On the Spectral Radius of Graphs with Cut Vertices

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We study the spectral radius of graphs with $n$ vertices and $k$ cut vertices and describe the graph that has the maximal spectral radius in this class. We also discuss the limit point of the maximal spectral radius.

Key Words: cut vertex, spectral radius, limit point.

1. INTRODUCTION

The graphs in this paper are simple. The spectral radius, $\rho(G)$, of a graph $G$ is the largest eigenvalue of its adjacency matrix $A(G)$. For results on the spectral radii of graphs, the reader is referred to [3, 4, 6] and the references therein. When $G$ is connected, $A(G)$ is irreducible and by the Perron–Frobenius Theorem, e.g., [1], the spectral radius is simple and has a unique (up to a multiplication by a scalar) positive eigenvector. We shall refer to such an eigenvector as the Perron vector of $G$. If we add an edge to $G$, the spectral radius increases.

A cut vertex in a connected graph $G$ is a vertex whose deletion breaks the graph into two (or more) parts.

The following problem concerning spectral radii was proposed by Brualdi and Solheid [2]: Given a set of graphs, find an upper bound for the spectral radii of graphs in $S$ and characterize the graphs in which the maximal spectral radius is attained. In this paper, we study this question for $S = \mathcal{G}_{n,k}$, the set of connected graphs on $n$ vertices and $k$ cut vertices. In Section 3, we show that the maximum is achieved only at the graph $G_{n,k}$ which is obtained by adding paths $P_1, \ldots, P_{n-k}$ of almost equal lengths (by the length of a path, we mean the number of its vertices) to the vertices of the complete graph $K_{n-k}$; that is, the lengths $l_1, \ldots, l_{n-k}$ of $P_1, \ldots, P_{n-k}$ which satisfy $|l_i - l_j| \leq 1$; $1 \leq i, j \leq n-k$ (Fig. 1). For example, for $n = 6$,

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Given $G_{6,0} = K_6$ and $G_{6,4}$ is a path of length 6. The proof is based on several lemmas that are stated and proved in Section 2. Finally, in Section 4, we study the limit points of the spectral radii.

### 2. NOTATION AND LEMMAS

**Lemma 2.1.** Let $G$ be a connected graph with vertex set $V(G) = \{v_1, ..., v_n\}$ and $d_i$ be the degree of vertex $v_i$, $i = 1, ..., n$. Then

$$\rho(G) \leq \max_{i, j \in E(G)} \sqrt{d_i d_j},$$

where $E(G)$ is the edge set of $G$. Moreover, the equality in (2.1) holds if and only if $G$ is a regular or bipartite semiregular graph.

**Proof.** Let $x$ be a Perron vector of $G$, where $x_i$ corresponds to the vertex $v_i$. Let $x_s = \max_{v_j \in V(G)} \{x_j\}$ and $x_t = \max_{(v_i, v_j) \in E(G)} \{x_j\}$. From $A(G)x = \rho(G)x$, we have

$$\rho(G)x_s = \sum_{v_j \in N_G(v_i)} x_j \leq \sum_{v_j \in N_G(v_i)} d_j x_j, \quad (2.2)$$

$$\rho(G)x_t = \sum_{v_j \in N_G(v_i)} x_i \leq \sum_{v_j \in N_G(v_i)} d_i x_j, \quad (2.3)$$

where $N_G(S)$ denotes the neighbors in $G$ of $S$. Hence

$$\rho(G)^2 x_s x_t \leq d_s d_t x_s x_t. \quad (2.4)$$

So (2.1) holds.

If $G$ is $d$-regular, then $\rho(G) = d$. If $G$ is $(\lambda, \delta)$-bipartite semiregular, then the sum of each row in $A(G)^2$ is equal to $\lambda \delta$. So $\rho(G) = \sqrt{\lambda \delta}$. Conversely,
if the equality in (2.1) holds, then the equality in (2.4) holds; furthermore, the equalities in (2.2) and (2.3) hold. Now we consider the following two cases.

Case 1: $x_i = x_j$. Let $V_1 = \{v_i, x_i = x_j\}$. If $V_1 \neq V(G)$, there exist vertices $v_r, v_p \in V_1$, $v_q \notin V_1$ such that $(v_r, v_p) \in E(G)$ and $(v_p, v_q) \in E(G)$ since $G$ is connected. Hence, from $\rho(G) x_i = \sum_{v \in N_G(v_i)} x_i \leq d_p x_r$ and $\rho(G) x_p = \sum_{v \in N_G(v_p)} x_i < d_p x_r$, we have $\rho(G) < \sqrt{d_p d_r}$, which contradicts that the equality holds in (2.1). Thus $V_1 = V(G)$ and $G$ is regular.

Case 2: $x_i > x_j$. Then $x_i = x_j$ for $v_i \in N_G(v_j)$ and $x_j = x_i$ for $v_j \in N_G(v_i)$. Let $U = \{v_i, x_i = x_j\}$ and $W = \{v_j, x_j = x_i\}$. So $N_G(v_j) \subseteq U$ and $N_G(v_i) \subseteq W$. Further, for any vertex $v_k \in N_G(N_G(v_j))$ there exists a vertex $v_r \in N_G(v_j)$ such that $(v_k, v_r) \in E(G)$, $(v_r, v_i) \in E(G)$. Hence $x_r = x_j$ and $\rho(G) x_p \leq \sum_{(v_k, v_r) \in E(G)} x_i \leq d_s x_r$. Thus, in addition to (2.2), we have $\rho(G) \leq d_s d_r$. So $\rho(G) = d_s d_r$, which yields $x_r = x_j$. Hence $N_G(N_G(v_j)) \subseteq U$. By a similar argument, we can show that $N_G(N_G(v_i)) \subseteq W$. Continuing the procedure, it is easy to see, since $G$ is connected, that $V = U \cup W$ and that the subgraphs induced by $U$ and $W$ respectively are empty graphs. Hence $G$ is bipartite. Moreover, for any edge $(v_k, v_i) \in E(G)$, where $v_k \in U$ and $v_i \in W$, we have $\rho(G) x_k I (v_k, v_i) \in E(G), (v_k, v_i) \in E(G)$. So $\rho(G) x_i \leq I (v_k, v_i) \in E(G), (v_k, v_i) \in E(G)$, $\rho(G) \leq \sqrt{d_s d_r}$, which yields that the vertices in $U$ are the same. The degrees of the vertices in $W$ are also the same. Hence $G$ is a bipartite semiregular graph.

Notation. The notation $H = G_1 + G_2$ will mean that $G_1$ and $G_2$ are two connected graphs with one common vertex $v$. Here, $H = G_1 + G_2$ is defined by $V(H) = V(G_1) \cup V(G_2)$, $V(G_1) \cap V(G_2) = \{v\}$, and $E(H) = E(G_1) \cup E(G_2)$. If $G_2$ is a singleton ($\{v\}$), then $H = G_1$.

**Lemma 2.2.** Let $u$ and $v$ be two adjacent vertices of the connected graph $G$ and for positive integers $k$ and $l$, let $G(k, l)$ denote the graph obtained from $G$ by adding pendant paths of length $k$ at $u$ and length $l$ at $v$. If $k \geq l \geq 2$, then $\rho(G(k, l)) > \rho(G(k + 1, l - 1))$.

**Proof:** The result follows from Theorem 6 in [7].

**Lemma 2.3.** Let $P_1$ and $P_2$ be disjoint paths, $v_1, v_2, v_3$ be the vertices of $K_3$, and $v_3$ be one of the vertices of $K_3$ in $G$. Let $H = K_3 v_1 P_1 v_2 P_2 v_3 G$ (Fig. 2) and $H_1$ be the graph obtained from $H$ by deleting the edge $(v_1, v_2)$ and connecting $v_1$ to all the neighbors of $v_3$ in $G$. If $\rho(H_1) \geq 3$, then $\rho(H_1) > \rho(H)$. 

**Proof:**
Proof. We may assume that $\rho(H) \geq 3$ since $\rho(H_1) \geq 3$. Let $x = (x_1, x_2, ..., x_n)^T$ be a Perron vector of $H$, where $x_i$ corresponds to the vertex $v_i$. Now we prove that

$$\sum_{v_i \in N_G(v_3)} x_i > \max \{x_1, x_2\}. \tag{2.5}$$

If the lengths of paths $P_1$, $P_2$ are more than 1, we may assume that $(v_1, v_2)$ is an edge of the path $P_1$ and $(v_2, v_3)$ is an edge of the path $P_2$. Hence it is easy to show that $x_1 > x_3$ and $x_2 > x_5$. From the first three equations of $A(H) x = \rho(H) x$ we get that $x_2 + x_4 + x_3 = \rho(H) x_1$, $x_1 + x_1 + x_5 = \rho(H) x_2$, $x_1 + x_2 + \sum_{v_i \in N_G(v_3)} x_i = \rho(H) x_4$. Hence, $x_3 = \rho(H) x_1 - x_2 - x_4 > (\rho(H) - 1) x_1 - x_2$ and $x_5 > (\rho(H) - 1) x_2 - x_1$. If $x_1 > x_2$, then by $\rho(H) > 3$,

$$\sum_{v_i \in N_G(v_3)} x_i = \rho(H) x_3 - (x_1 + x_2)$$

$$> \rho(H) \{ (\rho(H) - 1) x_1 - x_2 \} - (x_1 + x_2)$$

$$\geq (\rho(H)^2 - 2\rho(H) - 3) x_1 + x_2$$

$$\geq x_1 = \max \{x_1, x_2\}.$$ 

If $x_3 > x_1$, then, similarly, we can prove that $\sum_{v_i \in N_G(v_3)} x_i > \max \{x_1, x_2\}$.

By similar arguments, we can prove that (2.5) holds when one of the paths $P_1$, $P_2$ is of length 1. Hence

$$x^T (A(H_1) - A(H)) x = 2 x_1 \left( \sum_{v_i \in N_G(v_3)} x_i - x_2 \right) > 0.$$ 

Thus

$$\rho(H_1) = \max_{y \neq 0} \frac{y^T A(H_1) y}{y^T y} \geq \frac{x^T A(H_1) x}{x^T x} > \frac{x^T A(H) x}{x^T x} = \rho(H).$$
3. THE MAIN RESULT

**Theorem 3.1.** Of all the connected graphs with \( n \) vertices and \( k \) cut vertices, the maximal spectral radius is obtained uniquely at \( G_{n,k} \).

**Proof.** We have to prove that if \( G \in \mathcal{G}_{n,k} \), then \( \rho(G) \leq \rho(G_{n,k}) \) with equality only when \( G = G_{n,k} \). The adjacency matrix of a connected graph is irreducible, so if we add an edge \( e \) to a connected graph \( G, \rho(G+e) > \rho(G) \). Thus we can assume that each cut vertex of \( G \) connects exactly two blocks and that all of these blocks are cliques. Order the cardinalities of these blocks \( a_1 \geq a_2 \geq \cdots \geq a_{k+1} \geq 2 \). If \( k = 0 \), then \( \rho(G) \leq n-1 \) with equality if and only if \( G = K_n = G_{n,0} \). If \( k = n-2 \), then \( G \) is the path \( G_{n-2} \). If \( k = n-3 \), then \( a_1 = 3, a_2 = a_3 = \cdots = a_{k+1} = 2 \). The result follows from a repeated use of Lemma 2.2. Thus we may assume that \( 1 \leq k \leq n-4 \). Moreover, we observe that \( a_1 = n+k-(a_2 + \cdots + a_{k+1}) \leq n-k \). By Lemma 2.1,

\[
\rho(G) \leq \sqrt{(a_1 + a_2 - 2)(a_1 + a_3 - 2)}
\]

\[
= \frac{a_1 + a_2 - 2 + a_1 + a_3 - 2}{2}
\]

\[
= \frac{n + k - (a_4 + \cdots + a_{k+1}) + a_1 - 4}{2}
\]

\[
\leq \frac{n-k+a_1}{2}.
\]

If \( a_1 \leq n-k-2 \), then \( \rho(G) \leq \rho(G_{n,k}) \). Thus we have to consider only the cases \( a_4 = n-k-1 \) and \( a_1 = n-k \).

If \( a_1 = n-k \), then \( a_2 = \cdots = a_{k+1} = 2 \). So \( G = K_{n-k} v_1 P_1 v_2 P_2 \cdots v_{n-k} P_{n-k} \), where \( P_1, \ldots, P_{n-k} \) are disjoint paths, \( P_j \) is a path of length \( l_j \), \( V(P_j) \cap V(K_{n-k}) = \{ v_j \} \), and \( \sum_{i=1}^{n-k} l_i = n \). Now the result follows from a repeated use of Lemma 2.2.

If \( a_1 = n-k-1 \), then \( a_2 = 3 \) and \( a_3 = \cdots = a_{k+1} = 2 \), so \( G = F u_3 H \), where \( F = K_3 v_1 P_1 v_2 P_2 v_3 P_3 \); \( H = K_{n-k-3} u_4 P_4 u_5 P_5 \cdots u_{n-k+1} P_{n-k+1} \); all \( P_i \) are disjoint paths of lengths \( l_i \); \( \sum_{i=1}^{n-k+1} l_i = n \); and \( u_3 \) is a vertex of the path \( P_1 \). Now we consider two cases.

**Case 1:** \( l_1 > 1 \). By Lemma 2.1, we have \( \rho(G) \leq n-k-1 < \rho(G_{n,k}) \).

**Case 2:** \( l_1 = 1 \) and \( n-k-1 \geq 3 \). Since \( \rho(G_{n,k}) \geq 3 \), the result follows from Lemma 2.3 and a repeated use of Lemma 2.2. \( \blacksquare \)
4. LIMIT POINTS

The study of the limit points of the eigenvalues of a graph was initiated by Hoffman in [5], where he posed the problem of finding the limits of eigenvalues of graphs. Now we consider the limits of the spectral radius of $G_{n,k}$.

Theorem 4.1. The spectral radius $\rho$ of the graph $G_{n,k}$ with $k \leq \frac{n}{2}$ cut vertices satisfies the equation

$$\rho^3 - (n - k - 2) \rho^2 - (n - k) \rho + (n - 2k - 1) = 0.$$

Proof. We assume that the vertex set of $G_{n,k}$ is $\{v_1, ..., v_n\}$ and the induced subgraph by the vertex set $\{v_1, ..., v_{n-k}\}$ is clique and $(v_i, v_{n-k+i}) \in E(G_{n,k})$ for $i = 1, ..., k$. Let $x$ be a Perron vector of $G_{n,k}$, where $x_i$ corresponds to the vertex $v_i$. By the symmetry of $G_{n,k}$ we have $x_1 = \rho x_{n-k+1}$, $(n-2k)x_{n-k} + (k-1)x_1 + x_{n-k+1} = \rho x_1$, $(n-2k-1)x_{n-k} + kx_1 = \rho x_{n-k}$. Hence

$$x_{n-k} = \frac{\rho - (k-1)}{n-2k} x_1 - \frac{\rho - (k-1)}{n-2k} x_{n-k+1}$$

and

$$x_{n-k} = k x_1 = \frac{kp}{\rho - (n-2k-1)} x_{n-k+1}.$$ 

So

$$\frac{\rho^2 - (k-1) \rho - 1}{n-2k} = \frac{kp}{\rho - (n-2k-1)},$$

and the result follows from the above equation. \[\square\]

Corollary 4.2. Let $\rho$ be the spectral radius of the graph $G_{n,k}$ with $k \leq \frac{n}{2}$ cut vertices. Then

(i) $\rho < (n - k - 1) + \frac{k}{(n - k - 1)}$.

(ii) If $k$ is fixed, then

$$\lim_{n \to \infty} \left\{ \rho - \frac{(n - k - 1) + \frac{k}{(n - k - 1)}}{n} \right\} = 0.$$
Proof. Since $G_{n,k}$ contains a complete subgraph of order $n - k$, $\rho > n - k - 1$. Denote $\rho = n - k - 1 + \delta$, where $\delta > 0$. By Theorem 4.1, we have

$$\delta^3 + (2n - 2k - 1) \delta^2 + \{(n - k - 1)^2 + (n - k - 1)\} \delta - k = 0.$$ 

Hence $\delta < k/(n - k - 1)^2$ and (i) holds. Part (ii) follows from the above equation.

Theorem 4.3. Let $\rho$ be the spectral radius of $G_{n,k}$ with $n - k \leq n - 3$ cut vertices. If $n - k$ is fixed to be a constant $m$, then

$$\lim_{n \to \infty} \rho = m - 1 + \frac{1}{m - 1}.$$ 

Proof. Since $n - k$ is constant $m$, the spectral radius of $G_{n,k}$ is an increasing function of the number $n$ of the vertices of $G_{n,k}$ and $\rho \leq m$ by Lemma 2.1. Thus $\lim_{n \to \infty} \rho$ exists. Hence $\lim_{n \to \infty} \rho = a \leq m$, so we may only consider the subsequence of $\rho$ for $n = mr$, where $r$ is an integer. In other words, we can assume that $G_{n,k}$ is obtained by adding paths $P_1, \ldots, P_{n-k}$ of equal lengths to the vertices of the complete graph $K_{n-k}$, and the vertex set of $P_1$ is $\{v_1, \ldots, v_r\}$. Let $x = (x_1, \ldots, x_n)^T$ be a Perron vector of $G_{n,k}$, where $x_i$ corresponds to the vertex $x_i$. From $A(G_{n,k}) x = \rho x$, we have $x_3 - \rho x_2 + x_1 = 0$, ..., $x_r - \rho x_{r-1} + x_{r-2} = 0$. Hence

$$x_2 = \frac{x_r \sinh t + x_1 \sinh(r - 2) t}{\sinh(r - 1) t},$$

$$x_{r-1} = \frac{x_r \sinh(r - 2) t + x_1 \sinh t}{\sinh(r - 1) t},$$

where $t = \ln(\rho + \sqrt{\rho^2 - 4})/2$. By the symmetry of the graph $G_{n,k}$, we have $x_{r-1} + (m - 1) x_r = \rho x_r$ and $x_2 = \rho x_1$. Combining the above equations, we have

$$\frac{\sinh t}{\rho - (m - 1)} \frac{\sinh(r - 1) t - \sinh(r - 2) t}{\sinh(r - 1) t - \sinh(r - 2) t} = \frac{\rho \sinh(r - 1) t - \sinh(r - 2) t}{\sinh t}.$$ 

So

$$\left(\frac{\rho}{\sinh(r - 2) t} - 1\right) \left(\frac{\sinh(r - 1) t}{\sinh(r - 2) t} - 1\right) = \frac{\left(\sinh t\right)^2}{\left(\sinh(r - 2) t\right)^2}.$$ 

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Since
\[
\lim_{n \to \infty} \frac{\{\sinh r\}^2}{\{\sinh(r-2)\}^2} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\sinh(r-1)t}{\sinh(r-2)t} = a + \sqrt{a^2 - 4}.
\]
\[
\{a - (m-1)\} \frac{a + \sqrt{a^2 - 4}}{2} - 1 = 0.
\]
Hence \(\lim_{n \to \infty} \rho = a = m - 1 + \frac{1}{m-1} \).

Combining Theorem 3.1, Corollary 4.2, and Theorem 4.3, we have the following result:

**Theorem 4.4.** Let \(\rho(G)\) be the spectral radius of a graph \(G\) with \(n\) vertices and \(k\) cut vertices. Then

\[
\rho(G) < \begin{cases} 
  n - k - 1 + \frac{k}{(n-k-1)^2} & \text{for } k \leq \frac{n}{2}, \\
  n - k - 1 + \frac{1}{n-k-1} & \text{for } \frac{n}{2} < k \leq n - 3.
\end{cases}
\]

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**References**