A Two-mass Cantilever Beam Model for Vibration Energy Harvesting Applications

Qing Ou, XiaoQi Chen, Stefanie Gutschmidt, Alan Wood and Nigel Leigh

Abstract—While vibration energy harvesting has become a viable means to power wireless sensors, narrow bandwidth is still a hurdle to the practical use of the technology. For conventional piezoelectric or electromagnetic harvesters, having multiple proof masses mounted on a beam is one way to widen the effective bandwidth. This is because the addition of proof masses increases the number of resonant modes within the same frequency range. Based on the assumptions of the Euler-Bernoulli beam theory, this paper presents a continuum-based model for a two-mass cantilever beam. First, the equation of motion is derived from Hamilton’s principle. Next, the modal analysis is presented and a steady state solution for harmonic base excitation is derived. The two-mass beam is considered as two serially connected beam segments. In the derivation, emphasis is given to the transition conditions, which would otherwise not appear in the traditional single mass beam model. Experimental validation on a stainless steel beam confirms that the model can accurately predict both natural frequencies and the frequency response of an arbitrary point along the beam. The derivation procedure presented in this paper is applicable to a beam with any number of proof masses. Lastly, it is demonstrated how the model can be applied to a piezoelectric energy harvester.

I. INTRODUCTION

Throughout the world, most wireless sensors are powered by a finite battery source. Dependence on batteries not only requires frequent maintenance, but also has adverse environmental effects associated with battery disposal. For these reasons, large scale deployment of wireless sensors in the industry remains problematic. With recent advances in semiconductor technology, power consumption of wireless sensors has been greatly reduced [1], [2]. Consequently, energy harvesting has become a viable technology for self-powered wireless sensors, attracting the attention of researchers worldwide [3], [4]. The three most commonly employed schemes for vibration energy harvesting are piezoelectric, electromagnetic and electrostatic transductions. However, regardless of the chosen transduction mechanism, a fundamental issue that remains unresolved is the narrow bandwidth of energy harvesters. This is due to the fact that most vibrations existing in the industry are of relatively low displacement amplitude and strong force. For these types of vibration, energy harvesters need to rely on resonance in order to extract energy efficiently. The problem with resonant energy harvesters is that a small deviation from the resonant frequency will result in a significant power reduction. It is this limitation that renders vibration energy harvesters impractical. One solution to widen the effective bandwidth is having multiple proof masses mounted on a beam, which has been proposed and attempted by several researchers [5]. It has been demonstrated by Soobum et al [6] that for a tapered two-mass piezoelectric beam, the usable bandwidth of the harvester is significantly improved due to the two resonant peaks in the frequency range between 20 to 120 Hz.

Mathematical models and simulations play an important role in the design of energy harvesters. While models aim to predict both resonant frequencies and frequency response of a harvester, the accuracy of the former is normally more demanding for design purposes. In the literature, many mathematical models have been presented for piezoelectric and electromagnetic harvesters. Based on different assumptions, the accuracy and complexity of these models can vary significantly. The most widely used models are the lumped-mass model, the continuum-based model and the finite element approach. The lumped-mass model is quite often used as a convenient way of assessing the system dynamics. In the simplest way, a mass-spring-damper system is used as an approximation to the mass beam structure. The drawbacks of lumped-mass model are that it can only predict the fundamental frequency, and it has relatively low fidelity [7]. Furthermore, for beams with increasing number of masses, lumped-mass models become insufficient. Continuum-based models are subsequently adopted for higher accuracy. Sodano et al [8] developed an analytical model for a bimorph bender with no proof mass. The model has proven to be highly accurate by experimental validation. However, in energy harvesting application, having at least one proof mass is advantageous, as it lowers the resonant frequency and increases the strain of the bender. A continuum-based model for a bimorph with a mass at the free end was developed by Erturk and Inman [9]. The model is validated with experiments and show excellent accuracy. The high fidelity is also the result of incorporating the electromechanical coupling effect of piezoelectric material. Finite Element Method (FEM) is used to model plate-type piezoelectric energy harvesters [10], [11]. Being flexible and sufficiently accurate, FEM is not necessarily bound by assumptions of the Euler-Bernoulli beam theory, and it can be quickly adapted for complex geometries. However, solving dynamic problems by FEM is a computationally intensive process. More importantly,
In the absence of piezoelectric effects and external forces, the equation of motion for the homogeneous problem can be derived from Hamilton’s principle

$$\delta \int_{t_1}^{t_2} (T - U) \, dt = 0, \quad (1)$$

where $T$ is the total kinetic energy that includes both translational and rotational energy of each mass, and $U$ is the internal strain energy. In the succeeding equations, the notations “prime” and “dot” represent partial derivatives with respect to $x$ and $t$, respectively. The expression for the kinetic energy is

$$T = \frac{1}{2} \int_0^{L_1} m_1 \dddot{w}_1(x_1,t) \, dx_1 + \frac{1}{2} M_1 \dddot{w}_1^2(L_1,t)$$

$$+ \frac{1}{2} I_{yy1} \dddot{w}_1^2(L_1,t) + \frac{1}{2} \int_0^{L_2} m_2 \dddot{w}_2(x_2,t) \, dx_2$$

$$+ \frac{1}{2} M_2 \dddot{w}_2^2(L_2,t) + \frac{1}{2} I_{yy2} \dddot{w}_2^2(L_2,t), \quad (2)$$

and the internal strain energy of the beam $U$ is defined as

$$U = \frac{1}{2} \int_0^{L_1} Y_{xy1} \dddot{w}_1^2(x_1,t) \, dx_1 + \frac{1}{2} \int_0^{L_2} Y_{xy2} \dddot{w}_2^2(x_2,t) \, dx_2. \quad (3)$$

Two boundary conditions at $x_1 = 0$ and two transition conditions at $x_1 = L_1$ are imposed based on the particular mechanical configuration. In the case of a cantilever beam, the two boundary conditions are

$$w_1(0,t) = 0, \quad w_1'(0,t) = 0. \quad (4)$$

At the transition where the two segments are joined, the displacements and deflection angles are equal for all time. This leads to the following transition conditions

$$w_1(L_1,t) = w_2(0,t), \quad w_1'(L_1,t) = w_2'(0,t). \quad (5)$$

By substituting these four conditions (4) and (5) into (1), Hamilton’s principle yields the equations of motion

$$m_n \dddot{w}_n(x_n,t) + Y_n I_n \dddot{w}_n''''(x_n,t) = 0 \quad \forall n = 1, 2, \quad (6)$$

the transition conditions

$$Y_1 I_{yy1} \dddot{w}_1''''(L_1,t) - Y_2 I_{yy2} \dddot{w}_2''''(0,t) - M_1 \dddot{w}_1(L_1,t) = 0, \quad (7)$$

$$Y_1 I_{yy1} \dddot{w}_1''''(L_1,t) - Y_2 I_{yy2} \dddot{w}_2''''(0,t) + I_{yy1} \dddot{w}_1(L_1,t) = 0, \quad (8)$$

and the boundary conditions

$$Y_2 I_{yy2} \dddot{w}_2''''(L_2,t) - M_2 \dddot{w}_2(L_2,t) = 0, \quad (9)$$

$$Y_2 I_{yy2} \dddot{w}_2''''(L_2,t) + I_{yy2} \dddot{w}_2'(L_2,t) = 0. \quad (10)$$
C. Modal Analysis

To recapitulate, a total of ten partial differential equations (PDE) are obtained so far. They consist of two equations of motion (6), four boundary conditions (4), (9) and (10), and four transition conditions (5), (7) and (8). The aim is to find the solutions for the two equations of motion that satisfy all the eight boundary and transition conditions. This can be done analytically by separating the spatial and temporal domains and hence express the ten equations in ordinary differential equations (ODE). In modal analysis most attention is given to the spatial domain boundary and transition conditions, because they contain information of natural frequencies and coefficients of the mode shape function. Note that the modal analysis is based on the undamped and homogeneous problem of (6). The relative displacement of each beam segment can be represented by a convergent series of the eigenfunctions as

\[ w_n(x_n,t) = \sum_{r=1}^{\infty} \phi_{nr}(x_n) \eta_{nr}(t) \quad \forall n = 1, 2, \] \hspace{1cm} (11)

where \( \phi_{nr}(x) \) is the mass normalized mode shape function and \( \eta_{nr}(t) \) is the time dependent amplitude of the \( r \)th mode of vibration. For the sake of simplicity, the flexural rigidity of the two segments is assumed to be constant and identical, i.e.

\[ \frac{m_1}{Y_1 I_{y1}} = \frac{m_2}{Y_2 I_{y2}} = \frac{m}{Y}. \] \hspace{1cm} (12)

Furthermore, it is convenient to define \( \beta \) as

\[ \beta^4 = \frac{m}{Y} \omega^2_r. \] \hspace{1cm} (13)

Substituting (11) - (13) into (4) - (10) yields the ODEs (14) - (22)

\[ \phi_{nr}^{'''}(x_n) - \beta^4 \phi_{nr}(x_n) = 0 \quad \forall n = 1, 2, \] \hspace{1cm} (14)
\[ \phi_{1r}^{'}(0) = 0, \] \hspace{1cm} (15)
\[ \phi_{1r}(0) = 0, \] \hspace{1cm} (16)
\[ \phi_{2r}(L_2) + \beta^4 \frac{M_2}{m} \phi_{2r}(L_2) = 0, \] \hspace{1cm} (17)
\[ \phi_{2r}^{'}(L_2) - \beta^4 \frac{I_{y2}}{m} \phi_{2r}(L_2) = 0, \] \hspace{1cm} (18)
\[ \phi_{1r}(L_1) - \phi_{2r}(0) = 0, \] \hspace{1cm} (19)
\[ \phi_{1r}^{'}(L_1) - \phi_{2r}^{'}(0) = 0, \] \hspace{1cm} (20)
\[ \phi_{1r}^{'''}(L_1) - \phi_{2r}^{'''}(0) + \beta^4 \frac{M_1}{m} \phi_{1r}(L_1) = 0, \] \hspace{1cm} (21)
\[ \phi_{1r}^{'''}(L_1) - \phi_{2r}^{'''}(0) - \beta^4 \frac{I_{y1}}{m} \phi_{1r}(L_1) = 0, \] \hspace{1cm} (22)

Equations (15) - (18) are the boundary conditions and (19) - (22) are the transition conditions. Equation (14) derived from the equations of motion is the classic Euler-Bernoulli beam equation, whose general solution can be expressed as

\[ \phi_{nr}(x_n) = a_{nr1} \sin(\beta_n x_n) + a_{nr2} \cos(\beta_n x_n) + a_{nr3} \sinh(\beta_n x_n) + a_{nr4} \cosh(\beta_n x_n). \] \hspace{1cm} (23)

To find the specific solution, coefficients \( a = [a_{11} \ldots a_{24}]^T \) are determined to satisfy all boundary and transition conditions. Substituting (23) to (15) - (22) yields a linear system of equation

\[ K \eta = 0, \] \hspace{1cm} (24)

where \( K \) is a \( 8 \times 8 \) matrix. In order to have a non-trivial solution, the determinant of \( K \), i.e. the characteristic equation must be equal to zero. Solving this equation for \( \beta \), the undamped natural frequencies \( \omega_r \) can be obtained from (13). The mode shape function of the complete beam \( \phi_r(x) \) can be written in terms of the mode shape function of each individual beam segment and the Heaviside function as

\[ \phi_r(x) = \phi_{1r}(x)H(L_1 - x) + \phi_{2r}(x - L_1)H(x - L_1). \] \hspace{1cm} (25)

Since the determinant of \( K \) is zero, there are infinitely many solutions for \( a \). A unique solution can be determined to give a mass normalized mode shape function \( \phi_r(x) \). This unique solution fulfills the following orthogonality conditions

\[ \phi_{1r}(L_1)M_1 \phi_{1r}(L_1) + \phi_{2r}(L_2)M_2 \phi_{2r}(L_2) + \phi_{2r}(L_2)I_{y2} \phi_{2r}(L_2) \]
\[ + \int_0^{L_1+L_2} m \phi_s(x) \phi_r(x) \, dx = \delta_{rs}, \] \hspace{1cm} (26)

where \( \delta_{rs} \) is equal to unity for \( r = s \) and zero for \( r \neq s \).

D. Solution for Harmonic Base Excitation

For energy harvesting applications, it is a common practice to assume a harmonic base excitation in the transverse direction. The input vibration can be characterized by the acceleration amplitude \( A_0 \) and the excitation frequency \( \omega \) as

\[ \ddot{w}_b(t) = A_0 e^{j\omega t}. \] \hspace{1cm} (27)

Having determined the natural frequencies and the mass normalized mode shape function from modal analysis, internal strain rate damping is introduced to account for energy dissipation under external vibration. In this case, the equation of motion becomes

\[ m \ddot{w}(x,t) + c_s w'''(x,t) + Y w'''(x,t) = F_0 e^{j\omega t}, \] \hspace{1cm} (28)

where

\[ w(x,t) = \sum_{r=1}^{\infty} \phi_r(x) \eta_r(t), \] \hspace{1cm} (29)
\[ F_0 = -A_0 \left( m + M_1 \delta(x - L_1) + M_2 \delta(x - L_1 - L_2) \right), \] \hspace{1cm} (30)

and \( c_s \) is the equivalent strain rate damping constant. When subject to base excitation, the beam experiences forces due to the continuously distributed mass of the beam itself as well as the two rather concentrated proof masses, which are all incorporated in the external forcing term (30). Multiplying \( \phi_r(x) \) and integrating with respect to \( x \) for both sides of (28), the equation of motion is reduced to

\[ \ddot{\eta}_r(t) + 2 \xi_r \omega_r \dot{\eta}_r(t) + \omega_r^2 \eta_r(t) = F_r e^{j\omega t}, \] \hspace{1cm} (31)
where

\[ F_r = \int_0^{L_1 + L_2} F_0 \phi_r(x) \, dx, \tag{32} \]

and \( \xi_r \) is the \( r \)th mode damping ratio. Equation (31) is a familiar second order ODE, which has the steady state solution

\[ \eta_r(t) = \frac{F_r}{\omega_r^2 - \omega^2 + j2\xi_r\omega_r} e^{j\omega t}. \tag{33} \]

Therefore, the relative displacement \( w(x,t) \) under harmonic base excitation is obtained by substituting (33) and (25) into (29).

### III. EXPERIMENTAL VALIDATION

**A. Experimental Setup**

The accuracy of the analytical model was tested against experimental results. A rectangular stainless steel beam mounted with two identical iron cubes is used as the first investigation. The experimental setup is shown in Fig. 2. The key instruments involved in the experiment are: the laser vibrometer (A) that measures the velocity of an arbitrary point on the beam, the electromagnetic shaker (C) that generates base excitation, and the accelerometer (D) sits underneath of the clamp to measure the input acceleration. The damping ratios were estimated using the logarithmic decrement technique based on the time waveform of an impulse response. The parameters and material properties used in the experiment are given in Table I. The chosen beam has an aspect ratio of 13.2, hence it is reasonable to neglect the shear deformation and the rotary effects, which is in accordance with the assumptions of Euler-Bernoulli beam theory. In addition, the base acceleration was kept below 50 mg to preserve the linearity of the beam dynamics.

**B. Frequency Response**

The velocities of both iron cubes were measured when the beam was excited by swept-sine input. Due to hardware constraints, the measured velocity spectrum was limited to 1 Hz resolution. Fig. 3 and Fig. 4 show the velocity frequency response of \( M_1 \) and \( M_2 \), respectively. Note that the vibrometer measures the absolute velocity. In all subsequent plots, the y-axis is scaled by the input acceleration, so that it corresponds to the unit of (m/s/g). For both masses, the predicted resonant frequencies agree well with the experimental measurements.

**C. Mode Shapes**

Next, the mode shapes of the first two resonant modes are to be validated. This is done by separately exciting the

<table>
<thead>
<tr>
<th>Parameters/Properties</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>Elastic modulus, ( Y ) (GPa)</td>
<td>200</td>
</tr>
<tr>
<td>Beam width, ( w_{beam} ) (mm)</td>
<td>11.2</td>
</tr>
<tr>
<td>Beam thickness, ( t_{beam} ) (mm)</td>
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</tr>
<tr>
<td>Beam density, ( \rho_{beam} ) (kg m(^{-3}))</td>
<td>7800</td>
</tr>
<tr>
<td>Mass distance, ( L_1 ) (mm)</td>
<td>36</td>
</tr>
<tr>
<td>Mass distance, ( L_2 ) (mm)</td>
<td>50</td>
</tr>
<tr>
<td>Mass dimension, ( w \times h \times t ) (mm)</td>
<td>10 x 10 x 10</td>
</tr>
<tr>
<td>Mass density, ( \rho_m ) (kg m(^{-3}))</td>
<td>7900</td>
</tr>
<tr>
<td>1(^{st}) mode damping ratio, ( \xi_1 )</td>
<td>0.010</td>
</tr>
<tr>
<td>2(^{nd}) mode damping ratio, ( \xi_2 )</td>
<td>0.004</td>
</tr>
</tbody>
</table>
beam at each of the two resonant frequencies. The velocity was then measured at several points along the beam where reflective tape was fitted. The measurements of the 1st and 2nd vibration modes are shown in Fig. 5 and 6, respectively. In the mode shape measurements, the non-zero velocity at $x = 0$ corresponds to the absolute velocity of the vibrating base. In both figures, the modeling and experimental results agree well. A reason for the observed errors in Fig. 5 and 6 is the imprecise positioning of the laser spot onto the beam. Nevertheless, the model shows promises in giving satisfactory predictions of both natural frequencies and frequency response.

IV. PIEZOELECTRIC ENERGY HARVESTER

The two-mass beam structure can be turned into an energy harvester by using a piezoelectric beam. The effective bandwidth of such a harvester is widen because of the first two contiguous vibration modes. It should be stressed that the mode shape function derived earlier is still valid regardless of the material of the beam. The continuum-based model for a piezoelectric beam can be derived based on the model developed by Erturk and Inman [12]. When taking the electromechanical coupling effect into account, the equation of motion becomes

$$\ddot{\eta}_r(t) + 2\xi_r\omega_r\dot{\eta}_r(t) + \omega_r^2\eta_r(t) + \chi_re^{j\omega t} = F_re^{j\omega t},$$

where $V_0$ is the voltage output, and $\chi_r$ is a constant related to the geometries, material properties and the electrical connection and segmentation of the piezoelectric beam. The additional term $\chi_rV_0e^{j\omega t}$ acts as a force feedback from the induced voltage. If a resistive load $R$ is used as the electrical load, the solution for the voltage is given as $v(t) = V_0e^{j\omega t}$, where

$$V_0 = \frac{\sum_{r=1}^{\infty} \frac{j\omega F_r}{\omega_r^2 - \omega^2 + j2\xi_r\omega_r\omega}}{\sum_{r=1}^{\infty} \frac{1}{\omega_r^2 - \omega^2 + j2\xi_r\omega_r\omega} + \frac{1}{C_p\omega^2} + \frac{1}{R}}.$$  \hfill (35)

In the above equation, $C_p$ is the capacitance of the piezoelectric beam. $\chi_r$ and $\varphi_r$ are geometric related constants, whose explicit definitions can be found in the model developed by Erturk and Inman [12]. Once $V_0$ is known, the solution of (34) becomes

$$\eta_r(t) = \frac{F_r - \chi_rV_0}{\omega_r^2 - \omega^2 + j2\xi_r\omega_r\omega}e^{j\omega t}.$$  \hfill (36)

It should be noted that, for conciseness, all the steady state solutions derived in this paper are expressed as complex equations, which give the information of both amplitude and phase angle for the corresponding quantity.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this paper, a continuum-based model for a two-mass cantilever beam is presented. By applying Hamilton’s principle, the equations of motion together with all necessary boundary and transition conditions are derived. Undamped natural frequencies and the mass normalized mode shape function is determined through modal analysis. Furthermore, the steady state solution for harmonic base excitation is obtained. The derived model is validated experimentally on a stainless steel beam. In the experiment, both the mode shape and the velocity frequency response in the first two vibration modes were measured and compared with modeling results. The comparisons show excellent agreement between predictions of the model and the actual response of the beam. Although the presented model is for a two-mass beam, the derivation procedure is applicable for a beam with any number of masses. Based on the continuum-based piezoelectric beam model developed by Erturk and Inman, the analytical solutions for the relative displacement and output voltage of a two-mass piezoelectric beam is obtained.

B. Future Works

The derived boundary and transition conditions show that the resonant frequencies are a function of the second moment of area of each segment $I_n$, and the density and dimensions of each mass $M_n$. Moreover, the second moment of area depends upon the material, thickness and cross-sectional area of the beam segment. One of the future works would be to investigate the effectiveness of each of these parameters to the resulting resonant frequencies. More importantly, a method should be developed to find the combination of these parameters that maximizes power output while meeting all...
specified resonant frequencies. Another desirable future work would be to generalize the model for N masses.

VI. ACKNOWLEDGMENTS

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