Discrete-time queue with Bernoulli bursty source arrival and generally distributed service times

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Abstract

In this contribution, a discrete-time single-server infinite-capacity queue with correlated arrivals and general service times is investigated. Arrivals of cells are modelled as an on/off source process with geometrically distributed on-periods and off-periods, which is called Bernoulli bursty source. Based on the probability generating function technique, closed-form expression of some performance measures of system, such as average buffer content, unfinished work, cell delay and so on, are obtained. Finally, the effects of system parameters on performance measures are illustrated by some numerical examples.

Keywords: Discrete-time queue; Bernoulli bursty source arrivals; Generally distributed service times

1. Introduction

Recently there has been a rapid increase in the literature on discrete-time queueing system, which is particularly appropriate to describe the various queueing related phenomena in digital computer and communication systems, including mobile and B-ISDN networks based on asynchronous transfer mode (ATM) technology, due to the packetized nature of these transport protocols. In these models, the time axis is assumed to be divided into fixed-length time intervals, referred to as slots, and arriving and servicing of cells are synchronous, i.e. arrivals and departures of a cell can occur at the slot boundaries only.

Most studies of discrete-time queues assume service times is geometrically distributed [1,2] or constant [3–6]. Discrete-time queueing models with generally distributed service times have been studied in [7–9]. All these literature considers the discrete-time GI/G/1 queue system. Bruneel [7] presents explicit close-form expression of classical discrete-time GI/G/1 queue with infinite-capacity buffer, and Zhou [8] investigates a
discrete-time GI/G/1 queue with negative arrivals. A discrete-time GI/G/1 queue with interruptions is studied by Fiems [9].

In practice, measurements indicate that traffic in communication networks exhibits correlations, see for instance [10,11]. There are two techniques to model correlated arrivals – Markovian Processes [6,12] and Time Series [3]. In this paper we extend the work of Wittevrongel et al. [6]. As in the work of Wittevrongel, we use Bernoulli bursty source process (which is a special case of the discrete Markovian arrival processes: D-MAP [13,14]) to model correlated cell arrivals, and investigate a discrete-time queue with correlated arrivals and generally distributed service times.

The outline of the paper is as follows: In Section 2, the queue model under study is described. Section 3 gives steady-state analysis. In Section 4, the close-form expression is derived for the steady-state probability generating function (pgf) of the buffer occupancy. The pgf of unfinished work in the system is obtained in Section 5. Section 6 concentrates on the pgf of the cell delay. Finally, some numerical results which illustrate the effect of system parameters on the performance measures are presented in Section 7.

2. Queue model

We consider a discrete-time queueing system, i.e. the time axis is divided into fixed-length interval, called slots. During each of these slots, cells that arrive in the system are stored in a buffer with infinite-capacity, and are served on a FIFO basis. Further, the slots are marked by 0, 1, . . . , n, . . . in order. A potential arrival occurs in (n, n+) and a potential departure takes place in (n−, n). More specifically, we consider an early arrival system (EAS). For details, see Hunter [15]. Service of a cell takes only one slot if the server is available and is synchronized with respect to slot boundaries. This implies that the service of a cell cannot start before the beginning of the slot following its arrival slot. The service time of the consecutive cells – which is the number of slots that the cell would require for complete service – is modelled as a series of i.i.d discrete random variables, denoted by S. The common probability mass function (p.m.f) is \( P\{S = n\} = s_n, n = 1, 2, . . . \) (in order to avoid trivial cases, we assume \( P\{S = 0\} = 0 \)), and corresponding mean value is \( \bar{s} \) and pgf is defined as

\[
S(z) = \sum_{k=0}^{\infty} s_k z^k.
\]

Cells are generated according to a Bernoulli bursty source depicted in Fig. 1. That is to say, the source alternates between on-periods, during which exactly one cell is generated per slot, and off-period, during which no cells are generated. If source is ‘On’ (‘Off’), then it remains in on-periods (off-periods) with probability \( \alpha (\beta) \), and with transition probability \( 1 - \alpha (1 - \beta) \) transits to off-periods (on-periods). It is often convenient to use mean arrival rate in steady-state \( U \) and burstiness factor \( B \) instead of the distribution parameter \( \alpha \) and \( \beta \).

\[
U = \frac{E[T_{on}]}{E[T_{on}] + E[T_{off}]} = \frac{1 - \beta}{2 - \alpha - \beta},
\]

\[
B = \frac{E[T_{on}]E[T_{off}]}{E[T_{on}] + E[T_{off}]} = \frac{1}{2 - \alpha - \beta}.
\]

Given \( U \), the parameter \( B \) takes values between \( \text{Max}(U, 1 - U) \) and infinity and is a measure for the absolute lengths of on-periods and off-periods. The burstiness factor \( B \) equals 1 for uncorrelated arrivals.

![Fig. 1. Bernoulli bursty source arrival.](image-url)
For avoiding trivial cases, we assume $0 < \alpha < 1$, $0 < \beta < 1$. The traffic intensity is given by $\rho = \frac{(1-\beta)^{\alpha}}{2-\alpha-\beta}$.

To analyze the system, we consider three dimensional Markov Process \( \{ A_n, R_n, C_n \} \), where \( A_n \) represents the number of customer arrived in the \( n \)th slot, \( R_n \) is the remainder service time (i.e. remainder number of slots that the cell which is in service require for complete service) excluding the \( n \)th slot at the boundary of the \( n \)th slot, \( C_n \) is the buffer content in the \( n \)th slot, i.e. the total number of cells stored in the buffer including the possible cell in the service. To resolve the system, we define the following generating functions:

\[
Q_0(y, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(A_n = 0, R_n = i, C_n = j) y^i z^j,
\]

\[
Q_1(y, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(A_n = 1, R_n = i, C_n = j) y^i z^j.
\]

We assume the equilibrium \((\rho < 1)\) condition holds, and denote

\[
Q_0(y, z) = \lim_{n \to \infty} Q_0^n(y, z); \quad Q_1(y, z) = \lim_{n \to \infty} Q_1^n(y, z).
\]

3. Steady-state analysis

For convenience, we introduce a notation

\[
S_n(i, j, k) = \{ A_n = i, R_n = j, C_n = k \}.
\]

That is to say, \( S_n(i, j, k) \) is a state event. It is easy to see that \( P\{ S_n(1, i, 0) \} = 0 \), \( (i = 0, 1, \ldots) \), because of synchronization.

So, we can readily obtain the following equation:

\[
Q_1(0, 0) = 0.
\]

The one-step transition probabilities can then be given by

\[
P\{ S_n(0, i, j)|S_{n-1}(0, 0, 0) \} = \begin{cases} 
\beta, & \text{When } i = j = 0, \\
0, & \text{Otherwise},
\end{cases}
\]

\[
P\{ S_n(0, i, j)|S_{n-1}(0, l, k) \} = \begin{cases} 
\beta x^{l+1}, & \text{When } j = k - 1, l = 0, k \neq 0, \\
\beta, & \text{When } j = k, i = l-1, l \neq 0, \\
0, & \text{Otherwise},
\end{cases}
\]

\[
P\{ S_n(0, i, j)|S_{n-1}(1, l, k) \} = \begin{cases} 
(1 - \alpha) x^{l+1}, & \text{When } j = k - 1, l = 0, k \neq 0, \\
1 - \alpha, & \text{When } j = k, i = l-1, l \neq 0, \\
0, & \text{Otherwise},
\end{cases}
\]

\[
P\{ S_n(1, i, j)|S_{n-1}(0, 0, 0) \} = \begin{cases} 
(1 - \beta) x^{l+1}, & \text{When } j = 1, \\
0, & \text{Otherwise},
\end{cases}
\]

\[
P\{ S_n(1, i, j)|S_{n-1}(0, l, k) \} = \begin{cases} 
(1 - \beta) x^{l+1}, & \text{When } j = k \neq 0, l = 0, \\
1 - \beta, & \text{When } j = k + 1, i = l - 1, l \neq 0, \\
0, & \text{Otherwise},
\end{cases}
\]

\[
P\{ S_n(1, i, j)|S_{n-1}(1, l, k) \} = \begin{cases} 
2 x^{l+1}, & \text{When } j = k, l = 0, k \neq 0, \\
x, & \text{When } j = k + 1, i = l - 1, l \neq 0, \\
0, & \text{Otherwise},
\end{cases}
\]

\[
P\{ S_n(1, i, j)|S_{n-1}(1, l, k) \} = \begin{cases} 
2 x^{l+1}, & \text{When } j = k, l = 0, k \neq 0, \\
x, & \text{When } j = k + 1, i = l - 1, l \neq 0, \\
0, & \text{Otherwise},
\end{cases}
\]
According to total probability formula, we achieve

\[
Q_0^{+1}(y, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P\{S_n+1(0, i, j)\} y^i z^j
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P\{S_n(0, 0, 0)\} P\{S_n+1(0, i, j)\} P\{S_n(0, 0, 0)\} y^i z^j
\]

\[
+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P\{S_n(1, 0, 0)\} P\{S_n+1(0, i, j)\} P\{S_n(1, 0, 0)\} y^i z^j
\]

\[
+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} P\{S_n(0, 0, k)\} P\{S_n+1(0, i, j)\} P\{S_n(0, 0, k)\} y^i z^j
\]

\[
+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} P\{S_n(0, l, k)\} P\{S_n+1(0, i, j)\} P\{S_n(l, k, 0)\} y^i z^j
\]

\[
+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} P\{S_n(1, l, k)\} P\{S_n+1(0, i, j)\} P\{S_n(l, k, 0)\} y^i z^j
\]

\[
= \beta \left(1 - \frac{S(y)}{y} \right) Q_0^0(0, 0) + \beta \frac{S(y) - z}{y} \frac{Q_0^0(0, z)}{y} + \beta \frac{Q_0^0(y, z)}{y} + (1 - \alpha) \frac{S(y) - z}{y} \frac{Q_1^0(0, z)}{y}
\]

\[
+ \frac{1 - \alpha}{y} Q_1^0(y, z).
\]

Similarly, we have

\[
Q_1^{+1}(y, z) = \left(1 - \beta\right) \frac{S(y)}{y} (z - 1) Q_0^0(0, 0) + \frac{1 - \beta}{y} (S(y) - z) Q_0^0(0, z) + \frac{z(1 - \beta)}{y} \frac{Q_0^0(y, z)}{y}
\]

\[
+ \frac{z}{y} (S(y) - z) Q_1^0(0, z) + \frac{z}{y} Q_1^0(y, z).
\]

Let \(n \to \infty\) in Eqs. (1) and (2), and getting the derivative with respect to \(y\) and let \(y = 1\), then following the normalization condition, we get \(Q_0(0, 0) = 1 - \rho\).

In Eqs. (3) and (4), there are two functions \(Q_0(0, z)\) and \(Q_1(0, z)\), which are not given with explicit expressions. Next, we will solve them. Let \(H(y) = y^2 - (\alpha z + \beta) y - (1 - z - \beta) z^2\), and denote \(y^*\) as the only one root of \(H(y)\) in \(\{y | y^2 < 1, |z| \leq 1\}\), and it is the function of \(z\). So taking the derivative of both side of \(H(y) = 0\) with respect to \(z\), we get

\[
\frac{\partial y^*}{\partial z} = \frac{(1 - \alpha)y^* + \alpha(\beta - z) - (1 - z)(1 - \beta)}{\beta - y^*}.
\]
Let $y = y^*$ in the above two equations, it yields:

\[
Q_0(0, z) = \frac{(\rho - 1)(\bar{z} - \beta + \alpha y^*)[\beta(y^*z - S(y^*))(\bar{z} - y^*) - \bar{z}b(z - 1)zS(y^*)]}{(S(y^*) - z)[\bar{z}b((y^*)^2 + \bar{z} - \beta + \alpha y^*) - \beta(\bar{z} - y^*)(\bar{z} - \beta + \alpha y^*)]}
\]

\[
Q_1(0, z) = \frac{-\beta(1 - \rho)S(y^*)[z^2(\bar{z}(1 - y^*) + \beta(y^* - \beta)) + z(\bar{z}b(y^* - 1) + y^*\beta(\beta - y^*))]}{(S(y^*) - z)\{[\bar{z} - \beta + \alpha y^*)[\beta(\bar{z} - y^*) - \bar{z}b] + \bar{z}b(y^*)^2\}}
\]

\[
+ \frac{+\beta y^*(1 - \rho)[z(\bar{z} + \beta - 1) + y^*(y^* - \beta - \alpha z)]}{(S(y^*) - z)\{[\bar{z} - \beta + \alpha y^*)[\beta(\bar{z} - y^*) - \bar{z}b] + \bar{z}b(y^*)^2\}}.
\]

4. Buffer contents

In this section, we determine the pgf $C(z)$ of the steady-state buffer contents $C$ at the boundary of an arbitrary slot. Using the definitions and notations introduced above, it is easily seen that $C(z)$ can be derived as:

\[
C(z) = Q_0(1, z) + Q_1(1, z).
\]

Next, using Eqs. (3) and (4), we find the following explicit expression for $C(z)$

\[
C(z) = \frac{[(2 - 2\beta - \alpha)z - \beta][Q_0(0, z) - 1 + \rho] + [(1 - 2\alpha - \beta)z + 1 - \alpha]Q_1(0, z)}{(1 - \beta)z}.
\]

The average buffer occupancy $E[C]$ at the boundary of an arbitrary slot can be found from the formula

\[
E[C] = C'(1),
\]

yielding

\[
E[C] = \frac{\beta}{1 - \beta}Q_0(0, 1) - 1 + \rho + \frac{2 - \alpha - 3\beta}{1 - \beta}Q_0(0, 1) - 1 - \frac{1}{1 - \beta}Q_1(0, 1) + \frac{2 - 3\alpha - \beta}{1 - \beta}Q_1(0, 1).
\]

5. Unfinished work

The aim of this section is to obtain the pgf of the unfinished work of the queueing system at the boundary of an arbitrary slot, which is defined as the remaining number of slots needed to serve all the cells present in the system at the boundary of the slots, denoted by random variable $w$ with pgf of $W(z)$. For convenience, define $r$ as the remainder service time of the cell which is being served in server.

It is easily seen that $w = CS + r$, where $C$ is the buffer contents, $S$ is the service time of an arbitrary cell. So $W(z)$ is given by

\[
W(z) = C(S(z))r(z) = Q_0(z, S(z)) + Q_1(z, S(z)).
\]

6. Cell delay

The next step is to compute the statistics of the cell delay. The delay of an arbitrary cell is defined as the total number of slots between the end of the arrival slot of the cell and the departure instant of the cell. Since a cell can only leave the buffer at slot boundaries, the cell delay always consists of an integer number of slots and can hence be described as a discrete random variable denoted by $D$. $D(z)$ is the pgf of $D$, which is given by

\[
D(z) = \lim_{n \to \infty} E[z^n | a_n = 1] = \lim_{n \to \infty} \sum_{i=1}^{\infty} P\{w^*_n = i | A_n = 1\}z^i = \lim_{n \to \infty} \sum_{i=1}^{\infty} \frac{P\{w^*_n = i, A_n = 1\}}{P\{A_n = 1\}} z^i,
\]

where $\lim_{n \to \infty} w^*_n = (C + 1)S + r$, and for $\lim_{n \to \infty} P\{A_n = 1\} = U$, so we have

\[
D(z) = \frac{1}{U} Q_1(z, S(z)).
\]
The average cell delay can then be found as $E[D] = D'(1)$. It has been verified that $E[D] = E[C]/U$, in accordance with Little’s result.

If arrivals of cells are uncorrelated (that is to say, $\alpha + \beta = 1$), then the system reduce to the Geo/G/1 queue. We have the following compact expression:

$$C(z) = (1 - \alpha \bar{S}) \frac{(z - 1)S(1 + \alpha z - \alpha)}{z - S(1 + \alpha z - \alpha)},$$

$$D(z) = \frac{1 - \alpha \bar{S}(z)(z - 1)}{z - 1 + \alpha - \alpha \bar{S}(z)}.$$

7. Numerical example

In this section, we present some numerical examples on the performance measures derived in Sections 4 and 6. We assume that the service times are geometrical distributed, i.e., $S(z) = \frac{\alpha z}{z - 1}$.

![Fig. 2. Average buffer content vs. mean arrival rate.](image1)

![Fig. 3. Average cell delay vs. mean arrival rate.](image2)
Figs. 2 and 3 depict the behaviour of the average buffer content and cell delay of system against the mean arrival rate $U$, respectively. As was expected, both $E[C]$ and $E[D]$ are increasing as function of $U$. In Fig. 2, we present three curves corresponding to $B = 1, 5, 8$. As is to be expected, $E[C]$ increases with increasing values of burstiness factor $B$. Fig. 3 studies how the mean arrival rate affects the average cell delay. This graphic corroborates that the expectation $E[D]$ increases with increasing values of $U$.

The average buffer content and cell delay vs. the burstiness factor are depicted in Figs. 4 and 5, respectively. From these two figures, we observe that the larger the burstiness factor $B$, the worse the performance measure of the system. When the burstiness factor is large enough, the performance measure is very bad, even though mean arrival rate is very little (for example $B = 15$, $U = 0.1$ in Fig. 3). It is verified that bursty of arrivals will negatively affect the performance of the system (see Figs. 4 and 5).

**8. Conclusions**

We have studied a discrete-time queueing system with correlated arrivals and generally distributed service times. These arrivals are modelled using an on/off source process with geometrically distributed on-times and
off-times. Based on probability generating function approach, we obtain the explicit close-form expressions for some performance measures, such as average buffer content, average unfinished work, average cell delay and so on. The paper illustrates that effects of correlation of arrivals on performance measures of system could not be ignored.

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References