A Pseudo Bayesian Model in Financial Decision Making with implications to Market Volatility, Under- and Overreaction

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Abstract

This paper develops a model of weight assignments using a pseudo-Bayesian approach that reflects investors’ behavioral biases. In this parsimonious model of investor sentiment, weights induced by investors’ conservative and representative heuristics are assigned to observations of the earning shocks of stock prices. Such weight assignments enable us to provide a quantitative link between some market anomalies and investors’ behavioral biases. The seriousness of an anomaly can be quantitatively assessed by investigating into its dependency on weights. New results other than the short-run overreaction and long-run overreaction can be derived and new hypotheses can be formed.

Keywords: Bayesian model; Representative and conservative heuristics; Underreaction; Overreaction; Stock price; Stock return
1. Introduction

Among many market anomalies uncovered in the last two decades, three stand out as having a long history and receiving the most substantial empirical support. They are market excess volatility, overreaction, and underreaction. Together with other market anomalies, they pose a major challenge to financial economists. To meet these challenges, advocates of behavioral biases have constructed various behavioral models to explain these anomalies. Among the behavioral biases advocated, two stand out as having a long history and receiving much empirical supports from psychological literatures. These behavioral biases are investors’ usage of the conservatism heuristics and the representativeness heuristics in making decisions. The most notable model in this direction is an early paper by Barberis, Shleifer and Vishny (1998, henceforth BSV), in which they show that underreaction in the short-run and overreaction in the long-run is a consequence of the two mentioned heuristics. However, their paper did not show that excess volatility is also a result of the conservatism and representativeness heuristics.

BSV adopt a bounded rationalism approach in which some, but not all, assumptions under the traditional rational expectations asset-pricing theory are violated. Specifically, the “consistent beliefs” made by Sargent (1993) that agents possess correct knowledge of the economic structure is assumed to be violated. In their 1998 paper, BSV assume that while earning announcements follow a random walk, investors using conservative and representative heuristics believe that the earning announcements fall into one of two regimes, a trending regime and a mean reverting regime, and transition from one regime to the other follows a Markov chain. Assuming that the investors still use a correct Bayesian methodology for decision making, BSV then deduce that
such a wrong belief will lead to both short-term underreaction and long-term overreaction in the market.

This paper takes a different approach from that of BSV in modeling conservatism and representativeness. We assume that the investor knows the correct model but uses a wrong updating methodology. This approach has several advantages as follows: (1) psychological literature clearly states that the two psychological biases arise from investors’ attaching wrong weights to information, rather than from their adoption of a wrong model. In this paper, the weighting of information is emphasized and it is a more accurate description of the heuristics used by investors. (2) Since the wrong weights reflect the biases, different degree of biases can be assessed through considering a change in weights. As a result, the seriousness of an anomaly can be quantitatively assessed by investigating into its dependency on weights. (3) New results other than the short-run overreaction and long-run overreaction can be derived and new hypotheses can be formed. We will elaborate on these points below.

Let us elaborate on (1). According to DeBondt and Thaler (1995), a good finance theory must be based on psychological evidence of how people actually behave. Thus, it is important to look into the psychological literature on how the behavioral biases arise. Psychologists observe that investors pay too much attention to extreme information and less attention to its validity when making judgments and decisions about their investments (Griffin and Tversky 1992). When investors are overconfident about their analysis based on the past performance of stocks and underreact to recent information, thus updating their beliefs too slowly in the face of new evidence, they exhibit conservative heuristics (Edwards 1968; Grether 1980). On the other hand, if they are overconfident about the recent information on stocks and pay less attention to the past
information on stocks or extrapolate too readily from small samples, thus leading to belief revisions that are too dramatic, they demonstrate representative heuristics (Tversky and Kahneman 1971, 1974; Kahneman and Tversky 1973). Most studies of conservative heuristics involve large samples, whereas most studies on representative heuristics involve smaller samples. Misunderstanding the impact of sample size on the posterior mean leads investors to make conservative revisions with large samples and radical revisions with small samples. Thus, it is obvious that behavioral biases arise from an inappropriate treatment of information, rather than from a misjudgment on the model.

One of the earliest papers addressing conservatism is Edwards (1968), who reveals that people tend to make behavioral mistakes in their decisions, although they try to employ theoretical models or methodology. He observes that investors with conservative behavior might pay little attention or even ignore the full information from an earnings announcement. They may believe that this information is mainly temporary, and thus they still cling to their prior beliefs based on past earnings. As a result, they might incorporate only partial information from recent earnings announcement in their valuation of shares. In other words, they attach too little a weight to recent information. Edwards (1968) develops a Bayesian model in which individuals tend to underweigh useful statistical evidence relative to the less useful evidence used to form their priors. He observes that it takes two to five observations to do one observation’s worth of work in inducing a subject to change his/her opinions. Grether (1980) claims that individuals who exhibit conservatism update their beliefs too slowly in the face of new evidence. Klein (1990), Mendenhall (1991) and Abarbanell and Bernard (1992) further suggest that investors tend to underreact to new information. In terms of the Bayesian rule, conservatism means that
people tend to overweight the base rate (prior) and underweight new information. This is exactly
the approach of the proposed model in this paper.

On the representative heuristic, many experiments (see, for example, DeBondt and Thaler
1985; Lakonishok, Shleifer and Vishny 1994; Barberis, Shleifer and Vishny 1998) show that
individuals expect key population parameters to be “represented” in any recent sequence of
generated data (see Tversky and Kahneman 1971, 1974 for a detailed discussion). Tversky and
Kahneman (1971) suggest that local representativeness is a belief in the “law of small numbers,”
meaning that “the law of large numbers applies to small numbers as well.” Investors may find
that even small samples (rather than large samples) are highly representative of the populations
from which they are drawn. This simply shows that investors may place excessive weights on a
sample of small size and neglect distinct information unjustifiably. The model proposed in this
paper adopts this approach in modeling the law of small sample and in fact, the “smallness” of
the sample will be taken into account in our model.

We remark that some key measures of market anomalies like market volatility,
autocorrelation of market returns, trading profit of a self-finance long-short strategy etc, can be
expressed in terms of the weights and key financial variables like risk free interest rates. The
impact of the incorrect weights on the anomalous magnitudes can hence be quantitatively
assessed. In so doing, we can compare the impact of conservatism and representativeness on the
anomalous magnitudes. We can also study the interaction between the heuristics and the key
financial variable. For example, we can show that market’s excess volatility is essentially the
result of the “law of small number,” and under a reasonable assumption on smallness, volatility
can become 28 times that of the volatility attributable to pure information, see Section 3 for further details.

Our behavioral model gives rise to a richer body of consequences than BSV. Other than demonstrating short-run underreaction and long-run overreaction, we also derive excess volatility as a consequence of the behavioral model. Furthermore, we can attribute the excess volatility to the representative heuristic and show that excess volatility is more prominent when the discount rate is small. On overreaction and underreaction, we demonstrate that there exists a magnitude effect in the under- and overreaction phenomena. Specifically, our model provide theoretical support for the second part of the under- and overreaction hypotheses. Recall that the first part of the overreaction (underreaction) hypothesis in DeBondt and Thaler (1985) (Jegadeesh and Titman 1993) is “extreme movements in stock prices will be followed by subsequent price movements in the opposite (same) direction” and the second part of the overreaction (underreaction) hypothesis is “the more extreme the initial price movement, the greater will be the subsequent adjustment”. In other words, if \( n \) pieces of good/bad news announcements repeatedly occur \( n \) times, the overreaction that results increases with \( n \). Not only we can show that the autocorrelation magnitude and the trading profit increase with \( n \), our model actually shows that these anomalous magnitudes are a convex function of \( n \). Another consequence we can draw from the model is that the trading profit of the contrarian/momentum trading strategy increases when discount rate decreases.

The rest of the paper is organized as follows. In Section 2, we construct a pseudo-Bayesian framework to model investors’ conservative and representative heuristics and develop price dynamics under this model. In Section 3, we study how the heuristics will impact on market
volatility in equilibrium. We then outline in Section 4 the implications of our proposed model by using it to demonstrate the existence of short-run underreaction and long-run overreaction in the stock market. The trading profit resulting from the corresponding momentum/contrarian trading strategies is also derived and analyzed. In Section 5, we show that our model enables us to derive an additional result that there is a “magnitude effect” associated with the under-and-overreaction in the stock market. We show further that the magnitude effect is convex in nature. Section 6 wraps up this paper with a conclusion. Some proofs are provided in the appendices.

2. The Model

In BSV, a representative investor observes the earnings of an asset and updates his/her belief to price the asset. It is assumed that $N_t$, the earnings announcement of the asset at time $t$, follows a random walk, i.e., $N_t = N_{t-1} + y_t$, where $y_t$ is an earnings shock at time $t$. Using a discounting model, see, for example, Thompson and Wong (1991, 1996) and Wong and Chan (2004), the asset is priced at time $t$ as $P_t$ given by

$$P_t = E_t \left\{ \frac{N_{t+1}}{1+r} + \frac{N_{t+2}}{(1+r)^2} + \ldots \right\} = \frac{N_t}{r} + \frac{1+r}{r} \left\{ \frac{E_t y_{t+1}}{(1+r)^1} + \frac{E_t y_{t+2}}{(1+r)^2} + \ldots \right\},$$

where $r$ is the discount rate or the investor’s anticipated return. In (1), $E_t$ represents the investor’s expectation given the information set $\Phi_t$, which is the set of all information available to the investor at time $t$. In BSV, the following assumptions are made:
Assumption 1: The earnings shock $y_t$ is independent and follows a distribution with equal chance on discrete values $y_0$ or $-y_0$.

Assumption 2: The representative agent does not realize that the true process for earnings follows a random walk and uses a wrong model to update his or her beliefs. She or he assumes that the earnings shock $y_t$ transit between two regimes (or states) of a Markov chain: a momentum regime and a reversal regime. The transition probability (from $y_0$ to $-y_0$ or from $-y_0$ to $y_0$) is small (smaller than one-half) in the momentum regime and is large (larger than one-half) in the reversal regime.

Assumption 3: The representative agent uses a correct Bayesian approach to update his or her beliefs. In other words, the agent estimates the current state of the earnings shocks and uses them to determine the future expected price.

Notice that in (1), $E_t$ is the expectation taken under assumptions 1, 2 and 3. Hence, it incorporates the investor’s biased views toward pricing, as a result of a model mis-specification in assumption 2. Under these biased views, BSV deduce that both short-run underreaction and long-run overreaction exist. Notice that under-and-overreaction occurs as a result of the investor’s mis-specification of the model and not from his or her biased updating methods. In this paper, we propose an alternative approach by assuming that the representative investor is aware of the correct underlying model but fails to adopt a correct approach in the updating process. Brav and Heaton (2002) is the first paper to model an investor as one who misapplies Bayesian methodology. However, they model the representative heuristic only. In this paper, we
further extend their incorrect Bayesian method to also include the conservative heuristic. Specifically, BSV’s assumptions are modified as follows:

**Assumption 1’**: The earning $N_t$ follows a random walk model but the earnings shock $y_t$ is independent and follows a Gaussian distribution with mean $\mu$ and variance $\sigma^2_y$.

**Assumption 2’**: The representative agent knows the nature of the random walk process, except that the parameter $\mu$ in the model has to be estimated. In other words, the agent has to estimate the mean $\mu$ by employing observed data on the earnings shock $\{y_t\}$. For simplicity and tractability, we also assume that the representative agent knows $\sigma^2_y$ and adopts a vague prior for $\mu$.

**Assumption 3’**: The agent uses a wrong statistical method to update his or her belief. It is the wrong method that reflects the investor’s behavioral biases.

In BSV, the expectation $E_t$ incorporates investor’s assessments of the future based on a wrong model. Under the new assumptions, $E_t$ incorporates investors’ biases not as the result of a mis-specification of the model but as a result of an incorrect updating method. In this paper, the subjective expectation $E_t y_{t+1}$ reflects investors’ conservative and representative heuristics in estimating the parameter $\mu$. The term $E_t y_{t+1}$ can be explicitly computed after we describe the pseudo-Bayesian approach in detail in the following subsections.
2.1. Modeling Behavioral Biases by a Pseudo-Bayesian Approach

1. A quantitative behavioral model with general weights

Before we present the pseudo-Bayesian approach adopted by a behaviorally biased investor, we first describe the correct methodology to update information on the mean level $\mu$ of the earnings shock. We assume a vague prior for $\mu$, i.e. $P_0(\mu) \propto 1$ [see DeGroot (1970) and Matsumura, Tsui and Wong (1990)]. Let $y_i$ be the earnings shock observed at the end of period $i$, $i=1,2,\ldots,t$. Since the likelihood function is given by

$$L(\mu) = \prod_{i=1}^{t} L(y_{i-1}|\mu),$$

the posterior distribution of $\mu$ is given as follows:

$$P(\mu|y_1,\ldots,y_t) \propto \prod_{i=1}^{t} L(y_{i-1}|\mu) \cdot 1. \quad (2)$$

Notice that equal weight is placed on each observation in $y_1,\ldots,y_t$ under the Bayesian approach. Consistent with the predictions of traditional efficient markets, this rational expectations asset-pricing theory assumes that investors can have access both to the correct specification of the “true” economic model and to unbiased estimators of its coefficients; see, for example, Friedman (1979) for more information. Obviously, if the rational investor is endowed with an objectively correct prior and the correct likelihood functions, he/she will obtain the rational expectations equilibrium and thus any structural irrationally induced financial anomaly should disappear. Attainment of such structural knowledge on convergence to a rational
expectations solution has been studied widely in the literature. For example, Blume and Easley (1982) and Bray and Kreps (1987) observe that investors have to recognize and incorporate how their beliefs about the unknown essential features of the economy influence the structural model of the economy. However, the extreme knowledge required in these models is implausible. If investors do not recognize the effect of learning on prices to obtain equilibrium, Blume and Easley (1982) have shown that convergence of beliefs is not guaranteed within a general equilibrium learning model.

Nonetheless, as evidence has mounted against this traditional Bayesian model, theories of financial anomalies have to be developed by relaxing those assumptions. One approach is to assume that investors are plagued with cognitive biases (Slovic 1972), and they may incorrectly assign different weights to different observations. To model such behavioral biases, we assume that they place weight $\omega_1$ on the most recent observation $y_t$, $\omega_2$ on the second most recent observation $y_{t-1}$, and so on, with the possibility that $w_i$'s may not equal to 1. In other words, we modify the likelihood function as

$$L(\mu) = \prod_{i=1}^{t} L(y_{t-i+1} | \mu)^{\omega_i}. \tag{3}$$

The posterior distribution then becomes

$$P(\mu | y_1, ..., y_t) \propto \prod_{i=1}^{t} L(y_{t-i+1} | \mu)^{\omega_i} \cdot 1 \tag{4}$$

As a result, the posterior mean and posterior variance of the unknown mean can be obtained and the price dynamic under the behavioral model can be summarized in the following proposition:
Proposition 1 \ (Price dynamic under a pseudo-Bayesian approach).

1) Under a pseudo-Bayesian approach with a vague prior and an incorrect likelihood $L(\mu)$ as stated in (3), for any $k \geq 1$ the posterior mean $E_{t+k}^s y_{t+k}$ and posterior variance $\sigma_t^2$ of $\mu$ become

$$E_{t+k}^s y_{t+k} = \sum_{i=1}^{t+k} \omega_i y_i + \cdots + \omega_t y_t$$

and

$$\sigma_t^2 = \frac{\sigma_y^2}{s_t},$$

respectively where $s_t = \sum_{i=1}^{t} \omega_i$.

2) The price at time $t$ using the rational expectations pricing model in (1) becomes

$$P_t = \frac{N_t}{r} + \frac{(1+r) \sum_{i=1}^{t} \omega_i y_i + \cdots + \omega_t y_t}{r^2 s_t},$$

where $N_t = \sum_{i=1}^{t} y_i$.

2. Weight assignment schemes to reflect cognitive biases

In the model above, we incorporate general weights on observations into a simple asset-pricing setup. This allows us to examine the price formation process under a rational expectations approach with biased weights. This approach enables practitioners and academics to compare ways in which investors, with or without cognitive biases, incorporate their prior beliefs into the historical data to estimate valuation-relevant parameters that can lead to anomalous
asset-price behavior. We note that the idea of using different weights on evidences is not new in the finance literature. For example, Brav and Heaton (2002) consider weights given by

\[
\omega_1 = \cdots = \omega_{\text{t}} = 1, \quad \omega_{\text{t}+1} = \cdots = \omega_{\text{L}} = 0.
\]

Under this weighting scheme, investors simply ignore the distant half of the available data. Also, it is common in the psychological literature to assume that investors calculate the posterior mean, which is a weighted average rather than a simple average as suggested by a correct Bayesian approach. In this paper, we use a more general assumption that investors may use weights, \( \omega_1, \omega_2, \ldots, \) satisfying \( 0 \leq \omega_i \leq 1 \) for all \( i \). By allowing more flexibility in the choice of weights, investors’ various behavioral biases can be represented quantitatively. Specifically, in (A), (B), and (C) below, we spell out three weight assignment schemes to characterize the conservative and/or representative heuristics.

(A) **Investors using a conservative heuristic assign weights as:**

\[
0 \leq \omega_1 < \omega_2 < \cdots < \omega_{n_1} = \omega_{n_1+1} = \cdots = 1,
\]

(B) **Investors using a representative heuristic assign weights as:**

\[
1 = \omega_1 = \omega_2 = \cdots = \omega_{m_0} > \omega_{m_0+1} > \omega_{m_0+2} > \cdots \geq 0, \text{ and}
\]

(C) **Investors using both conservative and representative heuristics assign weights as:**

\[
0 \leq \omega_1 < \omega_2 < \cdots < \omega_{n_0} = \omega_{n_0+1} = \cdots = \omega_{m_0} = 1 > \omega_{m_0+1} > \cdots \geq 0.
\]
Figure 1A. Weights ($w_i$) assigned to evidence $i$-days ago for an investor using conservatism heuristics

Figure 1B. Weights ($w_i$) assigned to evidence $i$-days ago for an investor using representativeness heuristics
Figure 1C. Weights \((w_i)\) assigned to evidence \(i\)-days ago for an investor using conservatism heuristics, conservatism and representativeness heuristics.

For a graphical representation of the weights used by the conservative investors, see Figure 1A. Note that the weight assignment scheme of \(\omega_1 < \omega_2 < \cdots < \omega_{n_0} = 1\) is consistent with the psychological literature on conservative heuristics as reviewed in the introduction. Basically, people are overconservative in that they underweight recent information and overweigh prior information. The parameter \(n_0\) reflects the conservative heuristic that most recent \(n_0\) observations are underweighted. If Edwards (1968) is right in that it takes two to five observations to do one observation’s worth of work in inducing a subject to change his/her opinions, \(\omega_1, \omega_2, \ldots, \omega_{n_0}\) can be substantially less than 1 for \(n_0 \leq 5\). The smaller are the weights, the more conservative are the investors. Thus, the magnitudes of the weights \(\omega_1, \omega_2, \ldots, \omega_{n_0}\) can be used to measure the degree of conservatism. The evidence suggests that underreaction reflects the uncertainty regarding possible structural change in the data and a lack of knowledge that a change occurred. This will result in a failure to fully incorporate the price implications of this change into the estimation of the valuation-relevant parameters.
For a graphical representation of the weights used by the investor with the representative heuristic, see Figure 1B. Note that the weight assignment in Scheme B is consistent with the psychological literature on the representative heuristic, as reviewed in the introduction. The representative heuristic in behavioral finance is often described as the tendency of experimental subjects to overweigh recent clusters of observations and underweigh older observations that would otherwise moderate beliefs. Heavy weights on recent data could be a reaction to concern with structural change. Whenever such change occurs, the weight placed on recent data will be very high or similarly the weight placed on the older data will be very low, which will result in a pattern of overreaction caused by the representative heuristic. The representative heuristic is characterized by a parameter \( m_0 \) showing that the investor underweight the observations beyond the most recent \( m_0 \) data points. Here, the parameter \( m_0 \) arises from the “law of small numbers” (Tversky and Kahneman 1971) in the mind of the investor. Because of their representative heuristic, investors have the tendency to treat a small sample size, like \( m_0 \), as large enough to represent the whole population. So, they assign weights much smaller than 1 for observations beyond the most recent \( m_0 \) observations. Put differently, the weights \( \omega_{m_0+1}, \omega_{m_0+2}, \ldots \) are assigned to be much smaller than 1. Also, we assume here that \( \sum_{i=m_0+1}^{\infty} \omega_i < \infty \) because if the sum equals to infinity, the law of large numbers is still at work. For a genuine belief in the law of small numbers, the sum should be finite, meaning that the small sample of the most recent observations can play an overwhelming role in the inference process.

Our model formulation asserts that investors are influenced by the conservative and representativeness heuristics simultaneously. This is different from the formulation in BSV in which investors are under the influence of one heuristic and then suddenly shift to another.
regime of being influenced by the other heuristic. In other word, conservatism and representativeness are not mutually exclusive and investors can be simultaneously influenced by both heuristics at any point in time. A graphical representation of the weights used by investors with both heuristics is displayed in Fig 1C. When the investor is under the influence of both heuristics, the model has two parameters $n_o$ and $m_o$ as described above. Here, conservatism is reflected by the existence of $n_o > 0$ and the smallness of the sum $\omega_1 + \cdots + \omega_{n_o-1}$, and representativeness is reflected by the existence of $m_o < \infty$ and the smallness of the sum $\omega_{m_o+1} + \omega_{m_o+2} \cdots$.

Notice that a type (C) investors degenerate into a type (A) investors when $m_o = \infty$ and degenerate into a type (B) investors when $n_o = 0$. Also when $m_o = \infty$ and $n_o = 0$, all weights are equal to 1 and the investor has no behavioral bias. In this sense, the third type of investors embraces all other types. Thus, it suffices to consider investors of the third type. To fully understand the price anomalies that will be introduced when incorrect weights are assigned to the likelihood function, we investigate its implication to market volatility in Section 3 and investigate its implications for under- and overreaction in Sections 4 and 5.

3. Model’s Implications for Excess Volatility

3.1. Market volatility under the behavioral model

Since volatility in the market is one of the most interesting aspects of finance theory, in this section, we study the magnitude of market volatility under our behavioral model. Under our behavioral model with mis-specified weights $\omega_i$’s, the asset price $P_t$ measured in a log-scale
follows a stochastic process given by (6) in which the earnings $y_i$’s are i.i.d. $N(\mu, \sigma^2)$ random variables as stated in Assumption 1’. Similarly, the price, $P_{t+1}$, at $t+1$ can be expressed as

$$P_{t+1} = \frac{N_{t+1}}{r} + \frac{1+r}{r^2} \left( \frac{\omega_{t+1} y_1 + \ldots + \omega_{t+1} y_{t+1}}{s_{t+1}} \right).$$

Thus, the 1-period return, $R_{t,t+1} = P_{t+1} - P_t$, from time $t$ to time $t+1$ is given by

$$R_{t,t+1} = \frac{1}{r} y_{t+1} + \frac{1+r}{r^2} \left[ \left( \frac{\omega_{t+1} - \omega_t}{s_{t+1}} \right) y_1 + \ldots + \left( \frac{\omega_{t+1} - \omega_t}{s_{t+1}} \right) y_t + \left( \frac{\omega_{t+1} - \omega_t}{s_{t+1}} \right) y_{t+1} \right]$$

$$- \frac{1+r}{r^2} \left[ \left( \frac{\omega_{t+1} - \omega_t}{s_{t+1}} \right) y_1 + \ldots + \left( \frac{\omega_{t+1} - \omega_t}{s_{t+1}} \right) y_t \right] + \left( \frac{1+r}{r^2} \frac{\omega_t}{s_{t+1}} \right) y_{t+1}.$$

Hence, its variance can be expressed as

$$Var(R_{t,t+1}) = \left( \frac{1+r}{r^2} \right)^2 \left[ \left( \frac{\omega_{t+1} - \omega_t}{s_{t+1}} \right)^2 + \ldots + \left( \frac{\omega_{t+1} - \omega_t}{s_{t+1}} \right)^2 \right] \sigma_y^2 + \left( \frac{1+r}{r^2} \frac{\omega_t}{s_{t+1}} \right)^2 \sigma_y^2.$$

### 3.2. Market volatility at equilibrium

Notice that when $t$ is small, the investor is still learning about the economic structure. The learning process becomes complete when $t$ gets large. Since we want to distinguish whether the excess volatility is contributed by learning or by behavioral biases, we study the equilibrium situation when $t$ tends to infinity. We adopt this same treatment when we study over- and underreaction in the later sections as well.

When $s_i \to \infty$ as $t \to \infty$, one can show that $\lim_{t \to \infty} Var(R_{t,t+1}) = \frac{1}{r^2} \sigma_y^2$ (see Appendix 1), which can be regarded as the basic volatility due to information uncertainty. It is intuitive that
information uncertainty is a determinant of price volatility because the information process $N_t$ is a random walk, and hence, past information will have a permanent price effect. Thus, the most recent earnings shock $y_t$, with a variance equal to $\sigma_y^2$, will induce volatility in prices. In addition to information uncertainty, volatility may arise because of investors’ uncertainty about the values of the valuation-relevant parameters. However, as $t \to \infty$, more information is accumulated, and parameters can be estimated with greater accuracy if we assume that $s_t \to \infty$. As a consequence, under the condition that $s_t \to \infty$, uncertainty due to parameter estimation will vanish when the system reaches an equilibrium as $t$ tends to infinity.

However, the situation is different if there are substantial behavioral biases, characterized by the condition that $\lim_{t \to \infty} s_t = s_\infty < \infty$. Under this condition, true parameter values will never become known to market participants, and hence the condition $s_\infty < \infty$ introduces another kind of uncertainty to the price process. As a result of the cognitive bias, volatility can also arise from the uncertainty in the valuation-relevant parameter estimation. Proposition 2 below shows how market volatility will be affected by the presence of behavioral biases.

**Proposition 2.** If we assume that behavioral biases are severe, i.e., $\lim_{t \to \infty} s_t = s_\infty < \infty$, the limiting variance of the 1-period return is given by

$$
\left[ \frac{1}{r^2} + 2 \left( \frac{1}{r} \right) \left( \frac{1+r}{r^2} \right) \frac{s_1}{s_\infty} \right] \sigma_y^2 + \frac{(1+r)^2}{r^4} \left[ 1 - \frac{1}{s_\infty} \right] A_\infty \sigma_y^2
$$

where $A_\infty = \omega_1^2 + \sum_{i=1}^{\infty} (\omega_{i+1} - \omega_i)^2$. 

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In Proposition 2, market volatility is decomposed into two parts: the volatility due to information uncertainty and the volatility arising from behavioral biases. Specifically, the volatility due to information uncertainty is given by

\[ Var_{\infty} \text{ (information uncertainty)} = \frac{1}{r^2} \sigma_y^2, \quad (10) \]

and the volatility attributable to behavioral biases is given by

\[ Var_{\infty} \text{ (behavioral biases)} = \left[ 2 \left( \frac{1}{r} \right) \left( \frac{1 + r}{r^2} \right) \left( \frac{s_1}{s_{\infty}} \right) + \left( \frac{1 + r}{r^2} \right)^2 \left( \frac{1}{s_{\infty}^2} \right) \right] \sigma_y^2. \quad (11) \]

It is interesting to compare the volatilities arising from these two different sources by computing their ratio. Dividing (11) by (10), the volatilities ratio is equal to

\[ \frac{Var_{\infty} \text{ (behavioral biases)}}{Var_{\infty} \text{ (information uncertainty)}} = 2 \left( \frac{1 + r}{r} \right) \left( \frac{s_1}{s_{\infty}} \right) + \left( \frac{1 + r}{r} \right)^2 \left( \frac{1}{s_{\infty}^2} \right) A_{\infty}. \quad (12) \]

Expression (12) shows that the ratio depends on the mis-specified weights \( \omega_i \)'s. Since earnings announcements are usually made quarterly, the time period from \( t \) to \( t + 1 \) can be taken to represent one quarter of a year and hence \( r \) could be the discount rate for one quarter of a year.

To have a better idea of the magnitude of this ratio, we consider the following special case: \( \omega_1 = \cdots = \omega_{p_0} = 1 > \omega_{p_0+1} = \cdots = 0 \). Expression (12) then reduces to

\[ 2 \left( \frac{1 + r}{p_0 r} \right) + 2 \left( \frac{1 + r}{p_0 r} \right)^2 \approx 2 \left( \frac{1}{p_0 r} \right) + 2 \left( \frac{1}{p_0 r} \right)^2. \]
Obviously, from this ratio, the percentage of excess volatility increases as $p_0$ decreases. Since $p_0$ represents a bias coming from the “law of small numbers” (Tversky and Kahneman 1971), for this kind of bias, the small sample can be as small as 15 or 30, representing a horizon of 3.75 to 7.5 years of data. If we assume a quarterly discount rate $r = 2\%$, the volatility ratio equals $2\left[\frac{1}{0.02(30)}\right] + 2\left[\frac{1}{0.02(30)}\right]^2 = 8.83$ when $p_0 = 30$. This result simply shows that the volatility attributable to cognitive bias can be much larger than the volatility attributable to information uncertainty. In fact, if the “law of small numbers” operates on an even smaller scale, say, $p_0 = 15$, then the volatility ratio can be as large as 28.8.

Other than the magnitude of the excess volatility, we can make three interesting observations about excess volatility.

**Observation 1.** *Excess volatility is a decreasing function of the discount rate or investor’s anticipated return $r$.*

This observation follows trivially from (12) because the volatility ratio depends on $\frac{1+r}{r}$ which increases as $r$ decreases.

**Observation 2.** *Conservative heuristics will reduce excess volatility.*

This observation follows from a simple computation of the partial derivative $\frac{\partial}{\partial \omega_1}(Var_x(behavioral \ bias))$, which ends up with a positive result. In other words, when $\omega_1$
gets smaller, the excess volume becomes less. Since a smaller \( \omega_j \) means that the investor is more conservative, the conservative heuristic actually reduces excess volatility.

**Observation 3.** Representative heuristics will increase excess volatility.

This observation follows from a simple computation of the partial derivative

\[
\frac{\partial}{\partial \omega_j} \left( \text{Var}_n \left( \text{behavioral bias} \right) \right) \quad \text{when} \quad j \quad \text{is large.}
\]

It can be shown that the partial derivative is less than zero when \( j \) is large. This means that when \( \omega_j \) decreases, excess volatility increases. In other words, greater representativeness contributes to an increase in excess volatility.

4. **Model’s Implications for Under- and Overreaction**

4.1. **Measures of Under- and Overreaction**

Overreaction refers to the predictability of good (bad) future returns from bad (good) past performance, while underreaction refers to the predictability of good (bad) future returns from good (bad) past performance (DeBondt and Thaler 1985; LSV). In this paper, we further investigate how investors’ cognitive biases affect the resulting prices of an asset in the hope of explaining the underreaction and overreaction phenomena in stock markets. In many empirical studies, under- and overreaction are addressed in a portfolio context (DeBondt and Thaler 1985, 1987, 1990; Jegadeesh and Titman 1993) in which winner and loser portfolios are constructed by picking the best and worst performers in the formation period, respectively. Underreaction and overreaction are then defined according to the subsequent performance of the winner and loser
portfolios in the test period. In the literature, alternative approaches are also employed to study the under- and overreaction phenomena by examining the time series properties of the prices of a single asset. In the context of a single asset, under- and overreaction could be demonstrated either through return autocorrelations or through the abnormal return under an event approach (BSV). In this paper, both approaches are employed to illustrate the under- and overreaction phenomena documented by psychologists. Section 4.1.1 adopts a correlation approach and Section 4.1.2 deals with the same concept using an event approach.

1. Under-and-overreaction in terms of correlation coefficients

Consider the $k$-period return $R_{t,t+k}$ from time $t$ to time $t+k$ and the $k$-period return, $R_{t-k,t} = P_t - P_{t-k}$ from time $t-k$ to time $t$. The correlation coefficient between these two returns can be interpreted as the lag-one autocorrelation of the $k$-period return. Since underreaction is associated with positive autocorrelation and overreaction is associated with negative autocorrelation, we define short-term underreaction and long-term overreaction as follows:

**(I).** Prices of a single asset exhibit a short-term underreaction (in terms of return correlation) if the $k$-period return has a positive lag-one autocorrelation for sufficiently small $k$.

**(II).** Prices of a single asset exhibit a long-term overreaction (in terms of return correlation) if the $k$-period return has a negative lag-one autocorrelation for sufficiently large $k$.

We note that the above definition of under- and overreaction is consistent with the mean reversion phenomena reported by Fama and French (1988), who show that long-holding-period returns are significantly negatively serially correlated. Because of the existence of such negative serial correlation, a large percentage of the variance of large-horizon returns is predictable from
past returns. This phenomenon is called mean reversion in the finance literature. Notice that the short-term underreaction and long-term overreaction as defined in (I) and (II) above are by no means mutually exclusive. Put differently, \( K_1 \) and \( K_2 \) can exist simultaneously so that the \( k \)-period returns are positively autocorrelated for \( k < K_1 \) and are negatively autocorrelated if \( k > K_2 \).

2. Under-and-overreaction under an event approach

Analogous to the definition of underreaction and overreaction in terms of correlation coefficients at the start of the previous subsection, we now discuss an alternative way to measure under- and overreaction as in the event approach used by BSV. Under this approach, the market is said to have underreacted when the average return on the company’s stock in a period following an announcement of good news is higher than that in a period following an announcement of bad news. Quoting BSV, “The stock under-reacts to the good news, a mistake which is corrected in the following period, giving a higher return at that time.” However, when pieces of news come in continuing strings, the opposite phenomenon may occur. Put differently, the average return following a series of good news announcements turns out to be lower than that following a series of bad news announcements. This is described as the long-term overreaction phenomenon documented in psychology. To quantify such under- and overreaction in the sense used by BSV, we note that the earnings shock, \( y_t \), provides a measure of how good or bad the earning is. Since the earnings shock \( y_t \) follows a \( N(\mu, \sigma^2) \) distribution, a piece of good (bad) news can be viewed as one in which the earnings shock is larger (smaller) than \( \mu + s\sigma_y \).
\( (\mu - s\sigma_y) \). Now consider the difference in average returns after a string of good or bad news and define \( U_i(s, j) \) as follows:

\[
U_i(s, j) = E \left\{ R_{t+1} \left| y_t > \mu + s \cdot \sigma_y, \ldots, y_{t-j+1} > \mu + s \cdot \sigma_y \right\} - E \left\{ R_{t+1} \left| y_t < \mu - s \cdot \sigma_y, \ldots, y_{t-j+1} < \mu - s \cdot \sigma_y \right\} \right. \\
\]

(13)

In (13), \( j \) represents the time length of the string of good or bad news, \( s > 0 \) represents the intensity of the news content, and the quantity \( U_i(s, j) \) represents the expected profit of a momentum trading strategy that dictates buying when there is a string of good news and selling when there is a string of bad news. If \( U_i(s, j) \) is positive, the momentum trading strategy is profitable resulting from a market underreaction. Otherwise, the contrarian trading strategy is profitable, signifying the existence of an overreaction. A formal definition of short-term underreaction and long-term overreaction can now be given as follows:

**(III).** Prices exhibit a short-term underreaction if \( U_i(s, j) > 0 \) for sufficiently small \( j \).

**(IV).** Prices exhibit a long-term overreaction if \( U_i(s, j) < 0 \) for sufficiently large \( j \).

Notice that the short-term underreaction and long-term overreaction as defined above are by no means mutually exclusive. Just like the existence of constants \( K_1 \) and \( K_2 \) in the previous section, constants \( J_1 \) and \( J_2 \) can both exist, so that \( U_i(s, j) > 0 \) for \( j < J_1 \) and \( U_i(s, j) < 0 \) for \( j \geq J_2 \).
4.2. **Under- and-overreaction in the presence of behavioral biases**

In this section, we assume that the representative investor possesses both conservative and representative heuristics and assigns weights to data as described by (C) in Section 3. We will show in Proposition 3 that asset prices will exhibit underreaction in the short run and overreaction in the long run, where under- and overreaction is measured by return autocorrelations. Proposition 3 is important because it shows that our behavioral model implies that returns are predictable, a well-documented market anomaly in the finance literature. In Proposition 3, predictability results even after the system has reached equilibrium and hence it does not arise only from the investors’ learning process. We then demonstrate short-term underreaction and long-term overreaction phenomena using an event approach in Proposition 4. Specifically, for both under- and overreaction, we further define in Section 5 what we mean by a “magnitude effect,” alternatively known as the second part of the under- or overreaction hypothesis. We then prove in Proposition 5 that when investors exhibit both types of behavioral biases, the under- and overreaction observed in Proposition 4 displays a “magnitude effect.”

**Proposition 3.** If investors possess both conservative and representative heuristics, then prices exhibit short-term underreaction and long-term overreaction in terms of return autocorrelations (see I and II in Section 4.1.1). Specifically, there exist positive integers $K_1$ and $K_2$ such that for sufficiently large $t$, we have

\[
\begin{align*}
(i) \quad & \text{Corr}(R_{t-k,t}, R_{t,t+k}) > 0 \quad \text{for} \quad k \leq K_1, \\
& \text{Corr}(R_{t-k,t}, R_{t,t+k}) < 0 \quad \text{for} \quad k > K_2.
\end{align*}
\]
Furthermore, the correlation coefficients in (i) is non-trivial for sufficiently large \( t \), i.e.

\[
(ii) \quad \begin{cases} 
\lim_{t \to \infty} \text{Corr}(R_{i-k,t}) > 0 & \text{for } k \leq K_1, \\
\lim_{t \to \infty} \text{Corr}(R_{i-k,t}) < 0 & \text{for } k > K_2.
\end{cases}
\]

As explained in Section 4.1.2, under- and overreaction can also be treated using an event approach. Under this approach, under- and overreaction is measured by the expected momentum profit \( U_t(s,j) \) defined in (13). We will show in Proposition 4 below that when an investor possesses both types of behavioral biases, \( U_t(s,j) \) is positive when \( j \) is small and is negative when \( j \) is large. In other words, momentum trading is profitable on a short run of good or bad news but contrarian trading is profitable on a long-run of good or bad news. This signifies short-term underreaction and long-term overreaction.

**Proposition 4.** If investors possess both conservative and representative heuristics, prices exhibit short-term underreaction and long-term overreaction using an event approach (see III and IV in Section 4.1.2). Specifically, we have

(i) there exist integers \( J_1 \) and \( J_2 \) such that for given \( s > 0 \) and for large \( t \), we have

\[
U_t(s,j) > 0 \quad \text{for} \quad j \leq J_1,
\]

\[
U_t(s,j) < 0 \quad \text{for} \quad j > J_2;
\]

(ii) the expected momentum trading profit \( U_t(s,j) \) is non-trivial when \( t \) tends to infinity, i.e.,

\[
\lim_{t \to \infty} U_t(s,j) > 0 \quad \text{for} \quad j \leq J_1,
\]

\[
\lim_{t \to \infty} U_t(s,j) < 0 \quad \text{for} \quad j > J_2.
\]
We first state a lemma from which Proposition 3 follows naturally.

**Lemma 1.** \( U_i(s, j) = 2\sigma_y \frac{1 + r}{r^2} \left[ \Delta(t, j) \right] D(s) \)

where \( \Delta(t, j) = \frac{s_{j+1} - s_1}{s_{t+1}} - \frac{s_j}{s_r} \) and \( D(s) = E(Z|Z > s) \). (14)

in which \( Z \) represents a standard normal random variable with mean zero and unit standard deviation.

This proposition links investors’ irrational cognitive biases to financial anomalies of overreaction and underreaction. Empirically, overreaction and underreaction could arise in different kinds of environments. Overreaction occurs after long-run periods of good or bad performance, whereas underreaction happens after short-run periods of good or bad performance. These environments fit well with our proposition. In addition to demonstrating the existence of overreaction in the long run, Proposition 4 also provides good insights into how the contrarian/momentum profits arise. Since

\[ U_i(s, j) \rightarrow 2\sigma_y \frac{1 + r}{r^2} \frac{\omega_{j+1} - \omega_1}{s_{r^2}} D(s), \] (15)

a contrarian trading strategy will derive a profit when \( \omega_{j+1} < \omega_1 \) as \( t \rightarrow \infty \). Note that a small \( \omega_j \) signifies a heavy bias in representativeness and a small \( \omega_i \) signifies a heavy bias in conservatism. Thus, it is obvious that the representative heuristic has to overpower the conservative heuristic for a contrarian profit to surface. The long-run assumption is necessary for a contrarian profit because under a long-run situation the representativeness bias will become noticeable. On the
other hand, the momentum profit when \( j \) is small arises from the conservative heuristic and the representative heuristic plays no role in determining the momentum profit.

Another interesting observation is that both momentum/contrarian profits are sensitive to the discount rate \( r \). The smaller the discount rate, the larger the momentum/contrarian profits. This is because when \( r \) is small, future cash flows become important, and a mis-estimation of future cash flows will intensify the over- or underreaction phenomena.

5. **Model’s Implications for the Magnitude Effect**

5.1. **Existence of a magnitude effect**

In this section, we will provide theoretical support for the second part of the under- and overreaction hypotheses. Recall that the first part of the overreaction (underreaction) hypothesis in DeBondt and Thaler (1985) (Jegadeesh and Titman 1993) is “extreme movements in stock prices will be followed by subsequent price movements in the opposite (same) direction” and the second part of the overreaction (underreaction) hypothesis is “the more extreme the initial price movement, the greater will be the subsequent adjustment.” Even though the empirical tests and/or theoretical explanations of the first part of the overreaction hypothesis have been heavily studied and uncovered by financial economists, the second part of the overreaction hypothesis is seldom systematically addressed by researchers. In this paper, we show that our quantitative behavioral model can provide theoretical support to the second part of the under- and overreaction hypotheses by showing that the magnitude effect exists under our behavioral specification. Recall that in the definition of \( U_c(s,j) \) in (13),
\[ U_i(s, j) = E \left\{ R_{t+1} \middle| y_t > \mu + s\sigma_y, \ldots, y_{t-j+1} > \mu + s\sigma_y \right\} - E \left\{ R_{t+1} \middle| y_t > \mu - s\sigma_y, \ldots, y_{t-j+1} > \mu - s\sigma_y \right\}. \]

\( U_i(s, j) \) stands for the expected profit of the momentum trading strategy. Observe that both parameters \( s \) and \( j \) represent an “event magnitude.” For the parameter \( s \), the larger is \( s \), the more extreme is the earnings shock, and the more extreme is the event under study. On the other hand, the parameter \( j \) represents another dimension of “event magnitude.” If \( j \) is large, the event consists of a bigger clustering of good or bad news and the event becomes more extreme as \( j \) gets larger. Thus, the “magnitude effect” associated with the under- or overreaction may have two meanings:

(1) the momentum (contrarian) profit \( U_i(s, j)(-U_i(s, j)) \) increases as \( s \) increases,

(2) the momentum (contrarian) profit \( U_i(s, j)(-U_i(s, j)) \) increases as \( j \) increases.

We demonstrate that both magnitude effects exist in Propositions 5 and 6.

**Proposition 5** (a magnitude effect in \( s \)). If investors possess both conservative and representative heuristics, both the long-term overreaction and the short-term underreaction established in Proposition 4 will exhibit a magnitude effect in \( s \). In other words, both the momentum and contrarian strategies of trading on a string a good or bad news will have a profit that will increase with the magnitude of the impact of the news. Specifically, there exist integers \( J_1 \) and \( J_2 > 0 \) such that

(a) for sufficiently small \( t \), and for \( j \leq J_1 \), the momentum profit \( U_i(s, j) \) is positive and is monotonically increasing with \( s \);
(b) for sufficiently large $t$, and for $j \geq J_2$, the contrarian profit $-U_t(s, j)$ is positive and is monotonically increasing with $s$.

**Proposition 6.** (a magnitude effect in $j$).

1. When $j$ is sufficiently large, the contrarian profit based on $j$ consecutive good or bad news increases as $j$ increases.

2. When $j$ is sufficiently small, the momentum profit based on $j$ consecutive good or bad news increases as $j$ decreases.

### 5.2. Convexity in the magnitude effect

In Section 4.1, we demonstrate that there is a magnitude effect in the under-or-overreaction phenomena, in the sense that momentum/contrarian trading profit increases with the magnitude of the earnings shock. In this Section, we go one step further to show in Proposition 7 that when $s$ is used as a magnitude measure, the magnitude effect is convex in nature. For example, when magnitude doubles, the momentum/contrarian trading profit is more than doubled.

**Proposition 7.** If investors possess both conservative and representative heuristics, the momentum/contrarian trading profit $U_t(s, j)$ is a convex function in $s$. 
6. Concluding Remarks

We posit that some investors possess conservative and/or representative heuristics that leads them to underweight recent observations and/or underweight past observations in the earnings shocks of stock prices. We introduce a quantitative pseudo-Bayesian approach to model such investors’ behavior.

Compared with other behavioral models in which investors possess either conservative heuristics at one time or representative heuristics at another time but not both, our specification captures the essential feature of either conservative or representative biases in one parsimonious model that allows investors to possess conservative or representative heuristics at the same time.

This paper develops a model of weight assignments using a pseudo-Bayesian approach to reflect investors’ behavioral biases. In this parsimonious model of investors’ sentiment, weights induced by investors’ conservative and representative heuristics are assigned to observations of the earnings shocks of stock prices. Our behavioral model provides a quantitative link between some market anomalies and investors’ behavioral biases. While learning may contribute to market anomalies, anomalies still exist even after the learning process has been completed. In particular, we can deduce the following: (1) Excess market volatility will result from investors’ biased heuristics. The representative heuristic, rather than the conservative heuristic, contributes to excess volatility in the market. Excess volatility is more prominent when the discount rate is small. (2) Through a misapplication of Bayes’ rule, investors’ behavioral biases lead to short-term underreaction and long-term overreaction in the markets. The more conservative/representative the heuristic, the larger is the magnitude of the return auto-correlation. Further
analysis shows that the representative heuristic contributes to the contrarian trading profit and the conservative heuristic contributes to the momentum profit. The smaller the discount rate, the larger the contrarian/momentum profit. (3) Investors’ behavioral biases induce a magnitude effect in the under- and overreaction phenomena documented in psychology, i.e., the more severe the earning shock, the larger the market autocorrelation and the larger the momentum/contrarian trading profit. (4) The magnitude effect described in (3) is convex in nature.

Understanding Investors’ behaviors will be useful in investment decision making. The information of companies (Thompson and Wong, 1991, 1996; Nishihara and Fukushima, 2008), the Economic and Financial environment (Fong, Wong and Lean, 2005; Broll, Wahl and Wong, 2006), technical analysis (Wong, Chew and Sikorski, 2001), fundamental analysis (Samaras, Matsatsinis and Zopounidis, 2008) and other advanced econometrics techniques (Li and Lam, 1995; So, Li and Lam, 1997; So, Lam and Li, 1998) can also be utilized to make better investment decisions. Another extension to improve the investment decision making is to study other behaviors of investors (Wong, 2007; Wong and Chan, 2008) or to incorporate Mean–variance portfolio optimization (Buckley, Saunders, Seco, 2008; Leung and Wong, 2008; Josa-Fombellida and Rincón-Zapatero, 2008; Zhao and Ziemba, 2008) and stochastic dominance criterion (Gasbarro, Wong and Zumwalt, 2007; Post, 2008; Wong and Ma, 2008) to study investors’ conservative and representative heuristics.
References


