Convergence Analysis of Narrowband Active Noise Equalizer System under Imperfect Secondary Path Estimation

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Abstract— Active noise equalizer systems are used to adjust the noise level in an environment, based on the preference of retaining noise information. Several researches have been carried out to determine the maximum step size bound of narrowband active noise control system with perfect secondary path estimation without gain factor consideration. However, in practical environment, secondary path estimation error of the system exists. In this paper, a stochastic approach analysis is applied to determine the maximum step size of the system under imperfect secondary path estimation. Simulation results are conducted to verify the analysis. Results show that the gain factor, sampling frequency, and secondary path estimation errors are all major factors governing the maximum step size of the narrowband active noise equalizer system under imperfect secondary path estimation.

Index Terms—— Active noise control (ANC), active noise equalizer (ANE), filtered-X least mean square (FXLMS), secondary path, convergence analysis.

I. INTRODUCTION

Active noise control (ANC) [1], [2] systems based on the principle of superposition have evolved and developed in many applications to cancel undesirable noise. However in some applications, it is desired to independently control some frequency components of the noise signal, either to keep some portions of the noise signal or to reshape the sound field. This new demand leads to an extension of the ANC concept to include noise equalization.

In order to adjust the level of noise cancellation, a narrowband active noise equalizer (NBANE) was proposed by Kuo et al [3], [4] that used the filtered-X LMS (FXLMS) [5] algorithm to adapt the coefficients of a two-weight filter by minimizing a pseudo-error signal, instead of the residual noise. The main idea behind NBANE lies in its ability to control the tonal noise to a desired level. The NBANE system can also be configured in parallel to shape the residual noise spectrum [6]. In the practical environment, imperfect secondary path estimation exists. This imperfect estimation causes the asymmetric out-of-band overshoot in narrowband ANC (NBANC) systems [7]. It’s also the factor causing the misequalization problem [8] in NBANE systems, in which case the NBANE system converges to an undesired value.

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Furthermore, the estimation errors also reduce the max step size of the NBANE system. Due to its ability to control over the signal and simplicity to implement, NBANE and similar systems still continue to receive high interest from researchers [9], [10]. To overcome the drawback of NBANE system on sensitivity to secondary path estimation error, Rees et al proposed sound-profiling algorithms [11].

This paper looks into the convergence analysis of the NBANE system under practical situation. The remaining part of this paper is organized as follow: Section II briefly reviews the convergence analysis in NBANE system. Section III first introduces convergence analysis of NBANE, followed by the convergence analysis of NBANC as special case of NBANE system. Simulation results and discussion are presented in Section IV. Section V concludes the paper.

![Block diagram of NBANE system](image)

**Fig. 1.** Block diagram of NBANE system

II. CONVERGENCE ANALYSIS OF NBANE SYSTEM

Convergence analysis of NBANE system when the secondary path is perfectly estimated has been reported in [4]. The step size bound for this case is shown to be similar to the NBANC system. The system converges to the same filter weights regardless of the gain factor value. However, under an imperfect condition, Wang et al. [8] reported on the misequalization problem in the NBANE system. The results showed that the estimation error of the secondary path alters the poles of the system transfer function. This analysis is based on the assumption of the convergence of the NBANE system.

Most of the researchers analyze the convergence of the narrowband ANC/ANE system in frequency domain by finding the z-transfer function of the system. The theoretical basis that enables this analysis was proposed by Glover [12]. The behavior of the adaptive filter is exactly described by a linear time-invariant filter between the primary signal and the error signal. As a result, the system stability property can be obtained according to the poles placement of the system.
transfer function. However when the gain factor is close to zero, the stochastic behavior of the system is amplified by the secondary path estimation error, which leads to a significant reduce of maximum step size. In order to examine the effects of such behavior, the stochastic analysis of NBANE system with quadrature sinusoidal reference signals is presented in next section.

III. STOCHASTIC ANALYSIS OF NBANE SYSTEM

The algorithm for NBANE system is given as:

\[
x_0(n) = \cos(\omega n), \quad x_1(n) = \sin(\omega n),
\]

where \(\omega\) is the reference frequency. The error signal is:

\[
e(n) = d(n) - (1 - \beta) \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i),
\]

\[
e'(n) = e(n) - \beta \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i),
\]

where \(s_i\) and \(\hat{s}_i\) are the secondary path filter coefficients and their estimation, for \(i = 0, 1, \ldots, L_s-1\). \(w_0(n)\) and \(w_1(n)\) are the two adaptive filter weights. The adaptive filter output is given as:

\[
y(n) = x_0(n)w_0(n) + x_1(n)w_1(n).
\]

Substituting (1) in (4),

\[
y(n) = \cos(\omega n)w_0(n) + \sin(\omega n)w_1(n).
\]

The adaptive filter weight \(w_0\) is updated as (same for weight \(w_1\)):

\[
w_0(n+1) = w_0(n) + \mu \sum_{i=0}^{L_s-1} \hat{s}_i e'(n) \cos(\omega(n-i))
\]

\[
+ \mu \sum_{i=0}^{L_s-1} \hat{s}_i e'(n) \cos(\omega(n-i)) d(n)
\]

\[
- (1 - \beta) \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i) - \beta \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i).
\]

We denote \(w_0^*\) as the optimal weight after convergence, the weight error at time \(n\) is expressed as:

\[
w_0(n+1) - w_0^* = w_0(n) - w_0^* + \mu \sum_{i=0}^{L_s-1} \hat{s}_i \cos(\omega(n-i)) d(n)
\]

\[
+ \sum_{i=0}^{L_s-1} \hat{s}_i \cos(\omega(n-i)) [d(n) + \mu \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i) - (1 - \beta) \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i)].
\]

For a narrowband system with sinusoidal input, the secondary path can be modeled as a scaled delay with amplitude of \(S\) at the \(\Delta\) th sample, the estimation is modeled by \(\hat{S}\) and \(\hat{\Delta}\) correspondingly. Defining \(v_0(n+1) = w_0(n+1) - w_0^*\) and \(v_0(n) = w_0(n) - w_0^*\) as the weight error, the weight error updating is given as:

\[
v_0(n+1) = v_0(n)
\]

\[
+ \mu \hat{S} \cos(\omega(n-\hat{\Delta})) [d(n) + \mu \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i) - (1 - \beta) \sum_{i=0}^{L_s-1} \hat{s}_i y(n-i)].
\]

From (9), the expectation of the weight error can be shown to be (proof provided in Appendix),

\[
E[v_0(n+1)] = E[v_0(n)] + \mu \hat{S} [E[\cos(\omega(n-\hat{\Delta})) d(n)] - \frac{1}{2} (1 - \beta) \hat{S} E[\cos(\omega(\Delta - \hat{\Delta}))]
\]

\[
- \frac{1}{2} (1 - \beta) \hat{S} w_0^* \cos(\omega(\Delta - \hat{\Delta}))
\]

\[
+ \frac{1}{2} (1 - \beta) \hat{S} [E[v_0(n-\Delta)] \sin(\omega(\Delta - \hat{\Delta}))]
\]

\[
- \frac{1}{2} \beta \hat{S} E[v_0(n-\Delta)] - \frac{1}{2} \beta \hat{S} w_0^*.
\]

A. Characteristic equation under imperfect secondary path estimation

To derive the characteristic equation of the NBANE system under imperfect secondary path estimation, we can obtain following equation using (12)
$$E[\cos(\omega t(n-\hat{\Delta}))d(n)] = \frac{1}{2}(1-\beta)|\hat{S}|\nu^*\cos(\omega t(\Delta-\hat{\Delta}))$$
$$+\frac{1}{2}(1-\beta)|\hat{S}|\nu^*\sin(\omega t(\Delta-\hat{\Delta}))$$
$$+\frac{1}{2}\beta|\hat{S}|\nu^*.$$  
(13)

Substituting the result of (13) into (10), the following equation is obtained

$$E[v_t(n+1)] = E[v_t(n)]$$
$$-\frac{1}{2}z|\hat{S}||\hat{S}|(1-\beta)E[v_t(n-\Delta)]\cos(\omega t(\Delta-\hat{\Delta}))$$
$$+|\hat{S}|\beta E[v_t(n-\Delta)].$$  
(14)

The corresponding characteristic equation is

$$z - 1 + \frac{1}{2}z\beta z^{-1}\cos(\omega t(\Delta-\hat{\Delta})) + |\hat{S}|\beta z^{-1} = 0.$$  
(15)

To ensure stability of the system, all roots of the characteristic equation have to be inside the unit circle. Characteristic equation (15) applies to both narrowband ANC/ANE systems under perfect or imperfect secondary path estimation.

**B. Comparison with previous literature**

In order to compare with literature [13] reported for ANC system, perfect secondary path estimation ($|\hat{S}| = |S| = 1, \hat{\Delta} = \Delta$) is assumed. (15) can be simplified to

$$z - 1 + \frac{1}{2}\mu z^{-1} = 0.$$  
(16)

It can be seen that, under perfect estimation, the equation is independent from gain factor $\beta$. This coincides with the result shown in previous literature [1]. An approximation solution of the maximum step size bond can be obtained by following the standard root locus theory as explained in [13], [14].

The poles of the system are given by the values of $z$ satisfying

$$\mu = 2(z^\Delta - z^{-1}).$$  
(17)

According to the control theory, the pole break points can be obtained by setting the differential below to zero,

$$\frac{\partial \mu}{\partial z} = 2[\Delta z^\Delta - (\Delta + 1)z^\Delta] = 0.$$  
(18)

Therefore,

$$z = 0 \text{ or } z = \frac{\Delta}{\Delta + 1}.$$  
(19)

Substituting (19) into (17), the value of optimized step size $\mu_e$ for large $\Delta$ is

$$\mu_e \approx \frac{2}{e}\left(\frac{1}{\Delta + 1}\right),$$  
(20)

where $e$ is the base of the natural logarithm. Note that, according to the root locus, this approximate value of step size corresponds to the break away of poles. As a result this step size is slight smaller than the maximum step size keeping the poles in unit circle and the system stable. For large $\Delta$, this result leads to $\mu_e \approx \frac{2}{e}\frac{1}{\Delta}$. This optimized step size value shows good agreement with the result reported by Elliot et al in [13] for ANC system. Therefore, the characteristic equation (15) under imperfect secondary path estimation can also be applied under perfect case.

**IV. SIMULATION RESULTS**

Simulations are conducted to verify the previous analysis. The results given below are based on the theoretical analysis in previous section. These results are also verified against empirical simulations. The results are categorized into different factors.

**A. Gain factor**

The simulation shown in Figure 2 presents the maximum step size with regard to the gain factor value under secondary path amplitude estimation error of $|\hat{S}| = |S| = 2$. The dash-dot line is obtained by using Glover’s method [12]. The dashed straight line is the maximum step size with perfect secondary path estimation. It shows the independence of gain factor value as found in [3]. The solid line and discrete points are the maximum step size found according to analysis in previous section equation (15) and empirical simulations respectively. It is clear that the maximum step size is dependent on the gain factor.
B. Amplitude estimation error

As stated in section II, for smaller gain factor ($\beta < 1$), the stochastic behavior of the NBANE system is further amplified by the secondary path estimation error. Following simulations present the results for $\beta = 0.1$ and 0.7. As shown in Figure 3 and 4, the maximum step size decreases dramatically with the increase of secondary path amplitude estimation error defined by $|\hat{S}|/|S|$. The solid line and discrete points are from equation (15) and empirical simulations respectively.

C. Phase error/ Delay error

In this simulation, the period of the reference signal used is 20 samples. The delay in the secondary path is 40 samples and the estimated delay ranges from 40 to 100 samples, covering a phase error from 0 to $6\pi$. As shown in Figure 5, the theoretical curve shows the envelope of the maximum step size given by (15). This envelope is controlled by both phase error and delay. For same phase error of $2\pi$, $4\pi$ and $6\pi$, the maximum step size envelope drops with the increase of delay. With phase errors of $\pi$, $3\pi$ and $5\pi$, the system still converges, though the phase error exceeds $\pi/2$. This agrees with the result reported in [8].

D. Sampling frequency

In previous literatures, sampling frequency was not considered as a factor in determining the stability or step size bond problem in narrowband ANC/ANE system. On the other hand, the number of samples of delay in the secondary path is a crucial factor. In practical cases, this number of samples is direct proportion to the system sampling frequency. Consequently, using a higher sampling rate, the number of
samples of the impulse response would be extended by the same ratio. This decreases the step size bound of the system. Experimental results reported in [15] confirmed this conclusion.

Using the characteristic equation (15), the effect of changing sampling frequency can be examined by using the corresponding delay. The approximated value of the optimum step size in (20) is

\[
\mu_e \approx \frac{2}{e} \left( \tau \times F_s + 1 \right),
\]

where \( \tau \) is the secondary path delay in seconds, and \( F_s \) is the sampling frequency. To distinguish the effect from the sampling frequency alone, the simulation is conducted under perfect secondary path estimation. The following parameters is used in the simulation: \( \hat{S} = |S| = 1 \), \( \beta = 0.7 \), \( \Delta = \hat{\Delta} = 0.05F_s \).

**Fig. 6.** Maximum step size under different sampling frequencies

It can be seen from Figure 6 that the step size bond decreases with the increase of system sampling frequency. The solid curve is calculated according to the system characteristic equation, while the stars are from empirical simulations. The inverse law of step size and sampling frequency shown in (21) can also be observed from the simulation result in Figure 6. Considering the narrowband ANC/ANE systems are usually implemented to control noise signal below 500 Hz, a lower sampling frequency is recommended for a larger bond of step size, and in turn faster convergence. In [7] it is also suggested that sampling frequency should be chosen so that the normalized frequency of the expected interference normalized frequency is near 0.25.

**V. CONCLUSION**

In this paper, the stochastic approach convergence analysis for NBANE system is presented for the first time. In order to find out the maximum step size for NBANE system, we derived the characteristic equation from the analysis. Based on this equation, we are able to examine the system using root locus theory. It shows that, for imperfect secondary path estimation, the maximum step size of the NBANE systems is governed by factors including gain factor, secondary path estimation errors and sampling frequency. The results are supported by simulations and agree with previous literatures on NBANC systems which can be seen as a special case NBANE system.

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**APPENDIX**

**PROOF OF EQUATIONS (10)**

Substituting (5) into (9), leads to

\[
v_o(n+1) = v_o(n) + \mu [S \cos(o_\omega(n - \hat{\Delta}))d(n)
- (1 - \beta) [S \cos(o_\omega(n - \hat{\Delta}))w_o(n - \Delta) 
+ \sin(o_\omega(n - \Delta))w_i(n - \Delta)]
- \beta [S \cos(o_\omega(n - \hat{\Delta}))w_o(n - \hat{\Delta})
+ \sin(o_\omega(n - \hat{\Delta}))w_i(n - \hat{\Delta})].
\]

(A.1)

Note that the signal used in training secondary path is white noise. Consequently, the estimated secondary path impulse response is independent from the input narrowband noise samples. Taking expectation of (A.1) results

\[
E[v_o(n+1)] = E[v_o(n)] + \mu [S \{ 
+ E[\cos(o_\omega(n - \hat{\Delta}))d(n)]
- E[\cos(o_\omega(n - \hat{\Delta}))](1 - \beta) [S \cos(o_\omega(n - \Delta))w_o(n - \Delta)]
- E[\cos(o_\omega(n - \hat{\Delta}))][S \cos(o_\omega(n - \Delta))w_i(n - \Delta)]
- E[\cos(o_\omega(n - \Delta))][S \sin(o_\omega(n - \hat{\Delta}))w_o(n - \hat{\Delta})]
- E[\cos(o_\omega(n - \Delta))][S \sin(o_\omega(n - \hat{\Delta}))w_i(n - \hat{\Delta})].
\}
\]

\]

(A.2a - A.2e)

The term (A.2b), (A.2c) and (A.2d) can be further expressed as

\[
- E[\cos(o_\omega(n - \hat{\Delta}))](1 - \beta) [S \cos(o_\omega(n - \Delta))w_o(n - \Delta)]
= \frac{1}{2} (1 - \beta) [S E[w_o(n - \Delta)]E[\cos(o_\omega (2n - \Delta - \hat{\Delta}) + \cos(o_\omega (\Delta - \hat{\Delta}))]
= \frac{1}{2} (1 - \beta) [S E[w_o(n - \Delta)]E[\cos(o_\omega (\Delta - \hat{\Delta}))].
\]

(A.3)
Similarly,

\[-E[\cos(\omega_n(n-\Delta))(1-\beta)]\sin(\omega_n(n-\Delta))w_l(n-\Delta)]

\[= \frac{1}{2}(1-\beta)S[E[w_l(n-\Delta)]\sin(\omega_l(\Delta-\Delta)), \quad (A.4)\]

and

\[-E[\cos(\omega_n(n-\Delta))]\beta S[\cos(\omega_l(n-\Delta))w_l(n-\Delta)]

\[= -\frac{1}{2}\beta S[E[w_l(n-\Delta)]. \quad (A.5)\]

Take note that the term in (A.2c) equals zero. As a result, (A.2) can be simplified as

\[E[v_i(n+1)] = E[v_i(n)] + m\mu S[E[\cos(\omega_l(n-\Delta))d(n)]

\[-\frac{1}{2}(1-\beta)S[E[w_l(n-\Delta)]\cos(\omega_l(\Delta-\Delta))

\[+ \frac{1}{2}(1-\beta)S[E[w_l(n-\Delta)]\sin(\omega_l(\Delta-\Delta))

\[-\frac{1}{2}S[\beta E[w_l(n-\Delta)]]. \quad (A.6)\]

From (A.6), construct the weight error again and (10) can be obtained.

\[E[v_i(n+1)] = E[v_i(n)] + m\mu S[E[\cos(\omega_l(n-\Delta))d(n)]

\[-\frac{1}{2}(1-\beta)S[E[w_l(n-\Delta)]\cos(\omega_l(\Delta-\Delta))

\[-\frac{1}{2}(1-\beta)S[w_l(n-\Delta)]\cos(\omega_l(\Delta-\Delta))

\[+ \frac{1}{2}(1-\beta)S[E[w_l(n-\Delta)]\sin(\omega_l(\Delta-\Delta))

\[-\frac{1}{2}S[\beta E[w_l(n-\Delta)]]. \quad (A.7)\]

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