Controllability of the Ornstein–Uhlenbeck equation

DIOMEDES BÁRCENAS AND HUGO LEIVA†
Departamento de Matemáticas, Universidad de Los Andes,
Mérida 5101, Venezuela
AND
WILFREDO URBINA
Departamento de Matemáticas, Facultad de Ciencias, Universidad Central
de Venezuela, Apartado 47195, Los Chaguaramos,
Caracas 1041-A, Venezuela

In this paper we study the controllability of the following controlled Ornstein–Uhlenbeck equation

$$z_t = \frac{1}{2} \Delta z - \langle x, \nabla z \rangle + \sum_{n=1}^{\infty} \sum_{|\beta|=n} u_\beta(t) \langle b, h_\beta \rangle \gamma_d, \quad t > 0, \ x \in \mathbb{R}^d,$$

where $h_\beta$ is the normalized Hermite polynomial, $b \in L^2(\gamma_d)$, $\gamma_d(x) = \frac{e^{-|x|^2}}{\pi^{d/2}}$ is the Gaussian measure in $\mathbb{R}^d$ and the control $u \in L^2(0, t_1; l^2(\gamma_d))$, with $l^2(\gamma_d)$ the Hilbert space of Fourier–Hermite coefficient

$$l^2(\gamma_d) = \left\{ U = \{U_\beta|\beta|=n\}_{n \geq 1}: U_\beta \in \mathbb{C}, \sum_{n=1}^{\infty} \sum_{|\beta|=n} |U_\beta|^2 < \infty \right\}.$$

We prove the following statement: If for all $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{N}^d$

$$\langle b, h_\beta \rangle \gamma_d = \int_{\mathbb{R}^d} b(x)h_\beta(x)\gamma_d(dx) \neq 0,$$

then the system is approximately controllable on $[0, t_1]$. Moreover, the system can never be exactly controllable.

Keywords: Ornstein–Uhlenbeck equation; approximate controllability; compact semigroup.

1. Introduction

In this paper we study the controllability of the following controlled Ornstein–Uhlenbeck equation

$$z_t = \frac{1}{2} \Delta z - \langle x, \nabla z \rangle + \sum_{n=1}^{\infty} \sum_{|\beta|=n} u_\beta(t) \langle b, h_\beta \rangle h_\beta, \quad t > 0, \ x \in \mathbb{R}^d,$$  \hfill (1.1)

where $h_\beta$ is the normalized Hermite polynomial of order, $b \in L^2(\gamma_d)$, $\gamma_d(x) = e^{-|x|^2/\pi^{d/2}}$ is the Gaussian measure in $\mathbb{R}^d$ and the control $u \in L^2(0, t_1; l^2(\gamma_d))$, with $l^2(\gamma_d)$ the Hilbert space of

†Email: Leiva@ula.ve

© The author 2005. Published by Oxford University Press on behalf of the Institute of Mathematics and its Applications. All rights reserved.
Fourier–Hermite coefficient,

$$l_2(\gamma_d) = \left\{ U = \{ \{ U_\beta \}_{\beta = n} \}_{n \geq 1} : U_n \in \mathbb{C}, \sum_{n=1}^{\infty} \sum_{|\beta|=n} |U_\beta|^2 < \infty \right\}.$$ 

We prove the following statement: If for all $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{N}^d$

$$\langle b, h_\beta \rangle_{\gamma_d} = \int_{\mathbb{R}^d} b(x) h_\beta(x) \gamma_d(dx) \neq 0,$$

then the system is approximately controllable on $[0, t_1]$. Moreover, the system can never be exactly controllable.

As a special case we can consider the scalar Ornstein–Uhlenbeck equation with a single control

$$z_t = z_{xx} - x z_x + b(x) u, \quad t \geq 0, \quad x \in \mathbb{R},$$

where $b \in L^2(\gamma_1)$ and the control $u$ belong to $L^2(0, t_1; \mathbb{R})$. This system is approximately controllable iff

$$\left(-\frac{1}{\beta!}\right)^\frac{\beta}{2} \int_{\mathbb{R}} b(x) e^{x^2} \frac{\partial^\beta}{\partial x_1^\beta} \left(e^{-x^2} \gamma_1(dx) \right) \neq 0, \quad \beta = 1, 2, 3, \ldots.$$ 

The Ornstein–Uhlenbeck differential operator

$$L = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle$$

is a well-known operator in harmonic analysis, probability and quantum mechanics (see Meyer, 1983; Muckenhoupt, 1969; Nelson, 1973; Sjögren, 1997; Urbina, 1998; Watanabe, 1984). As far as we know, the controllability of this equation has not been studied.

2. Notations and preliminaries

In this section we shall choose the space where this problem will be set and give the definition of exact and approximate controllability. Also, we will present some results to be used in the next section.

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ be a multi-index, $\alpha! = \prod_{i=1}^{d} \alpha_i!$, $|\alpha| = \sum_{i=1}^{d} \alpha_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. Then the normalized Hermite polynomial of order $\alpha$ in $d$ variables is given by

$$h_\alpha(x) = \frac{1}{(2^{d!} d!)^{1/2}} \prod_{i=1}^{d} (-1)^{\alpha_i} \frac{\partial^\alpha}{\partial x_i^{\alpha_i}} \left(e^{-x_i^2}\right).$$

It is well known that the Hermite polynomials are eigenfunctions of $L$,

$$L h_\alpha(x) = -|\alpha|h_\alpha(x).$$

Given a function $f \in L^2(\gamma_d)$, its $\beta$-Fourier–Hermite coefficient is defined by

$$\langle f, h_\beta \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_\beta(x) \gamma_d(dx).$$
Let $C_n$ be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of \{h_\beta: |\beta| = n\}, $C_n$ is a finite-dimensional subspace of dimension $(n+d−1)_n$. By the orthogonality of the Hermite polynomials with respect to $\gamma_d$, it is easy to see that \{\mathcal{C}_n\} is an orthogonal decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n,$$

which is called the Wiener chaos.

The orthogonal projection $P_n$ of $L^2(\gamma_d)$ onto $C_n$ is given by

$$P_n f = \sum_{|a|=n} \langle f, h_a \rangle_{\gamma_d} h_a, \quad f \in L^2(\gamma_d),$$

and for a given $f \in L^2(\gamma_d)$, its Hermite expansion is given by $f = \sum_n P_n f$.

Using this notation one can prove the following spectral decomposition of $L$

$$Lf = \sum_n (-n) P_n f, \quad f \in L^2(\gamma_d),$$

and its domain $D(L)$ is

$$D(L) = \left\{ f \in L^2(\gamma_d): \sum_n n^2 \|P_n f\|_{L^2(\gamma_d)} < \infty \right\}.$$

Let $Z = L^2(\gamma_d)$ and $l^2(\gamma_d)$ the Hilbert space of Fourier–Hermite coefficient,

$$l^2(\gamma_d) = \left\{ U = \{U_\beta\}_{|\beta| \geq n} : \sum_{n=1}^{\infty} U_\beta \sum_{|\beta| = n} |U_\beta|^2 < \infty \right\},$$

with the inner product and norm defined as follow

$$\langle U, V \rangle_{l^2} = \sum_{n=1}^{\infty} \sum_{|\beta| = n} U_\beta V_\beta, \quad \|U\|_{l^2}^2 = \sum_{n=1}^{\infty} \sum_{|\beta| = n} |U_\beta|^2, \quad U, V \in l^2(\gamma_d).$$

Now, suppose that $b$ is a fixed element of $Z$ and consider the linear and bounded operator $B: l^2(\gamma_d) \to Z$ defined by

$$BU = \sum_{n=1}^{\infty} \sum_{|\beta| = n} U_\beta \langle b, h_\beta \rangle_{\gamma_d} h_\beta. \quad (2.2)$$

Then, the system (4.4) can be written as follows:

$$z' = Lz + Bu, \quad t > 0. \quad (2.3)$$

**THEOREM 2.1** The operator $L$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-nt} P_n z, \quad z \in Z, \quad t \geq 0, \quad (2.4)$$
where \( \{P_n\}_{n \geq 1} \) is a complete orthogonal projection in the Hilbert space \( Z \) given by
\[
P_n z = \sum_{|\alpha| = n} \langle z, h_\alpha \rangle h_\alpha, \quad n \geq 1, \quad z \in Z.
\] (2.5)

**Lemma 2.1** The semigroup given by (2.4) is compact for \( t > 0 \).

**Proof.** Since \( T(t) \) is given by
\[
T(t)z = \sum_{n=1}^{\infty} e^{-nt} P_n z, \quad t > 0,
\]
we can consider the following sequence of compact operator
\[
T_n(t)z = \sum_{k=1}^{n} e^{-kt} P_k z, \quad t > 0.
\]
Then
\[
\|T(t)z - T_n(t)z\|^2 = \sum_{k=n+1}^{\infty} \|e^{-kt} P_k z\|^2 \leq e^{-2nt} \|z\|^2.
\]
Therefore, the sequence of compact operator \( \{T_n(t)\} \) converges uniformly to \( T(t) \), for all \( t > 0 \). Then, from part e of Theorem A.3.22 of Curtain & Zwart (1995) we conclude the compactness of \( T(t) \).

Now, we shall give the definition of exact and approximate controllability in terms of system (2.3). To this end, for all \( z_0 \in Z \) and a control \( u \in L^2(0, t_1; l_2(\gamma d)) \), (2.3) has a unique mild solution given by
\[
z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)\,ds, \quad 0 \leq t \leq t_1.
\] (2.6)

**Definition 2.1** (Exact controllability) We shall say that the system (2.3) is exactly controllable on \([0, t_1]\), if and only if, for all \( z_0, z_1 \in Z \) there exists a control \( u \in L^2(0, t_1; l_2(\gamma d)) \) such that the solution \( z(t) \) of (2.6) corresponding to \( u \) verifies \( z(t_1) = z_1 \).

Consider the following bounded linear operator
\[
G: L^2(0, t_1; l_2(\gamma d)) \to Z, \quad Gu = \int_0^{t_1} T(t_1-s)Bu(s)\,ds.
\] (2.7)

Then, the following proposition is a characterization of the exact controllability of the system (2.3).

**Proposition 2.1** The system (2.3) is exactly controllable on \([0, t_1]\), if and only if the operator \( G \) is surjective, i.e. to say
\[
GL^2(0, t_1; l_2(\gamma d)) = GL^2 = \text{Range}(G) = Z.
\]

**Definition 2.2** We say that (2.3) is approximately controllable in \([0, t_1]\), if for all \( z_0, z_1 \in Z \) and \( \epsilon > 0 \) there exists a control \( u \in L^2(0, t_1; l_2(\gamma d)) \) such that the solution \( z(t) \) given by (2.6) satisfies
\[
\|z(t_1) - z_1\| \leq \epsilon.
\]

The following theorem holds in general and can be found in Curtain & Zwart (1995).

**Theorem 2.2** Equation (2.3) is approximately controllable on \([0, t_1]\) iff
\[
B^*T^*(t)z = 0 \quad \forall t \in [0, t_1], \quad \Rightarrow z = 0.
\] (2.8)
3. Main results

In this section we shall prove the main results of this work. To this end, we will use Lemma 3.14 from Curtain & Zwart (1995, p. 62).

**LEMMA 3.1** Let \( \{a_j\}_{j \geq 1} \) and \( \{\beta_{i,j}: i = 1, 2, \ldots, m\}_{j \geq 1} \) be two sequences of complex numbers such that \( a_1 > a_2 > a_3 \ldots \).

Then
\[
\sum_{j=1}^{\infty} e^{a_j t} \beta_{i,j} = 0 \quad \forall t \in [0, t_1], \ i = 1, 2, \ldots, m,
\]
iff
\[
\beta_{i,j} = 0, \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, \infty.
\]

**THEOREM 3.1** If for all \( n \in \mathbb{N} \) and \( |\beta| = n \) we have
\[
\langle b, h_\beta \rangle_{\gamma_d} = \int_{\mathbb{R}^d} b(x) h_\beta(x) \gamma_d(dx) \neq 0,
\]
then the system (2.3) is approximately controllable on \([0, t_1]\) but never exactly controllable.

**Proof.** Suppose condition (3.1). Next, we compute \( B^*: \mathbb{Z} \to l_2(\gamma_d) \). In fact,
\[
\langle BU, z \rangle_{\gamma_d} = \left( \sum_{n=1}^{\infty} \sum_{|\beta| = n} U_\beta \langle b, h_\beta \rangle_{\gamma_d} h_\beta, z \right)_{z,z} = \sum_{n=1}^{\infty} \sum_{|\beta| = n} U_\beta \langle b, h_\beta \rangle_{\gamma_d}(z, h_\beta)_{z,z} = \langle U, \{ \{ \langle b, h_\beta \rangle_{\gamma_d}(z, h_\beta) \} | |\beta| = n \} \rangle_{l_2(\gamma_d), l_2(\gamma_d)}.
\]
Therefore,
\[
B^*z = \{ \{ \langle b, h_\beta \rangle_{\gamma_d}(z, h_\beta) \} | |\beta| = n \} \}_{n \geq 1} = \sum_{n=1}^{\infty} \sum_{|\beta| = n} \langle b, h_\beta \rangle_{\gamma_d}(z, h_\beta)e_\beta,
\]
where \( \{e_\beta\}_{|\beta| = n} \) is the canonical base of \( l_2(\gamma_d) \).

On the other hand,
\[
T^*(t)z = \sum_{n=1}^{\infty} e^{-nt} P_n z, \quad z \in \mathbb{Z}, \ t \geq 0.
\]

Then,
\[
B^*T^*(t)z = \{ \{ \langle b, h_\beta \rangle_{\gamma_d}(T^*(t)z, h_\beta) \} | |\beta| = n \} \}_{n \geq 1}.
\]

According to Theorem 2.2 the system (2.3) is approximately controllable on \([0, t_1]\) iff
\[
\langle b, h_\beta \rangle_{\gamma_d}(T^*(t)z, h_\beta) = 0 \quad \forall t \in [0, t_1], \ |\beta| = n, \ n = 1, 2, \ldots, \infty, \Rightarrow z = 0. \tag{3.2}
\]

Since \( \langle b, h_\beta \rangle_{\gamma_d} \neq 0 \), for \( |\beta| = n, n \geq 1 \), then condition (3.2) is equivalent to
\[
\langle T^*(t)z, h_\beta \rangle = 0 \quad \forall t \in [0, t_1], \ |\beta| = n, \ n \geq 1, \Rightarrow z = 0. \tag{3.3}
\]
Now, we shall check condition (3.3):

$$\langle T^* (t) z, h_\beta \rangle = \sum_{m=1}^{\infty} e^{-mt} \langle P_m z, h_\beta \rangle = 0, \quad |\beta| = n, \; n = 1, 2, \ldots, \infty, \; t \in [0, t_1].$$

Applying Lemma 3.1, we conclude that

$$\langle P_m z, h_\beta \rangle = 0, \quad |\beta| = n, \; m, n = 1, 2, \ldots, \infty,$$

i.e.

$$\sum_{|\alpha|=m} \langle z, h_\alpha \rangle \langle h_\alpha, h_\beta \rangle = 0, \quad |\beta| = n, \; m, n = 1, 2, \ldots, \infty,$$

i.e.

$$\langle z, h_\beta \rangle = 0, \quad |\beta| = n, \; n = 1, 2, \ldots, \infty.$$

Since \(\{h_\beta\}_{|\beta|=n \geq 1}\) is a complete orthonormal base of \(Z\), we conclude that \(z = 0\).

From Lemma 2.1 we know that \(T(t)\) is compact for \(t > 0\); then applying Theorem 3.3 from Barcenas et al. (2003), we conclude that the system 2.3 is not exactly controllable on any interval \([0, t_1]\).

But for better understanding of the reader and completeness of this work we will include here the proof.

In fact, from Propositions 2.1 it is enough to prove that the operator

$$G: L^2(0, t_1; l_2(\gamma_d)) \to Z, \quad Gu = \int_0^{t_1} T(t_1 - s) Bu(s) \, ds$$

satisfied

$$\text{Range}(G) \neq Z.$$

To this end, we shall prove that the operator \(G\) is compact. In fact, for all \(\delta > 0\) small enough the operator \(G\) can be written as follows

$$G = G_\delta + S_\delta, \quad G_\delta, S_\delta \in L(L^2(0, t_1; l_2(\gamma_d)), Z),$$

where

$$G_\delta u = \int_0^{t_1-\delta} T(t_1 - s) Bu(s) \, ds \quad \text{and} \quad S_\delta u = \int_{t_1-\delta}^{t_1} T(t_1 - s) Bu(s) \, ds.$$  

Claim 1. The operator \(G_\delta\) is compact. In fact,

$$G_\delta u = \int_0^{t_1-\delta} T(\delta) T(t_1 - \delta - s) Bu(s) \, ds = T(\delta) \int_0^{t_1-\delta} T(t_1 - \delta - s) Bu(s) \, ds = T(\delta) H_\delta u.$$  

Since \(T(\delta)\) is compact and \(H_\delta \in L(L^2(0, t_1; l_2(\gamma_d)), Z)\), then \(G_\delta\) is compact.

Claim 2. For \(\epsilon > 0\) there exists \(\delta > 0\) such that \(\|S_\delta\| < \epsilon\). In fact,

$$\|S_\delta u\| \leq \int_{t_1-\delta}^{t_1} \|T(t_1 - s)\| \|B\| \|u(s)\| \, ds \leq \int_{t_1-\delta}^{t_1} M \|B\| \|u(s)\| \, ds,$$
where
\[ M = \sup_{0 \leq s \leq t \leq t_1} \|T(t - s)\|. \]
Applying the Hölder inequality, we obtain
\[ \|S_\delta u\| \leq M \|B\| \|\delta\| \|u\|_{L^2}. \]
Therefore, \( \|S_\delta\| < \epsilon \) if \( \delta < \frac{\epsilon}{M \|B\|} \).

Hence, for all natural numbers \( n \) there exist \( \delta_n > 0 \) such that
\[ \|G - G_{\delta_n}\| = \|S_{\delta_n}\| < \frac{1}{n}, \quad n = 1, 2, 3, \ldots. \]

So, the sequence of compact operators \( \{G_{\delta_n}\} \) converges uniformly to \( G \). Then applying part (e) of Theorem A.3.22 from Curtain & Zwart (1995), we obtain that \( G \) is compact. Now, from part (g) of the same theorem we obtain that \( \text{Range}(G) \neq Z \).

**Example 3.1** As a special case we can consider the scalar Ornstein–Uhlenbeck equation with a single control
\[ z_t = z_{xx} - xz_x + b(x)u, \quad t \geq 0, \quad x \in \mathbb{R}, \]
where \( b \in L^2(\gamma_1) \) and the control \( u \) belong to \( L^2(0, t_1; \mathbb{R}) = L^2(0, t_1; L^2(\gamma_1)) \). In this case, (3.4) is approximately controllable iff
\[ \langle b, h_{\beta} \rangle_{\gamma_d} = \int_{\mathbb{R}} b(x)h_{\beta}(x)\gamma_d(dx) = \frac{(-1)^\beta}{(2^\beta \beta!)^2} \int_{\mathbb{R}} b(x)e^{x^2} \frac{d^\beta}{dx^\beta} (e^{-x^2}) \gamma_1(dx) \neq 0, \quad \beta = 1, 2, \ldots. \]

### 4. Conclusions

Most of the results from Fattorini (1966, 1967), Russell (1978) and Triggiani (1976) are collected in Curtain & Zwart (1995); in these references, the following linear control system is considered in a separable Hilbert space \( Z \):
\[ z' = -Az + \sum_{i=1}^m b_i u_i(t), \quad t > 0, \]
where \( b_1, b_2, \ldots, b_m \in Z, u_i \in L^2[0, t_1] \) and \( A: D(A) \subset Z \to Z \) is an unbounded linear operator with the following spectral decomposition:
\[ Az = \sum_{j=1}^\infty \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^\infty \lambda_j P_j x, \]
where \( \langle \cdot, \cdot \rangle \) is the inner product in \( Z \) and
\[ P_j z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \]

The eigenvalues \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \) of \( A \) have finite multiplicity \( \gamma_j \) equal to the dimension of the corresponding eigen space and \( \{\phi_{j,k}\} \) is a complete orthonormal set of eigenvectors of \( A \). So, \( \{P_j\} \) is a complete family of orthogonal projections in \( Z \) and
\[ z = \sum_{j=1}^\infty P_j z, \quad z \in Z. \]
A generates a strongly continuous semigroup \( \{ e^{-At} \} \) given by
\[
e^{-At}z = \sum_{j=1}^{\infty} e^{-\lambda_j t} P_j z.
\]
In the above references, particularly in Curtain & Zwart (1995), the following statement is proved.

**Theorem 4.1** (Theorem 4.2.1 from Curtain & Zwart, 1995) The system (4.1) is approximately controllable on \([0, t_1]\), \(t_1 > 0\) iff each of the following finite-dimensional systems are controllable on \([0, t_1]\):
\[
y' = -\lambda_j y + B_j u(t), \quad y \in \mathcal{R}(P_j),
\]
where \(u \in L^2(0, t_1; \mathbb{R}^m)\) and \(B_j\) is the following matrix
\[
B_j = \begin{bmatrix}
\langle b_1, \phi_{j,1} \rangle & \langle b_2, \phi_{j,1} \rangle & \cdots & \langle b_m, \phi_{j,1} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle b_1, \phi_{j,\gamma_j} \rangle & \langle b_2, \phi_{j,\gamma_j} \rangle & \cdots & \langle b_m, \phi_{j,\gamma_j} \rangle
\end{bmatrix}.
\]
From classical finite-dimensional theory it is easy to see that the controllability of system (4.2) is equivalent to
\[
\text{Rank}(B_j) = \dim \mathcal{R}(P_j) = \gamma_j.
\]

**Remark 4.1** Theorem 4.1 shows that for these systems the number of controls needed for the controllability must be at least the maximal multiplicity of the eigenvalues \(\gamma_j \leq m, j = 1, 2, 3 \ldots\), which is in most of the examples in Curtain & Zwart (1995) and Curtain & Pritchard (1978). For that reason, this theorem can not be applied in the case that \(\lim_{j \to \infty} \gamma_j = \infty\). Particularly, in the case of the Ornstein–Uhlenbeck equation where the eigenvalues are the natural numbers \(\{ j \}\) with corresponding multiplicity, \(\gamma_j = (j + d - 1)\) and, therefore, \(\lim_{j \to \infty} \gamma_j = \infty\).

The method we used to prove the controllability of the Ornstein–Uhlenbeck equation can be applied to the following more general class of diffusion system with \(\lim_{j \to \infty} \gamma_j = \infty\):
\[
\dot{z} = -Az + \sum_{j=1}^{\infty} \sum_{k=1}^{\gamma_j} u(t)_k, j \langle b, \phi_{k,j} \rangle \phi_k, j, \quad t > 0, \quad z \in \mathbb{Z}.
\]

The control \(u \in L^2(0, t_1; l^2(A))\), with the Hilbert space of Fourier coefficient
\[
l^2(A) = \left\{ U = \{ [U_k]_{k=1}^{\gamma_j} \}_{j \geq 1} : U_k, j \in \mathbb{C}, \sum_{j=1}^{\infty} \sum_{k=1}^{\gamma_j} |U_k, j|^2 < \infty \right\}.
\]
In the same way, we can prove the following theorem.

**Theorem 4.2** If for all \(j = 1, 2, 3 \ldots\) and \(k = 1, 2, \ldots, \gamma_j\)
\[
\langle b, \phi_{k,j} \rangle \neq 0,
\]
then the system (4.4) is approximately controllable on \([0, t_1]\). Moreover, the system can never be exactly controllable.
Acknowledgements

This work has been supported by ULA and FONACIT No G-97000668.

REFERENCES


