On sharp bounds of the zero-order Randić index of certain unicyclic graphs

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A B S T R A C T
Let G be a simple connected graph and t be a given real number. The zero-order general Randić index αt(G) of G is defined as \( \sum_{v \in V(G)} d(v)^{t} \), where \( d(v) \) denotes the degree of \( v \).

In this paper, for any \( t \), we characterize the graphs with the greatest and the smallest \( \alpha_t \) within two subclasses of connected unicyclic graphs on \( n \) vertices, namely, unicyclic graphs with \( k \) pendant vertices and unicyclic graphs with a \( k \)-cycle.

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1. Introduction

Given a simple connected graph \( G = (V(G),E(G)) \), the Randić index of \( G \) is defined as [5]:

\[
R(G) = \sum_{u,v \in E(G)} (d(u)d(v))^{-\frac{1}{t}},
\]

where \( d(u) \) denotes the degree of the vertex \( u \) of \( G \). In [5] Randić demonstrated the close relation of this index with a variety of physical–chemical properties of various classes of organic compounds. The Randić index has received considerable attention and has been generalized in many ways. In this paper, we are concerned with finding sharp bounds of the generalized zero-order Randić index \( \alpha_t(G) = \sum_{u \in V(G)} d(u)^t \) of certain unicyclic graphs.

Several special cases of \( \alpha_t \) have been studied for different graphs. In [3], Li and Zhao characterized all trees with the first three smallest and largest \( \alpha_t \) values, where \( t \) is restricted to \( \pm m \) and \( \pm \frac{1}{m} \) for some positive number \( m \geq 2 \). Wang and Deng generalized the work of [3] to unicyclic graphs. In particular, they characterized unicyclic graphs with the largest \( \alpha_{\pm m} \) and \( \alpha_{\frac{1}{m}} \) [6]. More recently, Hua and Deng [1] obtained sharp lower and upper bounds of general \( \alpha_t \) for unicyclic graphs which includes the results of [6] as a special case.

In this paper, we investigate extreme values of general \( \alpha_t \) for two subclasses of simple connected unicyclic graphs on \( n \) vertices, namely, unicyclic graphs with \( k \) pendant vertices, and unicyclic graphs with a \( k \)-cycle. We give sharp lower bounds and upper bounds for these two classes of graphs. Our result shows that there is certain duality of the extremal graphs of these classes. The bounds given in [1] can be easily deduced from our result. Since \( \alpha_0(G) = |V(G)| \) and \( \alpha_1(G) = 2|E(G)| \), we only consider \( t \neq 0, 1 \).

Before proceeding, we introduce some notations. A pendant vertex of a graph is a vertex with degree 1. Let \( \mathcal{U}_n \) denote the set of all simple connected unicyclic graphs of order \( n \) and \( C(G) \) denote the unique cycle of any graph \( G \in \mathcal{U}_n \). Let \( \mathcal{U}_{n,k} \subset \mathcal{U}_n \)
denote the set of all unicyclic graphs of order \( n \) with \( k \) pendant vertices and \( \mathcal{U}_n^k \subset \mathcal{U}_n \) denote the set of all unicyclic graphs of order \( n \) with a \( k \)-cycle. Let \( n_i \) denote the number of vertices of degree \( i \) and \( \Delta(G) \) denote the maximum degree of \( G \). \( C_n \) is a cycle of length \( n \), \( P_n \) is a path with \( n \) vertices, and \( S_n \) is a star on \( n \) vertices.

The following inequalities are needed for our proofs. Since they all can be easily proved, thus the proofs are omitted.

**Lemma 1.1.** Let \( 0 < x \leq \frac{p}{2} \). Then \( f(x) = x^\prime + (p - x)^\prime \) is strictly decreasing if \( t < 0 \) or \( t > 1 \), and strictly increasing if \( 0 < t < 1 \).

**Corollary 1.2.** Let \( a, b, c, d \) be four positive numbers with \( a + b = c + d \) and \( |a - b| < |c - d| \). Then \( \alpha^\prime \) is a strictly decreasing function of \( t \) for \( t < 0 \) or \( t > 1 \), and \( \alpha^\prime \) is a strictly increasing function of \( t \) if \( 0 < t < 1 \).

**Lemma 1.3.** If \( x - 2 \geq y \geq 1 \) then
\[
\begin{align*}
(x - 1)^t + (y + 1)^t < x^t + y^t & \quad \text{if } t < 0 \text{ or } t > 1, \\
(x - 1)^t + (y + 1)^t > x^t + y^t & \quad \text{if } 0 < t < 1.
\end{align*}
\]

**Corollary 1.4.** If \( x \geq y \geq 2 \) then
\[
\begin{align*}
(x + 1)^t + (y - 1)^t > x^t + y^t & \quad \text{if } t < 0 \text{ or } t > 1, \\
(x + 1)^t + (y - 1)^t < x^t + y^t & \quad \text{if } 0 < t < 1.
\end{align*}
\]

2. Determination of extreme zero-order Randić index for unicyclic graphs with \( k \) pendant vertices

If \( k = 0 \), then \( \mathcal{U}_{n,0} = \{C_n\} \). Now we assume \( 1 \leq k \leq n - 3 \). The next two lemmas characterize non-extremal graphs within the class \( \mathcal{U}_{n,k} \).

**Lemma 2.1.** Let \( G \in \mathcal{U}_{n,k} \). If there are two vertices \( u \) and \( v \) such that \( d(u) - 2 \geq d(v) \geq 2 \), then there exists another graph \( G' \in \mathcal{U}_{n,k} \) satisfying
\[
\alpha_t(G) > \alpha_t(G') \quad \text{for } t < 0 \text{ or } t > 1, \\
\alpha_t(G) < \alpha_t(G') \quad \text{for } 0 < t < 1.
\]

**Proof.** Let \( \Delta \) be the cycle in \( G \). Since \( d(u) \geq 4 \), there are at least two vertices \( u_1, u_2 \) in \( N(u) \setminus \Delta(G) \). Since \( G \) has a unique cycle, at least one of \( u_1, u_2 \) is not in \( N(v) \cup \{v\} \). Let \( w \in \{u_1, u_2\} \) and \( w \notin N(v) \cup \{v\} \).

Let \( G' = G - uw + vw \). Note that the above operation only changes the degree of \( u \) from \( d(u) \) to \( d(u) - 1 \geq 3 \) and the degree of \( v \) from \( d(v) \geq 2 \) to \( d(v) + 1 \geq 3 \). Therefore, \( G' \) also has \( k \) pendant vertices. By Corollary 1.3 we have
\[
\alpha_t(G') - \alpha_t(G) = \left\{ \begin{array}{ll}
\alpha_t(G') - \alpha_t(G) < 0 & \text{for } t < 0 \text{ or } t > 1, \\
\alpha_t(G') - \alpha_t(G) > 0 & \text{for } 0 < t < 1.
\end{array} \right.
\]

**Lemma 2.2.** Let \( G \in \mathcal{U}_{n,k} \). If there are two vertices \( u \) and \( v \) such that \( d(v) \geq d(u) \geq 3 \), then there exists a graph \( G' \in \mathcal{U}_{n,k} \) satisfying
\[
\alpha_t(G) < \alpha_t(G') \quad \text{for } t < 0 \text{ or } t > 1, \\
\alpha_t(G) > \alpha_t(G') \quad \text{for } 0 < t < 1.
\]

**Proof.** Let the shortest path from \( u \) to \( v \) be \( P : u = v_0 v_1 \ldots v_l = v \). Since \( d(u) \geq 3 \), there exists \( w \in N(u) \setminus (P \cup \{v\}) \).

Let \( G' = G - uw + vw \). It is easy to see that \( G' \in \mathcal{U}_{n,k} \). Then from Corollary 1.4 we have
\[
\alpha_t(G') - \alpha_t(G) = \left\{ \begin{array}{ll}
\alpha_t(G') - \alpha_t(G) > 0 & \text{for } t < 0 \text{ or } t > 1, \\
\alpha_t(G') - \alpha_t(G) < 0 & \text{for } 0 < t < 1.
\end{array} \right.
\]

Let \( \mu_{n,k} = \left\lfloor \frac{k-1}{n-k} \right\rfloor \) and \( \lambda_{n,k} = (n-k)\mu_{n,k} \). It is easy to see that \( k - \lambda_{n,k} \geq 1 \) and \( n - 2k + \lambda_{n,k} \geq 0 \).

Consider the following subclass of \( \mathcal{U}_{n,k} \):
\[
\mathcal{A}_{n,k} := \{ G \in \mathcal{U}_{n,k} \mid n_2 + \mu_{n,k} = n - 2k + \lambda_{n,k}, n_3 + \mu_{n,k} = k - \lambda_{n,k} \}.
\]
For any graph $G \in \mathcal{A}_{n,k}$, letting $V'$ denote the set of vertices with degree equal to 1, or $2 + \mu_{n,k}$, or $3 + \mu_{n,k}$, we have

$$\sum_{u \in V'} d(u) = k + (n - 2k + \lambda_{n,k})(2 + \mu_{n,k}) + (k - \lambda_{n,k})(3 + \mu_{n,k})$$

$$= 2n$$

$$= 2|E(G)|$$

$$= \sum_{u \in V(G)} d(u).$$

So $V' = V(G)$, i.e., the degree of each vertex in $G$ is either 1, or $\Delta(G) = 3 + \mu_{n,k}$, or $\Delta(G) - 1 = 2 + \mu_{n,k}$. Actually we have the following lemma.

**Lemma 2.3.** If $G \in \mathcal{U}_{n,k}$ and the degree of each vertex in $G$ is either 1, or $\Delta(G)$, or $\Delta(G) - 1$, then $G \in \mathcal{A}_{n,k}$.

**Proof.** Let $x = n_{\Delta(G)}$, then $1 \leq x \leq n - k$, and $n_{\Delta(G) - 1} = n - k - x$. By considering the total degrees we have

$$k + x \Delta(G) + (n - k - x)(\Delta(G) - 1) = 2n$$

$$\Rightarrow x = 3n - 2k - (n - k)\Delta(G)$$

$$\Rightarrow 1 \leq 3n - 2k - (n - k)\Delta(G) \leq n - k$$

$$\Rightarrow 2 + \frac{k}{n - k} \leq \Delta(G) \leq 3 + \frac{k - 1}{n - k}$$

$$\Rightarrow 2 + \left\lfloor k \frac{n}{n - k} \right\rfloor = \Delta(G) \leq 3 + \left\lfloor k - 1 \frac{n}{n - k} \right\rfloor = 3 + \mu_{n,k}.$$  

Since $2 + \left\lfloor k \frac{n}{n - k} \right\rfloor = 3 + \left\lfloor k - 1 \frac{n}{n - k} \right\rfloor$, then $\Delta(G) = 3 + \left\lfloor k - 1 \frac{n}{n - k} \right\rfloor = 3 + \mu_{n,k}$. Hence

$$n_{3 + \mu_{n,k}} = x = 3n - 2k - (n - k)\Delta(G) = k - \lambda_{n,k}$$

$$n_{2 + \mu_{n,k}} = n - k - x = n - 2k + \lambda_{n,k}.$$  

Thus $G \in \mathcal{A}_{n,k}$.

In general, there are many different graphs in $\mathcal{A}_{n,k}$. To determine the exact number of different structures in $\mathcal{A}_{n,k}$ and generate all of them are interesting open problems. We can use the following procedure to generate a subclass of $\mathcal{A}_{n,k}$.

Let $C_p$ be a cycle of length $p \leq n - k$ and $L_1, L_2, \ldots, L_k$ be $k$ paths with $\sum_{i=1}^{k} |L_i| = n - p + k$ such that $|L_j| = |L_{j+1}| = \cdots = |L_{k}| = 2$ for some $j \leq k + 1$ and $|L_i| \geq 3$ for any $1 \leq i < j$ where $|L_i|$ is the number of vertices of $L_i$. Let $u_i$ be a pendant vertex of $L_i$. Let $u_1$ be a vertex from $G_{n-1}$ with $d(u_1) \geq 2$ as small as possible and identify $u_1$ with $u_i$. Let $G_i$ be the resulting graph. Repeat this process until $i = k$. Let $G = G_k$.

Now we give the outline of an induction proof on $i$ to show $G_i \in \mathcal{A}_{n,i}$ where $n_i = |V(G_i)|$. Clearly, $G_1$ is a unicyclic graph with one degree-3 vertex, one pendant vertex, and $n_1 - 2$ degree-2 vertices. By Lemma 2.3 we have $G_1 \in \mathcal{A}_{n_1,1}$. Now assume $G_{i-1} \in \mathcal{A}_{n_{i-1},j-1}$. Identifying $v_i$ with $u_i$ will increase the degree of $v_j$ by 1 while keeping the degrees of all other vertices unchanged.

If $|L_{i-1}| \geq 3$, then $G_{i-1}$ has at least one degree-2 vertex, so $\Delta(G_{i-1}) = 3$ and $d_{G_{i-1}}(v_i) = 2$. Hence the vertices of $G_i \in U_{n,i}$ have only three possible degrees, 1, 2, or 3. Then by Lemma 2.3, $G_i \in \mathcal{A}_{n,i}$.

If $|L_{i-1}| = 2$, then $|L_i| = 2$. If $d_{G_{i-1}}(v_i) = \Delta(G_{i-1})$, then every vertex of $G_{i-1}$ is either a pendant vertex or a maximum degree vertex. Hence the vertices of $G_i$ have only three possible degrees: 1, $\Delta(G_i) = \Delta(G_{i-1}) + 1$, or $\Delta(G_i) - 1 = \Delta(G_{i-1}) - 1$. In both cases we have $G_i \in \mathcal{A}_{n,i}$ by Lemma 2.3. Therefore by induction, $G = G_k \in \mathcal{A}_{n,k}$.

For any graph $G \in \mathcal{A}_{n,k}$, we have

$$\alpha_i(G) = k + (n - 2k + \lambda_{n,k})(2 + \mu_{n,k})^i + (k - \lambda_{n,k})(3 + \mu_{n,k})^i.$$  

Another important subclass of $\mathcal{U}_{n,k}$ is:

$$B_{n,k} = \{G \in \mathcal{U}_{n,k}|n_2 = n - k - 1, n_{k+2} = 1\}.$$  

For any graph $G \in B_{n,k}$, letting $V''$ denote the set of vertices with degree equal to 1, or 2, or $k + 2$, we have

$$\sum_{u \in V''} d(u) = k + 2(n - k - 1) + k + 2$$

$$= 2n$$

$$= 2|E(G)|$$

$$= \sum_{u \in V(G)} d(u).$$
Let $G$. Now because any unicyclic graph of $B_{n,k}$ has only one vertex with degree greater than 2, it is easy to see that $B_{n,k}$ consists of all unicyclic graphs on $n$ vertices with $k$ pendant paths attached to the same vertex of the cycle where a pendant path is a path such that the degree of each non-pendant vertex is 2.

The graphs in $B_{n,k}$ can be constructed as follows:

Let $C_p$ be a cycle of length $p \leq n - k$ and $L_1, L_2, \ldots, L_k$ be $k$ paths with $|L_i| \geq 2$ and $\sum_{i=1}^k |L_i| = n - p + k$. Let $u$ be any vertex of $C_p$ and $u_i$ be a pendant vertex of $L_i$. Identify $u_1, u_2, \ldots, u_k$ with $u$. We denote the resulting graph by $G$.

Clearly $G$ is obtained by attaching $k$ pendant paths to the vertex $u \in C_p$, by the previous discussion we have $G \in B_{n,k}$. On the other hand, for any $G \in B_{n,k}$, we let $C_p = C(G), u \in C(G)$ such that $d(u) = k + 2$ and $L_1, L_2, \ldots, L_k$ be the $k$ pendant paths at $u$. Then it is easy to see that applying the above procedure will exactly generate $G$.

For any $G \in B_{n,k}$, we have

$$\alpha_t(G) = k + (n - k - 1)2^t + (k + 2)^t.$$  

Now we can give our main results for $G \in \mathcal{U}_{n,k}$.

**Theorem 2.4.** Let $G \in \mathcal{U}_{n,k}$. Then

$$\alpha_t(G) \begin{cases} \geq k + (n - 2k + \lambda_{n,k})(2 + \mu_{n,k})^t + (k - \lambda_{n,k})(3 + \mu_{n,k})^t & \text{for } 0 < t \leq 0, \\ \leq k + (n - 2k + \lambda_{n,k})(2 + \mu_{n,k})^t + (k - \lambda_{n,k})(3 + \mu_{n,k})^t & 0 < t < 1. \end{cases}$$

The inequalities hold if and only if $G \in \mathcal{A}_{n,k}$.

**Proof.** For any $t$ such that $t < 0$ or $t > 1$, let $H \in \mathcal{U}_{n,k}$ have the minimum $\alpha_t(\cdot)$ among $\mathcal{U}_{n,k}$. Lemma 2.1 tells us that the difference between the degrees of any two non-pendant vertices of $H$ is at most 1. In other words, $H$ has two or three different degrees: 1, $\Delta(H)$, and $\Delta(H) - 1$. Then from Lemma 2.3 we know that $H \in \mathcal{A}_{n,k}$. The first inequality is proved.

The proof of second inequality is similar and thus omitted. ■

**Theorem 2.5.** Let $G \in \mathcal{U}_{n,k}$. Then

$$\alpha_t(G) \begin{cases} \geq k + (n - k - 1)2^t + (k + 2)^t & \text{for } 0 < t \leq 0, \\ \leq k + (n - k - 1)2^t + (k + 2)^t & 0 < t < 1. \end{cases}$$

The inequalities hold if and only if $G \in \mathcal{B}_{n,k}$.

**Proof.** For any $t$ such that $t < 0$ or $t > 1$, let $H \in \mathcal{U}_{n,k}$ have the maximum $\alpha_t(\cdot)$ among $\mathcal{U}_{n,k}$. Lemma 2.2 tells us that there is at most one vertex in $H$ with degree $\leq 3$. Since $k \geq 1$ and $H \in \mathcal{U}_{n,k}$, then there is exactly one vertex with degree $\geq 3$. Hence $n_2 = n - k - 1$ and $n_{i+1} = 1$, i.e. $H \in \mathcal{B}_{n,k}$. The first inequality is proved.

The proof of second inequality is similar and thus omitted. ■

3. Determination of extreme zero-order Randić index for unicyclic graphs with a $k$-cycle

If $k = 1$, then $\mathcal{U}^n_n = \{C_n\}$. Now we assume $3 \leq k \leq n - 1$. The following lemmas characterize the non-extremal graphs within the class $\mathcal{U}^k_n$.

**Lemma 3.1.** Let $G \in \mathcal{U}^k_n$. If $G$ has at least two pendant vertices, then there exists $G' \in \mathcal{U}^k_n$ satisfying

$$\begin{align*}
\alpha_t(G) > \alpha_t(G') & \text{ for } t < 0 \text{ or } t > 1, \\
\alpha_t(G) < \alpha_t(G') & \text{ for } 0 < t < 1.
\end{align*}$$

**Proof.** Let $u$ and $v$ be two pendant vertices. Let the shortest path from $u$ to $v$ be $P: u = v_0v_1\ldots v_l = v$. Then there exists $i$ such that $0 < i < l$, $d(v_j) \geq 3$ and $d(v_{j-1}) < 3$ for $j = 0, \ldots, i - 1$. Clearly $v_{i-1} \notin N(v)$.

Let $G' = G - v_{i-1}u_i + v_iu_{i-1}$. Since $u_i \notin C(G)$ and $d(v_{i-1}) \leq 2$, then $v_{i-1} \notin C(G)$, hence the cycle is unchanged. So $G' \in \mathcal{U}^k_n$. From Corollary 1.3 we have

$$\alpha_t(G') - \alpha_t(G) = (d(v_i) - 1)^t + (1 + 1)^t - d(v_i)^t - 1^t \begin{cases} < 0 & \text{for } 0 < t \leq 0, \\ > 0 & 0 < t < 1. \end{cases}$$

**Lemma 3.2.** Let $G \in \mathcal{U}^k_n$. If there are two vertices $u$ and $v$ such that $u \in C(G), v \in C(G)$, and $d(u) \geq d(v) \geq 3$, then there exists another graph $G' \in \mathcal{U}^k_n$ satisfying:

$$\begin{align*}
\alpha_t(G) & < \alpha_t(G') & \text{ for } t < 0 \text{ or } t > 1, \\
\alpha_t(G) & > \alpha_t(G') & 0 < t < 1.
\end{align*}$$

**Proof.** Since $v \in C(G)$ and $d(v) \geq 3$, there exists $w \in N(v) \setminus C(G)$. Because $u \in C(G)$, clearly $w \notin N(u)$. Let $G' = G - vw + wu$. Since $w \notin C(G)$, the cycle is unchanged. So $G' \in \mathcal{U}^k_n$. From Corollary 1.4 we have
\[ \alpha_t(G') - \alpha_t(G) = (d(u) + 1)^t + (d(v) - 1)^t - d(u)^t - d(v)^t \begin{cases} > 0 & \text{for } t < 0 \text{ or } t > 1, \\ < 0 & \text{for } 0 < t < 1. \end{cases} \]

**Lemma 3.3.** Let \( G \in \mathcal{U}_n \). If there is a vertex \( u \) such that \( u \not\in C(G) \) and \( d(u) = r + 1 \geq 2 \), then there exists another graph \( G' \in \mathcal{U}_n \) satisfying

\[
\begin{cases}
\alpha_t(G) < \alpha_t(G') & \text{for } t < 0 \text{ or } t > 1, \\
\alpha_t(G) > \alpha_t(G') & \text{for } 0 < t < 1.
\end{cases}
\]

**Proof.** Let the shortest path from \( u \) to \( C(G) \) be \( P : u = v_0v_1 \ldots v_t = v \in C(G) \). Then \( N(u) \cap P = \{v_1\} \). Since \( d(u) \geq 2 \) and \( u \not\in C(G) \), then \( N(u) \setminus P \neq \emptyset \) and \( (N(u) \setminus P) \cap N(v) = \emptyset \). Let \( N(u) \setminus P = \{w_1, \ldots, w_r\} \) and \( G' = G - \cup_{i=1}^r uw_i + \cup_{i=1}^r vw_i \). Since \( u \not\in C(G) \), the cycle is unchanged, so \( G' \in \mathcal{U}_n \). By Corollary 1.2 we have

\[
\alpha_t(G') - \alpha_t(G) = (d(v) + r)^t + 1^t - d(v)^t - (r + 1)^t \begin{cases} > 0 & \text{for } t < 0 \text{ or } t > 1, \\ < 0 & \text{for } 0 < t < 1. \end{cases}
\]

Let \( C_{n,k} \) denote the graph resulting from identifying one vertex of \( C_k \) with one pendant vertex of \( P_{n-k+1} \). Let \( D_{n,k} \) denote the graph resulting from identifying one vertex of \( C_k \) with the center of \( S_{n-k+1} \).

Clearly \( C_{n,k} \in \mathcal{U}_n \), \( D_{n,k} \in \mathcal{U}_n \), \( \alpha_t(C_{n,k}) = 1 + 3^t + (n - 2)2^t \), and \( \alpha_t(D_{n,k}) = n - k + (k - 1)2^t + (n - k + 2)2^t \).

Now we can give our main results for \( G \in \mathcal{U}_n \).

**Theorem 3.4.** Let \( G \in \mathcal{U}_n \) then

\[
\alpha_t(G) \begin{cases} \geq 1 + 3^t + (n - 2)2^t & \text{for } t < 0 \text{ or } t > 1, \\
\leq 1 + 3^t + (n - 2)2^t & \text{for } 0 < t < 1. \end{cases}
\]

The equalities hold if and only if \( G \cong C_{n,k} \).

**Proof.** For any \( t \) such that \( t < 0 \) or \( t > 1 \), let \( H \in \mathcal{U}_n \) have the minimum \( \alpha_t(\cdot) \) among \( \mathcal{U}_n \). Lemma 3.1 tells us that \( H \) has at most one pendant vertex. Since \( k \leq n - 1 \), \( H \) has exactly one pendant vertex. Then it is easy to see that \( H \cong C_{n,k} \). The first inequality is proved.

The second inequality can be similarly proved.

**Theorem 3.5.** Let \( G \in \mathcal{U}_n \). Then

\[
\alpha_t(G) \begin{cases} \leq n - k + (k - 1)2^t + (n - k + 2)2^t & \text{for } t < 0 \text{ or } t > 1, \\
\geq n - k + (k - 1)2^t + (n - k + 2)2^t & \text{for } 0 < t < 1. \end{cases}
\]

The equalities hold if and only if \( G \cong D_{n,k} \).

**Proof.** For any \( t \) such that \( t < 0 \) or \( t > 1 \), let \( H \in \mathcal{U}_n \) have the maximum \( \alpha_t(\cdot) \) among \( \mathcal{U}_n \). By Lemma 3.2, \( C(H) \) has at most one vertex with degree \( \geq 3 \). Since \( k \leq n - 1 \), there is at least one vertex \( \not\in C(H) \). Hence \( C(H) \) has exactly one vertex, say \( u \), with degree \( \geq 3 \).

If \( v \not\in C(H) \), then by Lemma 3.3, \( v \) is a pendant vertex and therefore \( v \) must be adjacent to \( u \). Hence it is easy to see that \( H \cong D_{n,k} \). The first inequality is proved.

The second inequality can be similarly proved.

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**References**