’s formula, that the maximum connectivity that any planar graph can have is 5 and, hence, that $\kappa_{\max}(S_0) = 5$. Moreover, it
is well known that there are infinitely many 5-connected planar graphs. Hence, there are infinitely many graphs in the plane attaining the genus connectivity of the plane. Similarly, the maximum connectivity bound for the torus is 6 and there are infinitely many 6-connected graphs embeddable in every orientable surface of genus at least 1 and every nonorientable surfaces of genus at least 2 (see Negami [8–10]). Hence, for both the torus and the Klein bottle there are infinitely many 6-connected graphs embeddable there which attain the genus connectivity of the surface.

Also in the case of the projective plane it is easy to construct infinitely many graphs having \( \gamma = 1 \) and having connectivity \( \kappa_{\text{gen}} = \kappa_{\text{max}} = 5 \). For example, suppose the configuration shown on the left hand side of Fig. 1 is an induced subgraph of a 5-connected graph \( G \). Form a larger graph \( H \) from \( G \) by replacing a copy of this configuration in \( G \) by the configuration shown on the right-hand side of Fig. 1. Clearly, \( H \) is then also 5-connected.

Before proceeding, we state the following useful corollary to Euler’s formula for graphs of arbitrary genus. (See, for example, [18, pp. 62, 179].) Repeated use of this result will be made in the rest of this paper.

**Corollary 2.1.** If \( G \) is a connected simple graph with \( q \) edges and \( p \) vertices, then \( \gamma(G) \geq q/6 - p/2 + 1 \) and \( \gamma(G) \geq q/3 - p + 2 \).

We pause at this point to introduce a family of graphs which will prove useful in several instances in this paper. Denote the graph obtained by deleting a maximum matching from the complete graph \( K_m \) by \( O(m) \) and call it a generalized octahedron. When \( m \) is even (say \( m = 2n \)) we shall conform to the literature and denote \( O(m) \) by \( K_{n+2} \). In the next theorem, part (a) is due to Jungerman and Ringel [7] and Alpert and Gross [1] (see also [18]), part (b) is due to Jungeman [5, 6] and Alpert (unpublished), and part (c) is clear.

**Theorem 2.2.**

(a) \( \gamma(K_{n+2}) = (n - 3)(n - 1)/3, n \not\equiv 2(\mod 3) \),

(b) \( \gamma(K_{n+2}) = 3 \) and \( \gamma(K_{m+2}) = 2 \lceil (n - 3)(n - 1)/3 \rceil, n \geq 5 \), and

(c) \( \kappa(O(m)) = m - 2, m \geq 2 \).

We shall have occasion to appeal to the next theorem repeatedly in what follows.

**Theorem 2.3.** Let \( \Omega(m) \) denote any supergraph of \( O(m) \) having \( m \) vertices, other than \( K_m \). Then
(a) \( \kappa(\Omega(m)) = m - 2 \) and

(b) for every \( i, \gamma(\Omega(m)) \leq i \leq \gamma(K_m - K_2) \), there exists a spanning supergraph \( \Omega(m) \) with \( \gamma(\Omega(m)) = i \) (resp. for every \( i, \gamma(\Omega(m)) \leq i \leq \gamma(K_m - K_2) \), there exists a spanning supergraph \( \Omega(m) \) with \( \gamma(\Omega(m)) = i \)).

**Proof.** Since \( \kappa(K_m - K_2) = m - 2 \), simply replace the matching missing from \( \Omega(m) \) one edge at a time.

It is a surprising fact that the nonuniqueness of “maximally” connected graphs in the surfaces of small genus discussed at the beginning of this section turns out to be the exception, rather than the rule. More particularly, we have the next theorem. A graph \( G \) is said to be \( g \)-unique (resp. \( g \)-unique) if \( G \) is the only graph with \( \kappa(G) = \kappa_{\text{max}}(S_g) \) (resp. \( \kappa(G) = \kappa_{\text{max}}(N_g) \)) which embeds in surface \( S_g \) (resp. \( N_g \)).

**Theorem 2.4.** (A) (The orientable case). Suppose \( n \geq 7 \) and that suppose that the complete graph \( K_{n+1} \) genus embeds in \( S_g \) and that \( \kappa(K_{n+1}) = n = \kappa_{\text{max}}(S_g) \). Then \( K_{n+1} \) is \( g \)-unique, unless \( n = 7, 8, 9, 10, 12, 13, \) or \( 16 \). Moreover, in the cases \( n = 7, 8, 10, 12 \) and \( 16 \), \( K_{n+1} \) is not \( g \)-unique.

(B) (The nonorientable case). Suppose \( n \geq 7 \) and that the complete graph \( K_{n+1} \) genus embeds in \( N_k \) and that \( \kappa(K_{n+1}) = n = \kappa_{\text{max}}(N_k) \). Then \( K_{n+1} \) is \( k \)-unique, unless \( n = 7 \) or 10. Moreover, in both cases \( n = 7 \) and \( n = 10 \), \( K_{n+1} \) is not \( k \)-unique.

**Proof.** Let \( G \) be any graph embeddable in \( S_g \) with \( \kappa(G) \geq n \). Of course, then \( G \) must have at least \( n + 1 \) vertices. Moreover, we know that if \( G \) has exactly \( n + 1 \) vertices, then \( G = K_{n+1} \). So suppose \( G \) has \( n + 1 + r \) vertices, where \( r \geq 1 \). Now

\[
\frac{(n - 2)(n - 3)}{12} = \gamma(K_{n+1}) = g \geq \gamma(G) \geq \frac{2G - 5}{2} + 1,
\]
where, as usual, $q_G$ denotes the number of edges in $G$ and $p_G$ denotes the number of vertices in $G$. The first equality is by the famous Ringel–Youngs theorem on the genus of the complete graphs [13, 14] and the last inequality is by Corollary 2.1. Now since for any graph $G$, $2q_G = \sum \deg v$, where the sum is over all the vertices in $G$, we have

$$\left\lceil \frac{(n-2)(n-3)}{12} \right\rceil \geq \frac{n}{12} (n+r+1) = \frac{(n+r+1)}{2} + 1. \quad (2.1)$$

So

$$\frac{(n-2)(n-3)}{12} \geq \frac{n(n+r+1)}{12} - \frac{n+r+1}{2} = \frac{(n-6)(n+r+1)}{12}.$$

But then solving this inequality for $r$, we obtain $r(n-6) \leq 11$ and since $n-6 > 0$, it follows that $r \leq 11/(n-6)$. But then for $n \geq 18$, $r < 1$ and hence for $n \geq 18$, $K_{n+1}$ is the only graph embeddable in $S_g$ which has $\kappa(G) \geq n$.

Suppose now that $n = 17$. Then substituting into inequality (2.1) with $p_G = 18 + r$, we obtain the result that $11r \leq 6$ and hence, since $r$ is an integer, $r = 0$. So once again we have that $K_{n+1} = K_{18}$ is 18-unique.

Similarly, one finds that $K_{n+1}$ is $g$-unique in its genus surface for each of the values $n = 11, 14, 15$.

Using the same inequality, we find for the values $n = 7, 8, 9, 10, 12, 13,$ and 16 that $r \leq 4, 3, 2, 1, 1, 1,$ and 1, respectively. These cases are now treated separately.

For $n = 7$, the 10-vertex graph of Huneke [4] has $\gamma = 2$ and it may be checked (somewhat laboriously) that it is 7-connected. Hence $K_8$ is not 2-unique.

For $n = 8$, the 12-vertex tripartite graph $K_{4,4,4}$ has genus 3 [15, 17] and connectivity 8. So the graph $K_8$ is not 3-unique.

When $n = 10$, the octahedron $K_{n(2)}$ has genus 5 and connectivity 10 by Theorem 2.2, so $K_{11}$ is not 5-unique. Similarly, when $n = 12$, the octahedron $K_{n(2)}$ has genus 8 and is 12-connected, so $K_{13}$ is not 8-unique and when $n = 16$, the octahedron $K_{n(2)}$ has genus 16, and has connectivity 16, so $K_{17}$ is not 16-unique. So far we have been unable to settle the two remaining cases $n = 9$ and $n = 13$.

Now consider the nonorientable case. Recall the nonorientable analog of the Ringel–Youngs theorem which was first proved by Ringel [11, 12] and which says that if $n \geq 5$, then $\gamma(K_n) = \lfloor (n-3)(n-4)/6 \rfloor$. Using this result, together with the “nonorientable” part of Corollary 2.1 and proceeding exactly as in the first half of this proof, we obtain that for $n = 8, 9, 11$ and 13 and if $n \geq 12$, $K_{n+1}$ is $g$-unique in its genus surface.

We claim that for the two remaining values $n = 7$ and $n = 10$, the complete graphs $K_8$ and $K_{11}$ are not $g$-unique in their genus surfaces.
First, let \( n = 7 \). By Theorem 2.2, \( \chi(K_7) = 4 \). We proceed to construct another 7-connected graph \( H_{10} \) having 10 vertices and having \( \chi(H_{10}) = 4 \).

Begin by embedding \( K_7 \) in the torus as shown in Fig. 2a.

Let the vertices of \( K_7 \) be denoted by \( \{0, 1, 2, 3, 4, 5, 6\} \). Delete the three edges \( \{1, 5\} \), \( \{1, 4\} \) and \( \{3, 6\} \) from this embedding of \( K_7 \). (See Fig. 2a.)

Now add a “twisted handle” to the surface as follows. Cut holes in the interiors of faces \( \{1, 3, 4, 6\} \) and \( \{2, 1, 0, 4, 5\} \). Attach a cylinder with one end identified with each hole with the orientation shown by the arrows.

Now insert three new vertices \( x, y, \) and \( z \) in the new twisted handle and add the edges shown in Fig. 2(b) to obtain a 10-vertex graph which we will call \( H_{10} \). We have constructed \( H_{10} \) as embedded in the surface \( N_4 \), so

![Diagram](image-url)
On the other hand, by Corollary 2.1, \( \bar{\nu}(H_{10}) \geq 4 \), so equality must hold and, hence, \( \bar{\nu}(H_{10}) = 4 \). It is an easy, but tedious, task to verify that \( H_{10} \) is 7-connected.

Thus \( K_8 \) is not 4-unique. Finally, suppose \( n = 10 \). The 12-vertex graph \( K_{10}(2) \) has \( \bar{\nu} = 10 \) by Theorem 2.2 and it is easily checked that \( K_{10}(2) \) is 10-connected. Thus \( K_{11} \) is not 10-unique.

This completes the proof of the theorem.

Open Question(s): Is \( K_{10} \) the only 9-connected graph which embeds in \( S_4 \)? Is \( K_{14} \) the only 13-connected graph which embeds in \( S_4 \)?

3. A CLASSIFICATION OF SURFACES VIA CONNECTIVITY BOUNDS

Suppose \( m \geq 7 \) and suppose that the complete graph \( K_m \) genus embeds in the orientable surface \( S_g \). Then \( K_{m+1} \) genus embeds in \( S_{g+x} \), for some \( x \geq 1 \). We now proceed to study the continuum of genera values between \( g \) and \( g+x \).

Define the collection of all surfaces \( S_g \) (resp. \( N_g \)) for which \( \kappa_{\text{max}}(S_g) = \kappa_{\text{gen}}(S_g) \) (resp. for which \( \kappa_{\text{max}}(N_g) = \kappa_{\text{gen}}(N_g) \)) to belong to Class A, denoted \( \mathcal{A} \). All other surfaces will be said to belong to Class B, denoted \( \mathcal{B} \).

As yet, we have not proved that there are any surfaces in \( \mathcal{B} \).

First, consider again the surfaces \( S_g \) and \( S_{g+x} \), where \( x \geq 1 \), as defined as in the first paragraph of this section. Both of these surfaces are in Class \( \mathcal{A} \).

As one proceeds through increasing genera in the interval \([g+1, g+x]\), let \( g_{\mathcal{A}} \) denote the first genus encountered such that \( S_{g_{\mathcal{A}}} \) is in \( \mathcal{A} \). Similarly, if \( K_m \) and \( K_{m+1} \) are genus embedded in the nonorientable surfaces \( N_g \) and \( N_{g+x} \), respectively, then, as one proceeds from \( g+1 \) to \( g+x \), \( g_{\mathcal{A}} \) will denote the least nonorientable genus to be found in the interval \([g+1, g+x]\) which belongs to \( \mathcal{A} \). We call \( g_{\mathcal{A}} \) (resp. \( g_{\mathcal{B}} \)) the breaking point of the interval. For small values of \( g \) and \( g \), it may happen that \( g = g+x \) (\( g = g+x \)); that is, the interval is either not defined. In such cases, the breaking point will not be defined. If the interval consists of only one value, i.e., \( g+1 = g+x \) (\( g+1 = g+x \)), then the breaking point will be \( g+1 \) or \( g+1 \), respectively. It will be shown below that if \( g \geq 59 \) (resp. if \( g \geq 35 \)), then there always exists an \( x > 1 \) such that there exists a Class \( \mathcal{B} \) surface in the interval \([g+1, g+x]\) (resp. in the interval \([g+1, g+x]\)).
Let us begin our study of the breaking point by providing a lower bound. For all integers $m \geq 12$, write $m = 12s + k$, where $s \geq 1$ and $0 \leq k \leq 11$.

Consider first the orientable case. Define $\gamma_{s,k} = \gamma(K_m) = \gamma(K_{12s+k})$. By the Ringel-Youngs genus formula [13, 14], $\gamma_{s,k} = 12s^2 + (2k-7)s + \lceil (k-3)(k-4)/12 \rceil$. Next, define $\delta_k$ and $\epsilon_k$ as

$$\delta_k = \begin{cases} -2, & \text{if } k = 1, 2, \text{ or } 5 \\ -1, & \text{otherwise} \end{cases}$$

and $\epsilon_k$ by

$$\epsilon_k = \begin{cases} -1, & \text{if } k = 2 \\ 0, & \text{if } k = 0, 1, 3, 5, \text{ or } 6, \\ 1, & \text{otherwise.} \end{cases}$$

Note that $\gamma_{s,k+1} - \gamma_{s,k} = 2s + \epsilon_k$. To see this, note that the difference

$$\gamma_{s,k+1} - \gamma_{s,k} = 12s^2 + (2k-5)s + \lceil (k-2)(k-3)/12 \rceil - (12s^2 + (2k-7)s + \lceil (k-3)(k-4)/12 \rceil) = 2s + \lceil (k-2)(k-3)/12 \rceil - \lceil (k-3)(k-4)/12 \rceil.$$  

Upon computing the difference of the two ceiling functions, observe that $\gamma_{s,k+1} - \gamma_{s,k} = 2s + \epsilon_k$, as claimed. We then have the following theorem.

**Theorem 3.1.** Suppose that $m = 12s + k$ and that $K_m$ genus embeds in surface $S_{s,k}$ so that $\kappa_{\text{max}}(S_{s,k}) = \kappa_{\text{gen}}(S_{s,k}) = m - 1$. Suppose $\delta_k$ is as defined above and suppose also that $s + \delta_k \geq 1$. Then if $i$ is an integer in the interval $I_{s,k} = [\gamma_{s,k} + 1, \gamma_{s,k} + s + \delta_k]$, $S_i$ is in Class $B$ (and hence $B \neq \emptyset$).

**Proof.** Suppose $G$ is another graph with $\kappa(G) = m - 1$ which embeds in $S_{s,k}$, but suppose $G$ has more than $m$ vertices. That is, suppose $|V(G)| = 12s + k + t$ for some $t \geq 1$. We proceed to obtain a lower bound on the genus of $G$.

Since $G$ is $(m-1)$-connected,

$$|E(G)| \geq \frac{(12s + k + t)(12s + k - 1)}{2}$$

and so, by Corollary 2.1,

$$\gamma(G) \geq \frac{|E(G)|}{6} - \frac{|V(G)|}{2} + 1$$

$$\geq \frac{(12s + k + t)(12s + k - 1)}{12} - \frac{(12s + k + t)}{2} + 1$$

$$= \frac{1}{12} (144s^2 + 24sk + k^2 - 84s - 7k + 12 + t(12s + k - 7)).$$

Denote the last term in this string of inequalities by $f_{s,k}(t)$. 

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CONNECTIVITY OF EMBEDDED GRAPHS

Note that since \( s \geq 1 \), \( f_{s,k}(t) \) is a monotone nondecreasing function of \( t \) and, hence, \( \gamma(G) \geq f_{s,k}(1) \) and, hence,

\[
i \geq \gamma(G) \geq \lceil f_{s,k}(1) \rceil.
\]

Also note that if a noncomplete graph \( H \) genus embeds in the surface \( S_{g_\varphi} \) and is \((m-1)\)-connected, then \( H \) must have at least \( m+1 \) vertices. But then by the above argument, \( g_\varphi \geq f_{s,k}(1) \). We shall need this fact later.

Computing \( f_{s,k}(1) \), one obtains

\[
f_{s,k}(1) = 12s^2 + 6k - 6s + \frac{1}{12}(k^2 - 6k + 5)
= 12s^2 + s(2k - 7) + s + \frac{1}{12}(k^2 - 6k + 5).
\]

So \( \lceil f_{s,k}(1) \rceil = 12s^2 + s(2k - 7) + s + [(k^2 - 6k + 5)/12] \) and, hence,

\[
\lceil f_{s,k}(1) \rceil - \gamma_{s,k} = s + \left\lfloor \frac{k^2 - 6k + 5}{12} \right\rfloor - \left\lfloor \frac{(k-3)(k-4)}{12} \right\rfloor = s + \delta_k + 1,
\]

where the last equality is obtained by computing the difference of the two functions of \( k \) for all 12 values of \( k \) and comparing the results with the definition of \( \delta_k \).

So \( \lceil f_{s,k}(1) \rceil = \gamma_{s,k} + s + \delta_k + 1 \) and so, by inequality (3.1), \( \gamma(G) \geq \gamma_{s,k} + s + \delta_k + 1 \). Now suppose \( i \in [\gamma_{s,k} + 1, \gamma_{s,k} + s + \delta_k] \). We claim that in this case, \( S_i \in \mathcal{B} \).

Suppose, to the contrary, that \( S_i \in \mathcal{A} \). Then there exists a graph \( G \) which genus embeds in \( S_i \) and \( k(G) = m - 1 = \kappa_{\max}(S_i) \). If \( |V(G)| \geq m + 1 \), it was shown above that \( i = \gamma(G) \geq \gamma_{s,k} + s + \delta_k + 1 \), a contradiction. Thus \( |V(G)| \leq m \). But \( G \) is \((m-1)\)-connected, so, necessarily, \( G = K_m \). But then \( i = \gamma(G) = \gamma(K_m) = \gamma_{s,k} \), a contradiction.

Note that for \( m = 12s + k \geq 30 \), Theorem 3.1 guarantees that the interval \([\gamma(K_m) + 1, \gamma(K_{m+1})]\) must contain at least one integer \( j \) such that \( S_j \) is in Class \( \mathcal{B} \). Moreover, by the Ringel–Youngs theorem, \( \gamma(K_{10}) = 59 \). We proceed to treat the remaining surfaces \( S_i \) for \( 1 \leq j \leq 58 \).

First, note that \( S_1 = S_2, S_3, S_{10}, S_{11}, S_{13}, S_{18}, S_{20}, S_{23}, S_{26}, S_{29}, S_{32}, S_{35}, S_{39}, S_{43}, S_{46}, S_{50}, \) and \( S_{53} \) are all in Class \( \mathcal{A} \) since there is a complete graph which genus embeds in each.

Consider \( S_3 \). Note that by Theorem 2.2, the octahedron \( K_{7,2} \) has genus 8 and \( k = 12 = \kappa_{\max}(S_3) \). But then by Theorem 2.3, there must be a spanning subgraph of \( K_{14} - K_2 \) with connectivity 12 and genus 9. So \( S_3 \) belongs to class \( \mathcal{A} \). Similar arguments applied in turn to \( K_{9,2}, K_{10,2}, K_{12,2}, K_{13,2}, \)
and \( K_{15(2)} \), show that \( S_{17}, S_{21}, S_{22}, S_{23}, S_{34}, S_{41}, S_{42}, S_{56}, S_{57}, \) and \( S_{58} \) all belong to Class \( \mathscr{A} \). On the other hand, by Theorem 3.1, each of \( S_{34}, S_{40}, \) and \( S_{53} \) is in Class \( \mathscr{B} \).

Now consider embeddings in nonorientable surfaces. For all integers \( m \geq 12 \) we continue to write \( m = 12s + k \), where \( s \geq 1 \) and \( 0 \leq k \leq 11 \).

Define \( \bar{\gamma}_{s,k} = \gamma(K_m) \). Hence, \( \bar{\gamma}_{s,k} = 24s^3 + 2s(2k - 7) + \gamma(k - 3)(k - 4)/6 \).

Next, define \( \delta_k \) and \( \bar{\delta}_k \) as

\[
\delta_k = \begin{cases} 
-2, & \text{if } k = 0, 1, 2, \text{ or } 5, \\
-1, & \text{if } k = 3, 4, 6, 7, 8, \text{ or } 11, \\
0, & \text{if } k = 9 \text{ or } 10,
\end{cases}
\]

and \( \bar{\delta}_k \) by

\[
\bar{\delta}_k = \begin{cases} 
-1, & \text{if } k = 0 \text{ or } 2, \\
0, & \text{if } k = 1, 3, \text{ or } 5, \\
1, & \text{if } k = 4, 6, \text{ or } 8, \\
2, & \text{if } k = 7, 9, \text{ or } 11, \\
3, & \text{if } k = 10.
\end{cases}
\]

Upon computing the difference of the two ceiling functions, note that as \( \bar{\gamma}_{s,k+1} - \bar{\gamma}_{s,k} = 4s + \bar{\delta}_k \). We then have the following theorem. The proof is completely parallel to that of Theorem 3.1 and, hence, is omitted.

**Theorem 3.2.** Suppose \( m = 12s + k \) and that \( K_m \) genus embeds in surface \( N_{\bar{\gamma}_{s,k}} \) so that \( \kappa_{\max}(N_{\bar{\gamma}_{s,k}}) = \kappa_{\gen}(N_{\bar{\gamma}_{s,k}}) = m - 1 \). Suppose \( \delta_k \) and \( \bar{\delta}_k \) are as defined above and suppose also that \( 2s + \delta_k \geq 1 \). Then if \( i \) is an integer in the interval \( I_{s,k} = [\bar{\gamma}_{s,k} + 1, \bar{\gamma}_{s,k} + 2s + \bar{\delta}_k] \), \( N_i \) is in Class \( \mathscr{B} \) (and hence \( \mathscr{B} \neq \emptyset \)).

Note that for \( m \geq 18 \), Theorem 3.2 guarantees that the interval of nonorientable genera \([\gamma(K_m) + 1, \gamma(K_{m+1})]\) contains at least one integer \( j \) such that \( N_j \) is in Class \( \mathscr{B} \). Moreover, \( \gamma(K_{18}) = 35 \). It remains to consider the surfaces \( N_{11} - N_{34} \).

Of these, \( N_{11}, N_{13}, N_{14}, N_{15}, N_{17}, N_{19}, N_{21}, N_{16}, N_{18}, N_{22}, N_{26}, \) and \( N_{34} \) are all in Class \( \mathscr{A} \) since there exists a complete graph which genus embeds in each. Moreover, by Theorem 3.2, each of \( N_{33} \) and \( N_{37} \) is in Class \( \mathscr{B} \). It was observed earlier in this paper that there are 6-connected graphs which genus embed in the Klein bottle, so \( N_{2} \in \mathcal{A} \).

By Theorem 2.2, the generalized octahedron \( K_{52} \) has \( \gamma = 6 \). Moreover, \( \kappa(K_{52}) = 8 = \kappa_{\max}(N_k) \), so \( N_k \) is in Class \( \mathcal{A} \). The generalized octahedron \( K_{53} \) has \( \gamma = 10 \). But then by Theorem 2.3, \( N_{14} \) is in Class \( \mathcal{A} \). Similar arguments
applied to $K_{7;2}$, $K_{8;2}$, and $K_{8;2}$ show that $N_{16}, N_{17}, N_{18}, N_{24}, N_{25}, N_{32}, N_{33}$, and $N_{34}$ all lie in Class $\mathcal{A}$.

4. THE TWO SUBINTERVAL THEOREM

Each interval of genera $I_{x,k} = [\gamma_{s,k} + 1, \gamma_{s,k+1}]$ or $\bar{I}_{x,k} = [\bar{\gamma}_{s,k} + 1, \bar{\gamma}_{s,k+1}]$ either consists entirely of genera belonging to Class $\mathcal{A}$ surfaces, or else contains at least one genus of a Class $\mathcal{B}$ surface. We now show that in the second case these Class $\mathcal{B}$ genera cannot be “scattered” among other genera which are Class $\mathcal{A}$.

**Theorem 4.1 (The two subinterval theorem).** If $m = 12s + k$ and $\gamma_{s,k}$, $\bar{\gamma}_{s,k}$, $g_{\mathcal{A}}$ and $\bar{g}_{\mathcal{A}}$ are as defined above, then the intervals of genera can be written as the union of two subintervals as follows:

(a) (Orientable case). $I_{x,k} = [\gamma_{s,k} + 1, \gamma_{s,k+1}] = [\bar{\gamma}_{s,k} + 1, g_{\mathcal{A}} - 1] \cup [g_{\mathcal{A}}, \bar{\gamma}_{s,k+1}]$, and

(b) (Nonorientable case). $\bar{I}_{x,k} = [\bar{\gamma}_{s,k} + 1, \bar{\gamma}_{s,k+1}] = [\bar{g}_{\mathcal{A}} + 1, \bar{g}_{\mathcal{A}} - 1] \cup [\bar{g}_{\mathcal{A}}, \bar{\gamma}_{s,k+1}]$.

where the first subinterval is either empty or consists entirely of Class $\mathcal{B}$ genera and the second subinterval consists entirely of Class $\mathcal{A}$ genera.

**Proof.** (a) (Orientable case). Suppose the first subinterval is not empty. Then, by the definition of $g_{\mathcal{A}}$, it consists entirely of Class $\mathcal{B}$ surfaces. The second subinterval is always nonempty because $S_{\gamma_{s,k+1}}$ is a Class $\mathcal{A}$ surface, since $K_{12s + k + 1}$ genus embeds in this surface.

If the second subinterval consists of the single genus $g_{\mathcal{A}}$, then, trivially, it consists entirely of Class $\mathcal{A}$ surfaces. So suppose that the second subinterval contains more than one genus, that is, $g_{\mathcal{A}} < \gamma_{s,k+1}$.

Now consider the surface $S_{g_{\mathcal{A}}}$. This surface has $\kappa_{\max}(S_{g_{\mathcal{A}}}) = m - 1$.

Let $G \not\approx K_m$ be any $(m - 1)$-connected graph which genus embeds in $S_{g_{\mathcal{A}}}$. Then $|V(G)| \geq m + 1$.

Now we treat two cases. First, suppose $|V(G)| = m + 1$. Then, since $G$ is $(m - 1)$-connected, the degree of every vertex is at least $m - 1$. But it then follows that $G$ must contain a spanning subgraph isomorphic to the generalized octahedron $O(m + 1)$. Hence by Theorem 2.3, all surfaces $S_i$ having genus $i$ in the subinterval $[g_{\mathcal{A}}, \gamma_{s,k+1}]$ must belong to Class $\mathcal{A}$.

So, therefore, suppose that $|V(G)| \geq m + 2$. More particularly, suppose that $|V(G)| = m + t = 12s + k + t$ for some $t \geq 2$. Then, proceeding exactly as in the proof of Theorem 3.1, one has $\gamma(G) \geq f_{s,k}(2) = 12s^2 + 2sk - 5s + (1/12)(k^2 - 5k - 2)$. So
\[
\begin{align*}
\gamma_{n,k+1} - \lceil f_{s,k}(2) \rceil &= 12x^2 + (2k - 5)s + \left(\frac{(k-2)(k-3)}{12}\right) \\
&\quad - \left(\frac{k^2 - 5k - 2}{12}\right) \\
&= \left(\frac{(k-2)(k-3)}{12}\right) - \left(\frac{k^2 - 5k - 2}{12}\right).
\end{align*}
\]

But computing the difference of the two ceiling functions for all 12 values of \( k \) shows that this difference is 0, when \( k = 2, 3, 6, \) or 11, and is 1, otherwise. In other words, either \( \gamma_{n,k+1} = \lceil f_{s,k}(2) \rceil \) or \( \gamma_{n,k+1} = \lceil f_{s,k}(2) \rceil + 1. \)

If \( \gamma_{n,k+1} = \lceil f_{s,k}(2) \rceil \), then \( \gamma_{n,k+1} + 1 = \lceil f_{s,k}(2) \rceil + 1. \) So equality must hold and \( g_{s,k} = \gamma_{n,k+1}. \) If \( \gamma_{n,k+1} + 1 = \lceil f_{s,k}(2) \rceil + 1 \), then equality must hold and \( g_{s,k} = \gamma_{n,k+1} + 1. \) But then in either case, interval \( [\gamma_{s,k}, \gamma_{n,k+1}] \) consists only of general of Class \( \mathcal{A} \) surfaces as claimed.

Now consider the nonorientable case. Let \( G \) be an \((m-1)\)-connected graph which genus embeds in the surface \( N_{s,k}. \) If first one assumes that \( |V(G)| = m + 1 \), then arguing exactly as in the orientable case, all \( N_{s,k} \) with \( f \in \{ g_{s,k}, \gamma_{n,k+1} \} \) are in Class \( \mathcal{A} \) as desired.

So assume that \( |V(G)| \geq m + 2. \) Again let \( |V(G)| = m + t \), where \( t \geq 2. \) Then again, \( \gamma(G) \geq f_{s,k}(2). \) Now since \( f_{s,k}(2) = 24s^2 + 4sk - 10x + (1/6)(k^2 - 5k - 2), \) it follows that

\[
\gamma_{n,k+1} - \lceil f_{s,k}(2) \rceil = \left(\frac{(k-2)(k-3)}{6}\right) - \left(\frac{k^2 - 5k - 2}{6}\right),
\]

and upon computing this difference for each of the 12 values of \( k \), note that either \( \gamma_{n,k+1} = \lceil f_{s,k}(2) \rceil + 1 \) or \( \gamma_{n,k+1} = \lceil f_{s,k}(2) \rceil + 2. \) If \( \gamma_{n,k+1} = \lceil f_{s,k}(2) \rceil + 1, \) then we are done as in the orientable case.

So assume \( \gamma_{n,k+1} = \lceil f_{s,k}(2) \rceil + 2. \) There are four cases to consider: \( k = 1, 4, 7, \) and 10:

First suppose \( k = 1. \) Thus \( \gamma_{n,2} = \lceil f_{s,2}(2) \rceil + 2. \) Consider the generalized octahedron on \( 12 + 2 \) vertices, \( K_{(6s+12)(2)}. \) It has nonorientable genus \( 24s^2 - 8s \) by Theorem 2.2. Hence \( N_{24s^2 - 8s} \) is in Class \( \mathcal{A} \) and so \( g_{s,k} \leq 24s^2 - 8s. \)

On the other hand, \( f_{s,1}(1) = 24s^2 - 8s \) as well. Moreover, by inequality (3.1), \( f_{s,1}(1) \leq g_{s,k}. \) Thus, we have \( g_{s,k} \leq \gamma(K_{(6s+12)(2)}) = f_{s,1}(1) \leq g_{s,k}. \) Thus, equality must hold and, hence, \( g_{s,k} \leq \gamma(K_{(6s+12)(2)}). \) But then by Theorem 2.3, all the genera in the interval \( [g_{s,k}, \gamma_{n,k+1}] \) must be in Class \( \mathcal{A}. \)

Next let \( k = 7. \) By Theorem 2.2, the octahedron \( K_{(6s+43)(2)} \) has nonorientable genus \( 24s^2 + 16s + 2 = f_{s,7}(1). \) So arguing just as in the preceding case, \( g_{s,k} \leq \gamma(K_{(6s+43)(2)}) \) and, again, all surfaces \( N_{s,k} \) have genera in \( [g_{s,k}, \gamma_{n,k+1}] \) must be in Class \( \mathcal{A}. \)
Suppose now that \( k = 10 \). Recall that \( \bar{g}_{\alpha, \gamma} = \bar{\gamma}(G) \geq \lceil \bar{f}_{n,k}(2) \rceil = \bar{\gamma}_{n,k+1} - 2 \) and \( \bar{g}_{\alpha, \gamma} \leq \bar{\gamma}_{n,k+1} \), since \( N_{\bar{\gamma}_{n,k+1}} \) is a surface in class \( \alpha \). So there are only three possible values for \( \bar{g}_{\alpha, \gamma} \). If \( \bar{g}_{\alpha, \gamma} = \bar{\gamma}_{n,k+1} \) or \( \bar{\gamma}_{n,k+1} - 1 \), then the interval \( [\bar{g}_{\alpha, \gamma}, \bar{\gamma}_{n,k+1}] \) clearly consists of all Class \( \alpha \) surfaces as claimed.

So suppose \( \bar{g}_{\alpha, \gamma} = \bar{\gamma}_{n,k+1} - 2 \). It remains to show that the surface \( N_{\bar{\gamma}_{n,k+1} - 1} \) is in Class \( \alpha \). Ringel\cite{13} exhibited a triangular embedding of \( K_{12s+11} - K_2 \) into a closed surface \( \Sigma \) and then, by adding back the missing edge and one cross-cap to accommodate it, obtained a genus embedding of the complete graph \( K_{12s+1} \) into the nonorientable surface \( \Sigma \) consisting of surface \( \Sigma \) with one cross-cap added. Computing the Euler characteristic \( \chi \) of surface \( \Sigma \), one obtains

\[
\chi(\Sigma') = 2 - \frac{(12s+8)(12s-7)}{6} = 2 - (24s^2 + 30s + 10)
\]

which is an even number. But \( \chi(\Sigma) = \chi(\Sigma') + 1 \), since \( \Sigma' \) was obtained from \( \Sigma \) by adding one cross-cap. Thus \( \chi(\Sigma) \) must be odd and, hence, \( \Sigma \) is a non-orientable surface. So \( \bar{\gamma}(K_{12s+11} - K_2) = \bar{\gamma}_{n,k+1} - 1 \) and, thus, \( N_{\bar{\gamma}_{n,k+1}-1} \) lies in Class \( \alpha \).

It remains only to treat the case when \( k = 4 \). First consider the special case when \( k = 4 \) and \( s = 1 \). By the Ringel-Youngs theorem, \( \bar{\gamma}(K_{14}) = 26 \) and \( \bar{\gamma}(K_{15}) = 31 \). By Theorem 3.2, \( N_{27} \) belongs to Class \( \beta \) and, hence, \( \bar{g}_{\alpha, \gamma} \geq 28 \). By \cite[3, p. 246]{3}, \( \bar{\gamma}(K_{17} - K_2) = 30 \) and \( K_{17} - K_2 \) has connectivity 15, so \( N_{30} \) belongs to Class \( \alpha \). But then if \( \bar{g}_{\alpha, \gamma} = 29 \), we are done. So suppose that \( \bar{g}_{\alpha, \gamma} = 28 \). Let \( G \) be a graph which genus embeds in \( N_{28} \) and has connectivity 15. Suppose, for the moment, that the stronger assumption is made that \( |V(G)| = m + 3 \). Then \( f_{s,4}(3) - \bar{\gamma}_{s,5} = 2s - 5/2 > 0 \), since \( s \geq 2 \). Thus \( \bar{g}_{\alpha, \gamma} = \bar{\gamma}(G) \geq f_{s,4}(3) > \bar{\gamma}_{s,5} \). But this is a contradiction since \( K_{12s+5} \) genus embeds in \( N_{\bar{\gamma}_{s,5}} \).

So there remains only the case in which \( |V(G)| = m + 2 \). But then \( \bar{\gamma}(G) \geq f_{s,4}(2) = 24s^2 + 6s - 1 \). On the other hand, \( \bar{\gamma}(K_{m+1}) = \bar{\gamma}(K_{12s+5}) = 24s^2 + 6s + 1 \). If \( \bar{g}_{\alpha, \gamma} = 24s^2 + 6s \), we are done. So suppose \( \bar{g}_{\alpha, \gamma} = 24s^2 + 6s - 1 \). So \( \bar{g}_{\alpha, \gamma} \geq 24s^2 + 6s - 1 \). But by \cite[3, p. 246]{3}, \( \bar{\gamma}(K_{12s+5} - K_2) = 24s^2 + 6s \). Hence, \( N_{24s^2 + 6s - 1} \), \( N_{24s^2 + 6s} \), and \( N_{24s^2 + 6s+1} \) are all in Class \( \alpha \). This completes the proof of the theorem.

5. THE LOCATION OF THE BREAKING POINT

So far we have seen that (with a few exceptions where the genus is small) the interval of genera between those of complete graphs \( K_m \) and \( K_{m+1} \) can
be known.)

Theorem 3.1, \(g_{\mathcal{A}}(s, k)\) belonging to the interval \([\gamma_{s, k} + 1, \gamma_{s, k} + 1]\) by \(g_{\mathcal{A}}(s, k)\). Similarly, if \(s = 0\) or if \(s = 1\) or if \(s = 0\) and \(k \geq 9\), denote the value of \(g_{\mathcal{A}}\) belonging to the interval \([\gamma_{s, k} + 1, \gamma_{s, k} + 1]\) by \(g_{\mathcal{A}}(s, k)\).

**Theorem 5.1.** If \(s \geq 2\) and if \(k = 1, 2, 3, 4, 5, 6, 7\) or 11, then \(g_{\mathcal{A}}(s, k) = \gamma(O(12s + k + 1))\).

**Proof.** First assume that \(s + \delta_k \geq 1\). Then by Theorem 3.1, \(g_{\mathcal{A}}(s, k) \geq \gamma_{s, k} + s + \delta_k + 1\). On the other hand, \(O(12s + k + 1) = O(m + 1)\) is \((m - 1)\)-connected and clearly \(\gamma(O(12s + k + 1)) \leq \gamma(K_{12s + k + 1})\). Also from the proof of Theorem 3.1, \(\gamma_{s, k} + 2 \leq \gamma(f_{s, k}(1)) \leq \gamma(O(12s + k + 1))\). So \(\gamma_{s, k} + 2 \leq \gamma(O(12s + k + 1)) \leq \gamma_{s, k} + 1\). We also know that the surface in which \(O(12s + k + 1)\) genus embeds is in Class \(\mathcal{A}\). Therefore, by the definition of \(g_{\mathcal{A}}(s, k)\), \(g_{\mathcal{A}}(s, k) \leq \gamma(O(12s + k + 1))\). So, once again by the proof of Theorem 3.1, \(\gamma_{s, k} + s + \delta_k + 1 \leq g_{\mathcal{A}}(s, k) \leq \gamma(O(12s + k + 1))\).

Observe next that for \(k = 1, 5, 7,\) and 11, \(12s + k + 1 = 2(6s + (k + 1)/2)\) and \(6s + (k + 1)/2 \equiv 2 \pmod{3}\) and, hence, using Theorem 2.2, it is routine to verify that in each of these cases, \(\gamma(O(12s + k + 1)) = \gamma_{s, k} + s + \delta_k + 1\) and the desired result follows.

Now consider the cases \(k = 2, 3,\) or 6. Let \(G\) be an \((m - 1)\)-connected graph which genus embeds in the surface \(S_{\epsilon, k}\). We know that \(g_{\mathcal{A}}(s, k) \leq \gamma(O(12s + k + 1))\).

First suppose that graph \(G\) has \(m + 1\) vertices. Then, regardless of the parity of \(m\), \(G\) must contain the generalized octahedron \(O(12s + k + 1)\) as a spanning subgraph. But then \(g_{\mathcal{A}} \geq \gamma(O(12s + k + 1))\). Hence, \(g_{\mathcal{A}} = \gamma(O(12s + k + 1))\), as claimed. (It is interesting to note here that the numerical value of \(\gamma(O(12s + k + 1))\) in the above equation may or may not be known.)

So suppose \(G\) has at least \(m + 2\) vertices. Then as in the proof of Theorem 3.1, \(\gamma(G) \geq \gamma(f_{s, k}(2))\). But from the proof of part (a) of Theorem 4.1, when \(k = 2, 3, 6,\) or 11, \(\gamma_{s, k, 1} = \gamma(f_{s, k}(2))\). So \(\gamma_{s, k, 1} = \gamma(f_{s, k}(2)) \leq \gamma(G) = g_{\mathcal{A}} \leq \gamma(O(12s + k + 1)) \leq \gamma_{s, k} + 1\). So equality must hold; that is, \(g_{\mathcal{A}}(s, k) = \gamma(O(12s + k + 1))\).

Finally, suppose \(k = 4\). If \(G\) has \(m + 1\) vertices, since it has connectivity \(k = m - 1\), it must contain the octahedron \(O(12s + 5)\) as a spanning subgraph.
So \( \gamma(O(12s + 5)) \leq \gamma(G) = g_{\omega} \leq \gamma(O(12s + 5)) \) and the desired equality holds.

Suppose, therefore, that \( G \) has at least \( m + 2 \) vertices. Then, by the proof of Theorem 4.1, \( \gamma(G) = g_{\omega} \geq \lceil f_{s,2}(2) \rceil = \gamma_{s,5} - 1 \). On the other hand, by [13, p. 180], \( \gamma(K_{12s+5} - K_2) = \gamma_{s,5} - 1 \). But the octahedron \( O(12s + 5) \) is a spanning subgraph of \( K_{12s+5} - K_2 \), so \( \gamma_{s,5} - 1 = \lceil f_{s,2}(2) \rceil \leq g_{\omega} = \gamma(G) \leq \gamma(O(12s + 5)) \leq \gamma(K_{12s+5} - K_2) = \gamma_{s,5} - 1 \). So equality holds and the proof is complete when \( s + \delta_1 \geq 1 \).

Now suppose \( s = 2 \). Then if \( k \neq 1, 2, \) or \( 5 \), we are done as in the first part of this proof above. That is, for \( s = 2 \) and \( k = 3, 4, 6, 7 \), and \( 11 \), the conclusion of the theorem holds.

It remains to deal with the cases when \( s = 2 \) and \( k = 1, 2, \) and \( 5 \).

But when \( k = 1 \), \( \gamma(K_{13s}) = 40 \) by Theorem 2.2 and, since \( \gamma_{2,1} = 39 \), it follows that \( g_{\omega} = 40 \). Similarly, when \( k = 5 \), \( \gamma(K_{15s}) = 56 \), and \( \gamma_{2,5} = 55 \), so \( g_{\omega} = 56 \).

So finally suppose \( s = k = 2 \). Let \( G \) be an \((m - 1)\)-connected graph which genus embeds in the surface \( S_{g_{\omega}} \).

First suppose \( G \) has \( m + 1 = 24 + 2 + 1 = 27 \) vertices. But then the octahedron \( O(27) \) must be a spanning subgraph of \( G \). So \( g_{\omega} \leq \gamma(O(27)) \leq \gamma(G) = g_{\omega} \), where the first inequality follows from the definition of \( g_{\omega} \) and the final equality holds by the choice of graph \( G \). So equality holds as desired.

Now suppose \( |V(G)| \geq m + 2 \). Then from Theorem 4.1, \( \lceil f_{s,2}(2) \rceil = \gamma_{2,3} \). So \( \gamma_{2,3} = \lceil f_{s,2}(2) \rceil \leq \gamma(G) = g_{\omega} \leq \gamma(O(27)) \leq \gamma_{2,3} \). Hence, \( g_{\omega} = \gamma(O(27)) \) as desired.

For the sake of completeness, consider now the possible extension of the above theorem to the cases when \( s = 0 \) and \( s = 1 \). When \( s = 0 \), \( g_{\omega} \) is only defined when \( k \geq 7 \). When \( k = 7 \), \( g_{\omega} = \gamma(K_7) = 2 \), but the genus of the octahedron \( O(9) \) is not known. When \( k = 11 \), \( g_{\omega} = \gamma(K_{11}) = 6 \), but \( \gamma(K_{12s}) = 5 \), so the conclusion of the theorem does not hold.

So finally suppose \( s = 1 \). In this case, the genus of \( O(12s + k + 1) = O(13 + k) \) is known only for the values \( k = 1, 5, 7, \) and \( 11 \). (See Theorem 2.2.) When \( k = 1 \), \( g_{\omega} = 9 \), but \( \gamma(O(14)) = 8 \), so the conclusion of the theorem does not hold. Similarly, when \( k = 5, 7, \) or \( 11 \), the conclusion of the theorem does not hold.

It remains to check the cases \( k = 2, 3, 4, \) and \( 6 \). When \( k = 2, g_{\omega} = 11 \), but it is only known that \( 10 \leq \gamma(O(15)) \leq 11 \), so we do not know whether the conclusion of the theorem holds or not.

Suppose \( k = 3 \). Then since \( \gamma(K_{18s}) = 11 \) and \( \gamma(K_{16s}) = 13 \), it is not known whether \( g_{\omega} = 12 \) or \( 13 \). Likewise, the genus of \( O(16) \) is not known. However, we still claim that \( g_{\omega} = \gamma(O(16)) \). Suppose that \( G \) is any \( 14 \)-connected graph embeddable in \( S_{11} \). If \( |V(G)| = 16 \), then \( O(16) \) is a spanning

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**CONNECTIVITY OF EMBEDDED GRAPHS**

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subgraph of $G$. But then $g_{A} \lessgtr \gamma(O(16)) \lessgtr g_{A}$ and, hence, equality holds. So suppose $|V(G)| \geq 17$. But then $\gamma_{A} \lessgtr \gamma_{I}(2) \lessgtr g_{A} \lessgtr g_{A}(O(16)) \lessgtr g_{A}$. (Here the first inequality in the string is easily verified and the last inequality holds since $O(16)$ is a subgraph of $K_{16}$.) So once again equality holds. Thus the conclusion of the preceding theorem holds when $s = 1$ and $k = 3$. A similar computation shows that it also holds when $k = 6$.

So finally suppose $k = 4$. Then exactly the same argument as was used for the $k = 4$ case in the preceding theorem shows that $g_{A} = \gamma(O(17))$.

Now, once again, consider the nonorientable case.

**Theorem 5.2.** If $s \geq 1$ and if $k = 1, 3, 5, 7, 9$, or $11$, then $g_{A}(s, k) = \gamma(O(12s + k + 1))$.

**Proof.** Since by Theorem 2.2 value of $\gamma(O(m + 1))$ is known when $s \geq 1$ and when $k = 1, 3, 5, 7, 9$, and $11$, it is straightforward to check that for these values, $\gamma(O(m + 1)) \geq \gamma_{I}(1)$. It is also easy to check that for these same values, $\gamma(O(12s + k + 1)) = \gamma_{I}(1)$. On the other hand, just as in the proof of Theorem 3.2, it may be shown that $[\gamma_{I}(1)] \leq g_{A}$. Finally, since $\gamma(O(m + 1)) \geq \gamma_{I}$, then by the definition of $g_{A}$, $g_{A} \lessgtr \gamma(O(m + 1))$. So assembling these inequalities, $[\gamma_{I}(1)] \leq g_{A} \lessgtr \gamma(O(12s + k + 1)) = [\gamma_{I}(1)]$ and the equality must therefore hold.

Consider now a possible extension of the preceding theorem to the case when $s = 0$. Since $\gamma(K_{1}) = \cdots = \gamma(K_{5}) = 0$ and $\gamma(K_{6}) = \gamma(K_{7}) = 1$, $g_{A}$ is not defined for $k = 1, 3$, and $5$. Suppose $k = 7$. Then $\gamma(K_{2}) = 3$ and $\gamma(K_{3}) = 4$, so $g_{A}$ is not defined for $k = 1, 3$, and $5$. Similarly, when $k = 11$, $\gamma(K_{11}) = 10$, $\gamma(K_{12}) = 12$, and $g_{A} \lessgtr 11$, but $\gamma(O(12)) = 10$, so again equality does not hold. However, when $k = 9$, $\gamma(K_{9}) = 5$, $\gamma(K_{10}) = 7$, and $\gamma(O(10)) = 6$. Thus in this case $\gamma(O(10)) = g_{A}$.

Note that there are four of the $12$ possible values of $k$ which are not covered by Theorem 5.1 and there are six of the $12$ possible values of $k$ not covered by Theorem 5.2.

**Conjecture.** In each of the $10$ unsettled cases above, the breaking point is the orientable (nonorientable) genus of $O(12s + k + 1)$.

In Section 2 we considered those surfaces in which the complete graphs genus embed and considered the problem of deciding when these complete graphs are the only graphs embeddable in these surfaces having connectivity $k_{\text{max}}$ for that surface.

Of course, if we drop the demand that the complete graphs genus embed in a surface, we can look at the more general question as to when for any surface is a complete graph the only graph which embeds there and has connectivity $k_{\text{max}}$. We now conclude by addressing this more general question. We begin with the orientable case.
Recall that $\gamma(K_{2s+k+1}) - \gamma(K_{2s+k}) = 2s + e_k$. Consider the surfaces $S_{r,s+k}$, for $0 \leq r \leq 2s + e_k - 1$. For all these surfaces, $\kappa_{\max}(S_{r,s+k}) = m - 1 = 12s + k - 1$. In the next theorem, it will be shown that for four of the 12 congruence classes modulo 12, one can completely decide when the graphs having connectivity $\kappa_{\max}$ are the only graphs embeddable in the corresponding surface.

**Theorem 5.3.** If the surfaces $S_{r,s+k}$ and the quantities $\delta_k$ are defined as above, and if $s + \delta_k \geq 1$, then

(a) if $0 \leq r \leq s + \delta_k$, $K_m$ is the unique graph with connectivity $m - 1$ which embeds in $S_{r,s+k}$, while

(b) if $s + \delta_k + 1 \leq r \leq 2s + e_k - 1$, and $k = 1, 5, 7$, or 11, then there is more than one graph with connectivity $m - 1$ which embeds in $S_{r,s+k}$.

**Proof.** Suppose $G \neq K_m$ is a second graph which embeds in $S_{r,s+k}$, and $G$ has connectivity $m - 1$. Then $G$ must have at least $m + 1$ vertices and so by inequality (3.1), $\gamma(G) \geq f_{s+k}(1) = \gamma_{s+k} + s + \delta_k + 1 \geq \gamma_{s+k} + r$, a contradiction, and (a) is proved.

To prove part (b), observe that, of course, $K_m$ is one graph which embeds in $S_{r,s+k}$, and, hence, also in $S_{r,s+k}$, for all $r \geq 0$. By Theorem 2.2, $\gamma(O(12s+k+1)) = [((6s+(k-1)/2)(6s+(k-5)/2)/3]$ and for $k = 1, 5, 7$, and 11 and for these four values, the reader can easily show that this quantity, in turn, equals $\gamma_{s+k} + s + \delta_k + 1$. So, for $r = s + \delta_k + 1$ there is a second graph with $\kappa = m - 1$.

If $s + \delta_k < 0$, then part (a) of Theorem 5.3 has no meaning. If $s + \delta_k = 0$ then the value of $r$ in Theorem 5.3 must be 0 and the uniqueness of $K_m$ is treated in Theorem 2.4.

Now continue to let $r$ be as defined in Theorem 5.3 and consider further what happens to part (b) in those cases when $s + \delta_k < 1$. If $s \geq 3$, then Theorem 5.3 applies. Also note that whenever $r = 0$, uniqueness is again covered by Theorem 2.4. Note, for example, that $r$ must be 0 when $s = 0$ and, hence, Theorem 2.4 treats part (b) whenever $s = 0$.

So suppose first that $s = 1$ and that $k = 1, 5, 7$ or 11. When $k = 1$, it follows that either $r = 0$ or 1. But when $r = 1$, we have that $\gamma(K_{2s+1}) = 8$ and $\kappa(K_{2s+1}) = 12 = \kappa_{\max}(S_3)$. So there is a second graph which embeds in $S_3$ and has $\kappa = 12 = \kappa_{\max}(S_3)$.

Suppose next that $k = 5$. Again, either $r = 0$ or 1. But when $r = 1$, arguing as above, note that the octahedron $K_{2s+1}$ has genus 16 and $\kappa = 16 = \kappa_{\max}(S_1)$. So there is a second graph which embeds in $S_1$ with $\kappa = \kappa_{\max}(S_1)$. \(\square\)
When \( k = 7 \), it follows that \( r = 1 \) or 2. But octahedron \( K_{10;2} \) has \( \gamma = 21 \) and \( \kappa = 18 = \kappa_{\text{max}}(S_{21}) \). Thus there is a second graph embeddable in \( S_{21} \) (and hence in \( S_{22} \) as well) having \( \kappa = \kappa_{\text{max}}(S_{22}) \).

If \( k = 11 \), again note that \( r = 1 \) or 2. But \( K_{12;2} \) has genus 33 and \( \kappa = 22 = \kappa_{\text{max}}(S_{33}) \), so there is a second graph for surface \( S_{33} \) (and, hence, also for \( S_{34} \)) with \( \kappa = \kappa_{\text{max}}(S_{34}) \).

Now suppose \( s = 2 \). Then for \( k = 7 \) and 11, Theorem 5.3 applies. Suppose \( k = 1 \). It then follows that \( 1 \leq r \leq 3 \). But octahedron \( K_{13;2} \) has genus 40 and, hence, each of the surfaces \( S_{40} \), \( S_{41} \), and \( S_{42} \) admits a second graph with \( \kappa = \kappa_{\text{max}} \).

Finally, suppose \( k = 5 \). Again, it follows that \( 1 \leq r \leq 3 \) and, applying the usual argument to the octahedron \( K_{15;2} \), it follows that each of \( S_{56} \), \( S_{57} \), and \( S_{58} \) admits a second graph having \( \kappa = \kappa_{\text{max}} \).

Finally, suppose \( k = 5 \). Again, it follows that \( 1 \leq r \leq 3 \) and, applying the usual argument to the octahedron \( K_{15;2} \), it follows that each of \( S_{56} \), \( S_{57} \), and \( S_{58} \) admits a second graph having \( \kappa = \kappa_{\text{max}} \).

For our last result, we present a nonorientable analog of Theorem 5.3.

### Theorem 5.4

If the surfaces \( N_{i;\kappa+r} \) and the values of \( \bar{\kappa} \) are as defined above and if \( 2s + \bar{\delta}_k \geq 1 \), then

(a) If \( 0 \leq r \leq 2s + \bar{\delta}_k \), \( K_m = K_{12s+k} \) is the unique graph with connectivity \( m - 1 \) which embeds in \( N_{i;\kappa+r} \), while

(b) if \( 2s + \bar{\delta}_k + 1 \leq r \leq \gamma_{s,k} + 1 \) and \( k = 1, 3, 5, 7, 9, \) or 11, then there is more than one graph with connectivity \( m - 1 \) which embeds in \( N_{i;\kappa+r} \).

**Proof.** Suppose \( G \) embeds in \( N_{i;\kappa+r} \) and has connectivity \( m - 1 \), but \( G \neq K_m \). Then \( G \) must have at least \( m + 1 \) vertices. But then by inequality (3.2),

\[
\gamma(G) \geq f_{s,k}(1) = \gamma_{s,k} + 2s + \bar{\delta}_k + 1 > \gamma_{s,k} + r, \quad \text{a contradiction.}
\]

This proves part (a).

By Theorem 2.2, \( \gamma(O(12s+k+1)) = 2 \left\lfloor \frac{(6s+(k-1)/2)(6s+(k-5)/2)}{3} \right\rfloor \) for \( k = 1, 3, 5, 7, \) and 11 and for these values of \( k \), it is again easy to verify that this quantity equals \( \gamma_{s,k} + 2s + \bar{\delta}_k + 1 \). So for \( r = 2s + \bar{\delta}_k + 1 \), there is a second graph with \( \kappa = \kappa_{\text{max}} = m - 1 = 12s + k - 1 \). This proves part (b) and hence the theorem.

Note that if \( 2s + \bar{\delta}_k < 0 \), part (a) of the preceding theorem has no meaning. If \( 2s + \bar{\delta}_k = 0 \), then the uniqueness of \( K_m \) is treated in Theorem 2.4.

We conclude by investigating how part (b) extends to the cases when \( 2s + \bar{\delta}_k \leq 0 \) and \( k = 1, 3, 5, 7, 9, \) and 11. Let \( r \) be defined as in Theorem 5.4.
Suppose first that \( s = 0 \). Then when \( k = 1, 3, \) or \( 5 \), part (b) has no meaning. Suppose \( k = 7 \). It should be pointed out here that although \( \gamma_{s, k+1} \) \( - \gamma_{s, k} = 4s + \delta_k \) when \( 2s + \delta_k \geq 1 \), this is not so when \( s = 0 \) and \( k = 7 \) for \( \gamma(K_4) - \gamma(K_s) = 4 - 3 = 1 \). In the case \( k = 7 \), however, the \( r \) interval in part (b) of Theorem 5.4 translates to \( 0 \leq r \leq 0 \), that is, \( r = 0 \). But recall that there are infinitely many graphs which, like \( K_5 \), embed in the nonorientable surface \( N_3 \) and have \( \kappa = 6 = \kappa_{\text{max}}(N_3) \), the maximum connectivity for \( N_3 \).

Suppose \( k = 9 \). Then part (b) of Theorem 5.4 translates to \( r = 1 \). But the octahedron \( K_{6,2} \) has \( \tilde{\gamma} = 6 = \tilde{\gamma}(K_4) \) and hence is a second graph embeddable in \( N_8 \) having \( \kappa = 8 = \kappa_{\text{max}}(N_8) \).

Similarly, if \( k = 11 \) then \( r = 0 \) or 1. But then, as before, if \( r = 0 \), Theorem 2.4 treats the uniqueness of \( K_{11} \), while if \( r = 1 \), the octahedron \( K_{6,2} \) provides a second graph with \( \tilde{\gamma} = 10 \) and \( \kappa = 10 = \kappa_{\text{max}}(N_{10}) \).

Now suppose \( s = 1 \). Since only the values \( k = 1, 3, 5, 9, \) and 11 are being considered, then by Theorem 5.4, one need only consider the cases when \( k = 1 \) and \( k = 5 \).

Suppose first that \( k = 1 \). Then \( r = 1, 2, \) or \( 3 \). But octahedron \( K_{7,2} \) has \( \tilde{\gamma} = 16 \) and thus there are at least two different graphs (the other being \( K_{13} \)) which embed in \( N_{16} \) (and therefore in \( N_{17} \) and \( N_{18} \) as well) and which have \( \kappa = 12 = \kappa_{\text{max}} \).

Finally, suppose \( k = 5 \). Again, then, \( 1 \leq r \leq 3 \). In this case, octahedron \( K_{9,2} \) has \( \tilde{\gamma} = 32 \) and \( \kappa = 16 \). Hence for each of the surfaces \( N_{32}, N_{33}, \) and \( N_{34} \) there is more than one graph embeddable therein which has \( \kappa = 16 \), the maximum connectivity for each of these surfaces.

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