Abstract—The 2-parameter family of probability distributions introduced by Birnbaum & Saunders characterizes the fatigue failure of materials subjected to cyclic stresses and strains. It is shown that the methods of accelerated life testing are applicable to the Birnbaum–Saunders distribution for analyzing accelerated lifetime data, and the (inverse) power law model is used due to its justification for describing accelerated fatigue failure in metals. This paper develops the (inverse) power law accelerated form of the Birnbaum–Saunders distribution, and explores the corresponding inference procedures—including parameter estimation techniques and the derivation of the expected Fisher information matrix. The model approach in this paper is different from an earlier work, which considered a log-linear form of a model with applications to accelerated life testing. Here, using an example data set, the fitted model is effectively used to estimate lower distribution percentiles and mean failure times for particular values of the acceleration variable. The benefits of having an operable closed form of the Fisher information matrix, which is unique to this article for this model, include interval estimation of model parameters and LCB on percentiles using relatively simple computational procedures.

Index Terms—Cycles to failure, fatigue life, laplace method, maximum likelihood, percentile estimation.

ABBREVIATIONS & ACRONYMS

ASD asymptotic standard deviation
B–S Birnbaum–Saunders (distribution)
GLM general linear model
LCB lower s-confidence bounds
LSE least squares estimate
MLE maximum likelihood estimator
MTTF mean time to failure

Notation

\[ \alpha, \beta \] B-S [shape, scale] parameter
\[ V \] level of applied accelerated (maximum) stress
\[ h(V) \] acceleration model
\[ h_{\text{ps}}(V) \] power law acceleration model
\[ t \] cycles-to-failure variable
\[ F(t), G(t) \] Cdf’s of \( t \) distributed as

\[ \hat{\theta}, \hat{\gamma} \] parameter estimates by least-squares methods
\[ k \] number of unique accelerated stress levels
\[ V_i \] level \( i \) of applied accelerated (maximum) stress
\[ V_i \] index for \( V_i, i = 1, 2, \cdots, k \)
\[ n_i \] number of items tested at \( V_i \)
\[ t_{i,j} \] index for failure times observed for a \( V_i, j = 1, 2, \cdots, n_i \)
\[ \text{gau}(\cdot) \] standard Gaussian Cdf
\[ \theta \] likelihood function
\[ L(\cdot) \] log(\( L(\cdot) \))

Other, standard notation is given in “Information for Readers & Authors” at the rear of each issue.

I. INTRODUCTION

In materials engineering, “fatigue” is essentially a decrease in the strength of the material when it is subjected to fluctuating/cyclic stresses. Fatigue has been recognized as the cause of failure in metals and concrete structures, where failure is usually crack-induced. For example, [1] considered the fatigue failure in steel gun-barrels and estimated the distribution of the “critical crack size” that would result in failure of the specimen under test as the stress increases. Other modern materials, like carbon fibrous composites, are assumed to fatigue by developing intangible “stress concentrations” within the material; these ultimately result in catastrophic failure [2]. The development of size-dependent probability distributions to model the failure of complex systems (like carbon fibrous composites) under tensile stress using “cumulative damage” arguments are considered in [3], [4]. This paper models crack-driven fatigue observed in materials such as metals.

A. Model Selection and Accelerated Testing

For modeling fatigue data, statistical distributions are often used because the failure times of a fatiguing material can vary due to manufacturing variation or random flaws in the material. Popular distributions for failure data are the right-skewed, unimodal 2-parameter families. These include lognormal, Weibull, and gamma; all of them fit failure data fairly well within the central region of the distribution. However, engineers are often concerned with predicting the “safe life” at lower percentiles in the strength distribution, and these 3 probability distributions are all quite different with respect to the lower (and upper) percentiles. So, any one of these distributions has the potential of providing drastically erroneous percentile estimates. For this reason, it is important to have a sufficient amount of data in...
the lower tails of the strength/life distribution to be able to discriminate among various distributions. This is not always possible due to the high cost (time, people, money) of data collection, especially when the items under test have a very high reliability. For these situations, one method that is often used to obtain useful life test data is accelerated testing which involves observing the failure of materials under higher levels of stress (e.g., temperature, voltage, force) to shorten the material life. For example, [5] has data on the fatigue life (in cycles) to failure of coupons (rectangular strips) cut from aluminum sheeting that were tested under three levels of maximum stress. Table I shows the data, but they are scaled differently from their original appearance [5]. Higher levels of stress have the effect (on average) of shortening the lifetimes of the coupons. This data set is investigated in Section III.

To extrapolate from the observed accelerated stresses to the normal-use stress, an acceleration model  is used to describe the functional relationship between the parameters of the failure distribution and the environmental stresses. Often, it is assumed that the increased stress has the effect of changing the mean or scale of the failure distribution, but the “family” of failure distributions always remains intact [6]. Furthermore, the functional relationships usually involve unknown parameters to be estimated.

Many acceleration models have been derived from a knowledge of the physical nature of the material placed under test at elevated stresses. Common acceleration models are the:

- (inverse) power law,
- Arrhenius,
- Eyring,

they are described in [6]. The power law model has applications in fatigue testing of metals and the aging of multi-component systems, and is used for this investigation. Typically, is substituted for a mean or scale parameter in a lifetime distribution for an accelerated model. The parameters of the acceleration model are unknown and are estimated from life-test data observed from a collection of (ordered) elevated stress levels, , with (required for parameter estimation). After parameter estimation, inferences can be made, e.g., on the MTTF for a given level of . Ref [7] provides an example of using the power law model coupled with the Weibull distribution for fitting tensile strength data from an experiment testing several gauge lengths of carbon fibers.

### B. The 2-Parameter B-S Distribution

In considering the physical properties of the fatigue process with respect to crack growth, Birnbaum & Saunders [8] derived a fatigue life distribution from assumptions on the extension of a dominant crack under a cyclic loading scheme. Ref [6] presents a review of their derivation along with some of its basic properties. The Cdf of that model is

\[ G(t; \alpha, \beta) = \exp \left[ \frac{-\beta}{\alpha} \left( \frac{t}{\beta} \right)^{1/2} \right], \]

\[ t > 0, \quad \alpha > 0, \quad \beta > 0. \]  

(1)

The measure of interest for a r.v. having (1) as its Cdf is the number of stress cycles until failure, or analogously, time to failure [8]. The distribution (1) is denoted as B–S(, ). In addition to being a scale parameter, is the median of the distribution. Eq (1) has some interesting properties, and many are summarized in [6]. Of interest here, for , are the moments:

\[ \mathbb{E}[T] = \beta \left( 1 + \frac{\alpha^2}{2} \right), \]

\[ \mathbb{E}[T^{-1}] = \beta^{-1} \left( 1 + \frac{\alpha^2}{2} \right). \]

The second property follows from the fact that the B–S distribution possesses the reciprocal property [9]. Ref [10] shows that the B-S distribution also describes a T that is

### TABLE I

**ALUMINUM-COUPONS DATA SET LIFETIMES ARE IN 10^5 CYCLES**

| Sample 1: 101 Observations. Maximum Stress/Cycle = 2.1 · 10^4 psi |
|---|---|

| Sample 2: 102 Observations. Maximum Stress/Cycle = 2.6 · 10^4 psi |
|---|---|
| 2.33, 2.58, 2.68, 2.76, 2.99, 3.10, 3.12, 3.15, 3.18, 3.21, 3.21, 3.29, 3.35, 3.36, 3.36, 3.36, 3.36, 3.42, 3.42, 3.42, 3.44, 3.49, 3.50, 3.50, 3.51, 3.51, 3.52, 3.52, 3.56, 3.56, 3.58, 3.60, 3.62, 3.63, 3.66, 3.67, 3.70, 3.70, 3.72, 3.74, 3.74, 3.75, 3.76, 3.79, 3.79, 3.80, 3.82, 3.89, 3.89, 3.95, 3.96, 4.00, 4.00, 4.00, 4.03, 4.04, 4.06, 4.08, 4.08, 4.10, 4.12, 4.14, 4.16, 4.16, 4.16, 4.20, 4.22, 4.23, 4.26, 4.28, 4.32, 4.32, 4.33, 4.33, 4.37, 4.38, 4.39, 4.39, 4.43, 4.45, 4.45, 4.45, 4.52, 4.56, 4.60, 4.64, 4.66, 4.68, 4.70, 4.70, 4.73, 4.74, 4.76, 4.76, 4.86, 4.88, 4.89, 4.90, 4.91, 5.03, 5.17, 5.40, 5.60 |

| Sample 3: 101 Observations. Maximum Stress/Cycle = 3.1 · 10^4psi |
|---|---|
| 0.70, 0.90, 0.96, 0.97, 0.99, 1.00, 1.03, 1.04, 1.04, 1.05, 1.07, 1.08, 1.08, 1.08, 1.09, 1.09, 1.12, 1.12, 1.13, 1.14, 1.14, 1.16, 1.19, 1.20, 1.20, 1.20, 1.21, 1.21, 1.23, 1.24, 1.24, 1.24, 1.24, 1.28, 1.28, 1.28, 1.29, 1.29, 1.30, 1.30, 1.30, 1.30, 1.31, 1.31, 1.31, 1.31, 1.32, 1.32, 1.32, 1.32, 1.33, 1.34, 1.34, 1.34, 1.34, 1.36, 1.36, 1.36, 1.38, 1.38, 1.39, 1.39, 1.41, 1.41, 1.42, 1.42, 1.42, 1.42, 1.42, 1.44, 1.44, 1.45, 1.45, 1.46, 1.46, 1.48, 1.49, 1.51, 1.51, 1.52, 1.55, 1.56, 1.57, 1.57, 1.57, 1.57, 1.58, 1.62, 1.63, 1.63, 1.64, 1.66, 1.66, 1.68, 1.70, 1.74, 1.96, 2.12 |
observed when any amount of cumulating damage exceeds a critical value. Ref [11] shows that the B–S distribution can be written as a mixture of 2 inverse-Gaussian distributions. Ref [12] shows a 3-parameter generalization of the B–S distribution; it applies to tensile strength models derived using cumulative damage arguments from [3], [4]. The models incorporate a “system size” or “material length” variable which acts as an acceleration variable (since longer specimens under test typically have lower tensile strengths), and this allows for model fitting to data observed over multiple system sizes.

Statistical analysis for (1) has been developed for single samples. Ref [5] derived the MLE for $\alpha$ & $\beta$, which must be obtained with an iterative, numerical root-finding technique. Ref [13] provided hypothesis tests and $s$-confidence intervals for each parameter (assuming the other is an unknown nuisance parameter) by using asymptotic results and Monte Carlo methods. Ref [14] used a Bayes approach to estimate the reliability function for (1) when $\alpha$ is known or unknown. Ref [15] provided Bayes inferences using noninformative priors for the parameters in (1) by obtaining the Fisher information matrix by using the Laplace approximation for complicated integrals [16]. Knowledge of the information matrix can provide joint large-sample $s$-confidence intervals and other inferences for the parameters by using the asymptotic normality theory for MLE.

Ref [17] considered a log-linear model for the B–S model that applies to accelerated life testing for fatigue. Since in (1), $\beta$ is a scale parameter, the r.v. $T \sim B–S(\alpha, \beta)$ can be rewritten as $T = \beta \cdot X$, where $X \sim B–S(\alpha, 1)$. If $\beta$ is replaced by an acceleration model (see Section I-A) then $T = h(V) \cdot X$, where $V$ is the level of stress and $h$ depends on some parameters. The acceleration model [17] suggests $h(V) = \exp[\alpha + b \cdot V]$, and after taking logarithms,

$$\log(T) = a + b \cdot V + \log(X),$$

so the form of the r.v. can be expressed as log-linear. Least squares estimates of $a$ & $b$ can be obtained very simply, but $\alpha$ can be estimated only using alternative methods. The error term $\log(X)$ has the sinh-Gaussian distribution; properties of this distribution are in [17, 18]. The specific sinh-Gaussian distribution describing $\log(X)$ is symmetric about 0 and is unimodal when $\alpha \leq 2$. Thus the log-linear model is very useful since the errors seem to follow an approximate Gaussian distribution, and often in practice, $\alpha < 1$ for “cycles to failure” data [17]. This fact can make LSE of $a$ & $b$ very good, and [17] shows that the LSE are quite efficient for small values of $\alpha$ as compared to MLE based on the sinh-Gaussian distribution.

While using the acceleration model that leads to (2) makes the log-linear model attractive in a GLM framework (and this acceleration model is functionally related to the power law model), this paper uses the more common parameterization, $h_{\text{pl}}(V)$ for an accelerated life test.

Section II presents the model, with its assumptions, along with parameter estimation using MLE and asymptotic theory.

Section III applies the power law accelerated B-S model to the aluminum-coupon fatigue data-set from [5], and compares the results with the approach of [17].

II. POWER LAW ACCELERATED B–S MODEL

As described in Section I, the problem of fitting observed “cycles to failure” fatigue data at several elevated stress levels is addressed using the concepts of accelerated test models and the B–S distribution (1). Substitution of the power law model $h_{\text{pl}}(V)$ for $\beta$ in (1) yields the 3-parameter life distribution:

$$F(t; V) = \Phi(t; \alpha, h_{\text{pl}}(V)) = \frac{1}{\alpha} \cdot \left( \frac{t}{\gamma \cdot V^{\eta}} \right)^{1/2} - \left( \frac{\gamma \cdot V^{\eta}}{t} \right)^{1/2},$$

$$t > 0, \quad \alpha > 0, \quad \gamma > 0, \quad \eta > 0,$$

(3)

To fit the model, the parameters must be estimated from experimental data from a collection of elevated stress levels.

For estimates of the parameters in (3), the model can be written as [1], [4]:

$$\text{gaunt}^{-1}[F(t; V)] = \frac{\sqrt{\eta}}{\alpha \cdot \sqrt{\gamma}} - \frac{\sqrt{\gamma}}{\alpha \cdot \sqrt{t} \cdot V^{\eta/2}},$$

so that the r.h.s. is nonlinear in the model parameters. Then, for each $i = 1, 2, \ldots, k$, by:

- arranging the observed $t_{i,j}$ in increasing order of $j$,
- using the empirical estimate of $F(t; V_i)$ from the $n_i$ observations at stress level $V_i$,

set:

$$\text{gaunt}^{-1} \left[ \frac{j - 0.5}{n_i} \right] = \frac{\sqrt{\eta \cdot t_{i,j}} \cdot V_i^{\eta/2}}{\alpha \cdot \sqrt{\gamma}} - \frac{\sqrt{\gamma}}{\alpha \cdot \sqrt{t_{i,j}} \cdot V_i^{\eta/2}}.$$

(4)

Thus, (4) can be used to obtain a pseudo-least-squares type estimate for $\alpha$, $\gamma$, $\eta$ by using a nonlinear regression routine such as PROC NLIN in SAS. This is one way to obtain parameter estimates, but it is necessary for the $n_i$ to be large enough so that the empirical Cdf can adequately approximate the true Cdf at each stress level.

With the power law accelerated B–S model, a log-linear model, as in [17], can also be derived. Similar to (2) but with acceleration model $h_{\text{pl}}(V)$, the log-linear form is:

$$\log(T) = \log(\gamma) - \eta \cdot \log(V) + \log(X).$$

(5)

Eq (5) is linear in $\log(V)$, and the error term has the same distribution as in (2). Estimates of $\gamma$, $\eta$ can easily be calculated; but again $\alpha$ can only be estimated by using likelihood methods. These two very different regression-type methods are compared by examining a real data set in Section III. Either method can be used for starting values in a more statistically sound numerical MLE procedure, which is investigated next.

The pdf for the Cdf (3) is:

$$f(t; V) = \frac{V^{\eta/2}}{2\sqrt{2\pi} \cdot \alpha \cdot \sqrt{\gamma} \cdot t} \cdot \left( 1 + \frac{\gamma}{t \cdot V^{\eta}} \right) \cdot \exp \left[ -\frac{1}{2\alpha^2} \cdot \left( \frac{t + V^{\eta}}{\gamma} - 2 + \frac{\gamma}{t \cdot V^{\eta}} \right) \right], \quad t > 0.$$

The MLE of the unknown model parameters in (4) are found by maximizing the likelihood function for the observed data:

$$L(\theta) = \prod_{i=1}^{k} \prod_{j=1}^{n_i} f(t_{i,j}; V_i),$$
or equivalently, by maximizing $\log[L(\theta)]$.

The partial derivatives of $\log[L(\theta)]$ with respect to $\alpha$, $\gamma$, $\eta$ yield nonlinear likelihood equations, which are given in (6). The equation for $\alpha$ can be solved in terms of $\gamma$, $\eta$, but the roots of the equations for $\gamma$, $\eta$ must each be found numerically, given the values of the other 2 parameters:

$$
\alpha = \left[ \frac{1}{m} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( \frac{t_{i,j} \cdot V_i^\eta}{\gamma} - 2 + \frac{\gamma}{t_{i,j} \cdot V_i^\eta} \right) \right]^{1/2},
$$

$$
h_2(\gamma) = \frac{m}{2\gamma} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{1}{t_{i,j} \cdot V_i^\eta + \gamma} - \frac{1}{2\alpha^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( \frac{t_{i,j} \cdot V_i^\eta}{\gamma} - \frac{\gamma}{t_{i,j} \cdot V_i^\eta} \right) = 0,
$$

$$
h_2(\eta) = \frac{1}{2} \sum_{i=1}^{k} n_i \cdot \log(V_i) - \gamma \cdot \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( \frac{t_{i,j} \cdot V_i^\eta}{\gamma} - \frac{\gamma}{t_{i,j} \cdot V_i^\eta} \right) = 0.
$$

Iteration over these 3 equations can be performed with a nonlinear root-finding procedure until convergence to an approximate solution for $\alpha$, $\gamma$, $\eta$. The starting values in an iterative procedure are required only for $\gamma$, $\eta$.

Engineers are often interested in values of the lower percentiles in the strength distribution, for the manufacture of reliable systems. The $p$ 100th percentile, $t_p(V)$, of the strength distribution (3) and a given value of $V$ can be calculated by equating (3) to $p$ and then solving for $t$. For a specified $p$,

$$
t_p(V) = \frac{\gamma}{4V_0^\eta} \cdot \left[ \alpha \cdot z_p + \sqrt{\alpha^2 \cdot z^2 + 4} \right],
$$

$$
z_p \equiv \text{gauss}^{-1}(p);
$$

the MLE of (7) can be obtained by substituting the MLE of $\alpha$, $\gamma$, $\eta$. To obtain the LCB on (7), the usual asymptotic likelihood theory can be used for large sample sizes. The method is described next, and is based on the asymptotic distribution of $\hat{\theta} = (\hat{\alpha}, \hat{\gamma}, \hat{\eta})^T$ with the $s$-expected Fisher information matrix or the observed Fisher information [19]. The $s$-expected Fisher information matrix entries are calculated because, for model (3), they are easier to manage & compute, given the data. It is not too difficult to obtain a good closed-form approximation to the $s$-expected Fisher information matrix.

Once the asymptotic distribution of $\hat{\theta}$ is calculated, it can be used for $s$-confidence intervals on $\hat{\theta}$. The asymptotic distribution of $\hat{t}_p(V)$ can be calculated using the asymptotic distribution of $\hat{\theta}$ with the Cramér delta theorem [20], which is illustrated in Section IV. For the data example in Section III, the inference results from the observed Fisher information and from the (closed-form approximate) $s$-expected Fisher information matrix are compared and shown to be nearly equivalent. Other approximate methods for $s$-confidence regions based on likelihood methods are discussed in [21].

For a large value of $m$, the approximate distribution of $\hat{\theta}$ can be based on the usual asymptotic normality theory for MLE, provided certain regularity conditions exist [22]. The sampling distribution of $\hat{\theta}$ is necessary for, eg, large sample $s$-confidence intervals for $\theta$. As $m \to \infty$ (and $n_i/m \to p_i > 0$ for all $i = 1, 2, \ldots, k$):

$$
[I_m(\theta; V)]^{1/2} \cdot (\hat{\theta} - \theta) \to Z,
$$

$Z$ has the trivariate Gaussian distribution with zero mean vector and identity covariance matrix. $I_m(\theta; V)$ is the $s$-expected Fisher information matrix for (3), based on $m$ observations.

Given $I_m(\theta; V)$, then the asymptotic distribution for $\hat{t}_p(V)$, a function of $\hat{\theta}$, is, by the Cramér delta method [20]:

$$
[I_m(\theta; V)]^{1/2} \cdot [\hat{t}_p(V) - t_p(V)] \to [\nabla t_p(V)]^T \cdot Z,
$$

$\nabla t_p(V)$ is the gradient vector for $t_p(V)$; the partials are taken with respect to the members of $\theta$.

To obtain $I_m(\theta; V)$, the negative $s$-expected values for the second partial derivatives of the log-likelihood function are complicated but not impossible.

$$
E \left[ \frac{\partial^2 \ell}{\partial \alpha^2} \right], \quad E \left[ \frac{\partial^2 \ell}{\partial \alpha \partial \gamma} \right], \quad E \left[ \frac{\partial^2 \ell}{\partial \alpha \partial \eta} \right]
$$

can be found directly. However,

$$
E \left[ \frac{\partial^2 \ell}{\partial \gamma^2} \right], \quad E \left[ \frac{\partial^2 \ell}{\partial \gamma \partial \eta} \right], \quad E \left[ \frac{\partial^2 \ell}{\partial \eta^2} \right]
$$

require an approximation for the $s$-expectations involved, since they contain integrals not in closed form. The procedure to obtain the

$$
E \left[ \frac{\partial \ell}{\partial \alpha} \right], \quad E \left[ \frac{\partial \ell}{\partial \gamma} \right], \quad E \left[ \frac{\partial \ell}{\partial \eta} \right]
$$

is presented. For calculating

$$
E \left[ \frac{\partial \ell}{\partial \alpha^2} \right], \quad E \left[ \frac{\partial \ell}{\partial \gamma^2} \right], \quad E \left[ \frac{\partial \ell}{\partial \eta^2} \right]
$$

the second partial derivative is:

$$
\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{m}{\alpha^4} - \frac{3}{\alpha^4 \cdot \gamma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} t_{i,j} V_i^\eta - \frac{6m}{\alpha^4} - \gamma \sum_{i=1}^{k} \sum_{j=1}^{n_i} t_{i,j} V_i^{\eta \cdot \eta}.
$$

To take the negative $s$-expected value of this equation, requires $E[\ell; V]$ and $E[\ell^{-1}; V]$. For these, use the results in Section I-B on $s$-expectations of B-S r.v. and substitute $I_m(V)$ for $\beta$, and simplify:

$$
E \left[ -\frac{\partial \ell}{\partial \alpha^2} \right] = \frac{2m}{\alpha^2}.
$$

The terms

$$
E \left[ -\frac{\partial \ell}{\partial \alpha \partial \gamma} \right] = E \left[ -\frac{\partial \ell}{\partial \alpha \partial \eta} \right] = 0,
$$
are found quite similarly. For

\[ E\left[ -\frac{\partial^2 \hat{y}}{\partial \gamma^2} \right] \quad \& \quad E\left[ -\frac{\partial^2 \hat{y}}{\partial \gamma \partial \eta} \right] \]

the result is

\[
\frac{\partial^2 \hat{y}}{\partial \gamma^2} = \frac{m}{2\alpha^2} - \frac{k}{\alpha^2 \gamma^2} \sum_{i=1}^{n_i} \left[ t_{i,j} \cdot V_i^\eta + \gamma \right]^{-2}
\]

\[
- \frac{1}{\alpha^2 \gamma^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} t_{i,j} \cdot V_i^\eta
\]

\[
\frac{\partial^2 \hat{y}}{\partial \gamma \partial \eta} = - \frac{k}{\alpha^2 \gamma^2} \sum_{i=1}^{n_i} \left[ \log(V_i) \cdot \sum_{j=1}^{k} \left( t_{i,j} \cdot V_i^\eta + \gamma \right)^2 \right]
\]

\[
+ \frac{1}{\alpha^2} \sum_{i=1}^{k} \frac{\log(V_i)}{V_i^\eta} \cdot \sum_{j=1}^{n_i} \frac{1}{t_{i,j}}
\]

\[
+ \frac{1}{\alpha^2 \gamma^2} \sum_{i=1}^{k} V_i^\eta \cdot \left( \log(V_i) \cdot \sum_{j=1}^{n_i} t_{i,j} \right)
\]

To calculate the negative \( s \)-expectations of these 2 derivatives, the

\[ E[(T \cdot V^\eta + \gamma)^{-2}; V] \]

and

\[ E[(T \cdot V^\eta) \cdot (T \cdot V^\eta + \gamma)^{-2}; V] \]

are needed in addition to \( E[T; V] \) and \( E[T^{-1}; V] \). They cannot
be found explicitly in closed form. However, Laplace’s method
for approximation of integrals can be used [4], [14], [15] to ap-
proximate these expressions. The accuracy of the Laplace
approximation has been studied [23]. For this case, the approxima-
tions improve as \( \alpha \to 0 \). Ref [17] suggests that often \( \alpha < 1 \) for
“cycles to failure” data; this condition is verified in Section III
when a real data set is investigated.

The Laplace approximations for the two \( s \)-expectations are

\[ E[(T \cdot V^\eta + \gamma)^{-2}; V] \approx (4\gamma)^{-1}, \]

\[ E[(T \cdot V^\eta) \cdot (T \cdot V^\eta + \gamma)^{-2}; V] \approx (4\gamma)^{-1}. \]

Using these, the negative expectations in the Fisher information
become

\[ -E\left[ -\frac{\partial^2 \hat{y}}{\partial \gamma^2} \right] \approx \frac{m}{3\gamma^2} \cdot \left( \frac{1}{\alpha^2 + \frac{1}{4}} \right), \quad (8b) \]

\[ -E\left[ -\frac{\partial^2 \hat{y}}{\partial \gamma \partial \eta} \right] \approx -\frac{1}{\gamma} \cdot \left( \frac{1}{\alpha^2 + \frac{1}{4}} \right) \cdot \sum_{i=1}^{k} n_i \cdot \log(V_i); \quad (8c) \]

also,

\[ -E\left[ -\frac{\partial^2 \hat{y}}{\partial \gamma \partial \eta} \right] \approx \left( \frac{1}{\alpha^2 + \frac{1}{4}} \right) \cdot \sum_{i=1}^{k} n_i \cdot \left( \log(V_i) \right)^2, \quad (8d) \]

By comparing (8a)–(8d) to their corresponding second partials, the \( s \)-expected Fisher information entries are much more tractable. Using these expressions in \( I_m(\theta; V) \), the asymptotic variances of the MLE of \( \alpha, \gamma, \eta \) are found from the diagonal elements of \( I_m^{-1}(\theta; V) \), and are estimated by substituting the MLE for the parameters involved. These variances are used to calculate approximate \( s \)-confidence intervals for the parameters using the asymptotic normality result. The methods are illustrated in Section III with the aluminum-coupon fatigue data of Table I.

### III. Estimation From Fatigue Failure Data

A machine developed by the Instrument Development Unit of the Physical Research Staff, Boeing Aircraft Company, was de-
digned to subject metal-coupons to repeated alternating stresses & strains [5], [24]. The coupon was clamped at the ends and contorted in the center to either side from the position of equi-
librium; the frequency of oscillation and stress amplitude was controlled. The amplitude was set either to constant (periodic loading) or random with a determined distribution. The nominal maximum stress was computed from the maximum amplitude. Elaborate precautions were taken to avoid extraneous damaging effects and clamping stresses. The machine was equipped with a mechanism to clock the number of cycles that caused fatigue failure of the coupons.

Three sets of data were obtained, each under a periodic loading scheme with:

- frequency of 18 cycles/sec,
- maximum stresses of \( V_1 = 2.1, V_2 = 2.6, V_3 = 3.1 \) (in \( 10^4 \) psi).

The experimental data are in Table I, which is displayed in \( 10^5 \) cycles to failure. The data were initially reported & investigated in [5] and partially in [24]. The metal was 6061-T6 aluminum sheeting, which was cut (parallel to the direction of loading) into the rectangular strips. The experiment yielded data for the 3 levels of maximum stress with sample sizes of \( n_1 = 101, n_2 = 102, n_3 = 101 \). Ref [5] fit model (1) separately to each of the three aluminum-coupon data-sets and achieved good fits. In this paper however, the accelerated power law B–S model (3) is fitted over all of the levels of maximum stress and the methods of estimation in Section II are compared.

For estimation of the parameters in (3), starting values were found by using the “pseudo least squares” method in Section II and given in (4). In the aluminum-coupon data-set, these starting values were:

\[ \alpha' = 0.2677, \quad \gamma' = 1124.6, \quad \eta' = 5.955. \]

For comparison, estimates of \( \gamma, \eta \) were also calculated using the log-linear model method similar to [17] and given in (5). These were:

\[ \gamma'' = 1214.1, \quad \eta'' = 5.96, \]

and closely agree with the “pseudo least squares” values which were used for starting values in the MLE search routine from (6) to yield the approximate solutions:

\[ \hat{\alpha} = 0.22565, \quad \hat{\gamma} = 1100.2, \quad \hat{\eta} = 5.938. \]

The MLE are fairly close to the regression-type estimates, which is reasonable due to the large sample sizes. By using the MLE for \( \alpha, \gamma, \eta \), the Fisher information matrix entries (8a)–(8d) can be estimated. These are, respectively,

\[ 11940.786, \quad 0.004914, \quad -5.1404, \quad 5529.485. \]

\( \gamma \) could not be estimated by the log-linear method.
TABLE II
POINT ESTIMATES AND APPROXIMATE 90% LCB ON THE 10th PERCENTILE OF FATIGUE FAILURE FOR THE ALUMINUM-COUPON DATA

<table>
<thead>
<tr>
<th>$V$ (10^4 psi)</th>
<th>$t_{0.1}(V)$</th>
<th>$t_{0.1}(V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>10.15</td>
<td>9.84</td>
</tr>
<tr>
<td>2.6</td>
<td>2.86</td>
<td>2.80</td>
</tr>
<tr>
<td>3.1</td>
<td>1.01</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Since $\hat{\alpha}$ is relatively small, it can be argued that the Laplace approximated entries for $f_m(\hat{\theta}; V)$ are reasonable. To verify this, the observed Fisher information can be calculated and compared to $I_m(\hat{\theta}; V)$. For the data in Table I, the estimated entries for (8a)–(8d) differ from the corresponding observed Fisher information entries by no more than 0.43%. Thus, due to the large sample sizes, either matrix seems appropriate for approximating asymptotic variances of the MLE.

The estimated ASD for each MLE is calculated as:

$$A\hat{S}D(\hat{\alpha}) = 0.0092, \quad A\hat{S}D(\hat{\gamma}) = 85.97, \quad A\hat{S}D(\hat{\eta}) = 0.0810.$$ 

From the asymptotic normality result, approximate 95% confidence intervals for the 3 parameters are:

$$\left(0.2076, 0.2437\right)$$ for $\alpha$,

$$\left(9.407, 127.77\right)$$ for $\gamma$,

$$\left(5.779, 6.007\right)$$ for $\eta$.

Asymptotic 90% LCB on $t_{p}(V)$ can be calculated using the asymptotic normality result for $\hat{\theta}$ and computing the distribution of $\hat{t}_p(V)$ via the Cramér delta method in Section II. Table II has point estimates and 90% LCB on the 10th percentile for the aluminum-coupon data-set for each of the levels of accelerated stress, estimated from the overall accelerated model (3) fit. Because increasing the amount of maximum stress has the effect of decreasing the fatigue life, the estimates for $t_p(V)$ are decreasing with $V$ as anticipated.

ACKNOWLEDGMENT

The authors are grateful to the Associate Editor and a referee for many constructive comments & suggestions for improving this paper.

REFERENCES


W. Jason Owen is Assistant Professor of Statistics in the Department of Mathematics and Statistics at the University of New Hampshire. He received the Ph.D. (1997) in Statistics from the University of South Carolina, and has published papers on reliability and life testing. He is a Member of the American Statistical Association.

William J. Padgett is Carolina Professor and Chair’n of the Department of Statistics at USC. He received the Ph.D. (1971) in Statistics from Virginia Polytechnic Institute & SU. He has published papers on nonparametric & parametric inference in reliability theory & applications, and on other topics in statistics & probability. He is an Associate Editor for the *J. Nonparametric Statistics*, *Lifetime Data Analysis*, and *J. Applied Statistical Science*, and has served as an Associate Editor of *Technometrics* and a Coordinating Editor for *J. Statistical Planning and Inference*. He is a Member of the American Society for Quality, and is a Fellow of the American Statistical Association and of the Institute of Mathematical Statistics.