The infinity norm bound for the inverse of nonsingular diagonal dominant matrices

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Received 6 November 2006; received in revised form 8 March 2007; accepted 13 March 2007

Abstract

In this note, we bound the inverse of nonsingular diagonal dominant matrices under the infinity norm. This bound is always sharper than the one in [P.N. Shivakumar, et al., On two-sided bounds related to weakly diagonally dominant \(M\)-matrices with application to digital dynamics, SIAM J. Matrix Anal. Appl. 17 (2) (1996) 298–312].

Keywords: Diagonal dominant matrix; Weakly chained diagonal dominant matrix; Infinity norm; Inverse; Bound

1. Introduction

By \(C^{n \times n}\) (\(C^n\)) we denote all complex matrices (\(n\)-dimension vectors) of order \(n\). Let \(A = (a_{ij}) \in C^{n \times n}\). By \(|A|\) we denote that \(|A| = (|a_{ij}|)\). \(A\) is called a \(Z\)-matrix if \(a_{ij} \leq 0\) for any \(i \neq j\); a nonsingular \(M\)-matrix if \(A\) is a \(Z\)-matrix with \(A^{-1}\) nonnegative (denoted by \(A^{-1} \geq 0\)). The comparison matrix of \(A\) is denoted by \(\langle A \rangle\), i.e., \(\langle A \rangle = (\tilde{a}_{ij})\) with

\[
\tilde{a}_{ij} = \begin{cases} 
|a_{ii}|, & i = j \\
-|a_{ij}|, & i \neq j 
\end{cases}
\]

Let \(A = (a_{ij}) \in C^{n \times n}\). Throughout this note we always assume that \(A = D - L - U\), where \(D\), \(-L\) and \(-U\) are nonsingular diagonal, strict lower and strict upper triangular parts of \(A\). Notice that \(\langle A \rangle = |D| - |L| - |U|\).

Let \(B = (b_{ij}) \in C^{n \times m}\). By \(A_i(B)\) and \(r_i(B)\) we denote that

\[
A_i(B) = \sum_{i \neq k \in (n)} |b_{ik}| \quad \text{and} \quad r_i(B) = \sum_{k \in (n)} |b_{ik}|
\]

respectively, where \((n) = \{1, 2, \ldots, n\}\).

Let \(e = (1, \ldots, 1)^T\) with appropriate dimension. Then \(\langle A \rangle e = (|a_{11}| - A_1(A), \ldots, |a_{nn}| - A_n(A))^T\). We define

\[
|L|e = (l_1, \ldots, l_n), \quad |U|e = (u_1, \ldots, u_n).
\]

Then \(A_i(A) = l_i + u_i\). Let \(y \in C^n\). By \((y)_i\) we denote the \(i\)th entry of \(y\).

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0893-9659/$ - see front matter © 2008 Published by Elsevier Ltd
doi:10.1016/j.aml.2007.03.018
Definition 1.1. Let \( A = (a_{ij}) \in C^{n \times n} \). Then \( A \) is said to be

1. a diagonally dominant matrix (d.d.) if \( |a_{ii}| \geq \sum_{j \neq i} |a_{ji}| \) for each \( i \in \{n\} \);
2. a strictly diagonally dominant matrix (s.d.d.) if \( |a_{ii}| > \sum_{j \neq i} |a_{ji}| \) for each \( i \in \{n\} \);
3. a weakly chained diagonally dominant matrix (w.c.d.d.) if \( A \) is a d.d. matrix, and for all \( i \in \{n\}, i \notin \beta(A) = \{j \mid |a_{jj}| > \sum_{j \neq k \in \{n\}} |a_{jk}|\} \) there exist indices \( i_1, \ldots, i_k \in \{n\} \) with \( a_{i_0,i_{r+1}} \neq 0, 0 \leq r < k - 1 \), where \( i_0 = i \) and \( i_k \in \beta(A) \).
4. a generalized diagonally dominant matrix (g.d.d.) or an \( H \)-matrix if there is a positive diagonal matrix \( D \) such that \( DA \) is a s.d.d. matrix.

It is noted that a d.d. matrix \( A \) is an \( H \)-matrix if and only if \( A \) is a w.c.d.d. matrix (see Theorem 3.3 of [3]).

In [5] the author obtained a bound of \( \|A^{-1}\|_\infty \) for a strictly diagonally dominant matrix \( A \), i.e.,

\[
\|A^{-1}\|_\infty \leq \max_{i \in \{n\}} \left\{ \frac{1}{|a_{ii}| - \Lambda_i(A)} \right\} .
\] (1.1)

However, some application problems such as in digital circuit dynamics are related to w.c.d.d. matrices; the authors in [4] first provided a finite bound for the infinity norm of the inverse of w.c.d.d. \( M \)-matrices with \( a_j < a_{jj}, \forall j \in \{n\} \):

\[
\|A^{-1}\|_\infty \leq \sum_{i=1}^{n} \left[ a_{ii} \prod_{j=1}^{i} \left( 1 - \frac{u_j}{a_{jj}} \right) \right]^{-1} .
\] (2.1)

The bounds (1.1) and (2.1) can be applied to estimate the condition number of a matrix and the lower bound of the minimal eigenvalue of a w.c.d.d. \( M \)-matrix [4,5].

In this note, the bound for the infinity norm of the inverse of w.c.d.d. matrices is further discussed. Our bound given in the Section 2 is always sharper than the bound in (2.1); see Theorem 2.4.

2. A bound on \( \|A^{-1}\|_\infty \)

Let \( \alpha_1 \) and \( \alpha_2 \) be two subsets of \( \{n\} \) such that \( \{n\} = \alpha_1 \cup \alpha_2 \) and \( \alpha_1 \cap \alpha_2 = \emptyset \). By \( A_{ij} = A[\alpha_i \mid \alpha_j] \) we denote the submatrix of \( A \) whose rows are indexed by \( \alpha_i \) and columns by \( \alpha_j \). For simplicity, we use \( A[\alpha_i] \) instead of \( A[\alpha_i \mid \alpha_i] \). If \( A[\alpha_i] \) is nonsingular, by \( S_{\alpha_i} \) we mean the Schur complement of \( A[\alpha_i] \) in \( A \), i.e., \( S_{\alpha_i} = A[\alpha_2 \mid \alpha_1]A[\alpha_1]^{-1}A[\alpha_1 \mid \alpha_2] \). By \( A(k) \) we denote \( A(k) = A[\alpha^{(k-1)}] \), where \( \alpha^{(k)} = \{k + 1, \ldots, n\} \).

Let \( A = (a_{ij}) \in C^{n \times n} \). We define \( s_k(A) \) by the following recursive equations:

\[
s_n(A) = A_n(A), \quad s_k(A) = \sum_{i=1}^{k-1} |a_{ki}| + \sum_{i=k+1}^{n} |a_{ki}| \frac{s_i(A)}{|a_{ii}|}, \quad k = n - 1, \ldots, 1.
\] (2.1)

It is noted that \( s_k(A) \) can be computed easily using the iterative formula (2.1) for \( k = n, \ldots, 1 \).

Lemma 2.1. Let \( A = (a_{ij}) \in C^{n \times n} \). Then

\[
|D| (|D| - |U|)^{-1} |L| e = (s_1(A), \ldots, s_n(A))^T.
\] (2.2)

Proof. Let \( (|D| - |U|)^{-1} |L| e = (x_1, \ldots, x_n)^T \). Then \( |L| e = (|D| - |U|) x \), i.e.,

\[
|D| x = |L| e + |U| x.
\]

Notice that \( -|U| \) is a strictly lower triangular part of \( A \). Then we have

\[
x_n = \frac{1}{|a_{nn}|} A_n(A), \quad x_k = \frac{1}{|a_{kk}|} \left[ \sum_{i=1}^{k-1} |a_{ki}| + \sum_{i=k+1}^{n} |a_{ki}| x_i \right], \quad k = n - 1, \ldots, 1.
\]

Hence \( s_n(A) = |a_{nn}| x_n, \ldots, s_k(A) = |a_{kk}| x_k, \quad k = n - 1, \ldots, 1 \), which implies that the lemma holds. \( \blacksquare \)
Lemma 2.2 ([2]). Let $A, B \in C^{n \times n}$, and let $(A)$ be a nonsingular $M$-matrix. Then

$$|A^{-1}B| \leq (A)^{-1}|B|.$$  

Now we partition $A$ into the following block form:

$$A = \begin{pmatrix} a_{11} & x^T \\ y & A_{(1)} \end{pmatrix}. \quad (2.3)$$

Then it is easy to check that

$$A^{-1} = \begin{pmatrix} S_1^{-1} & -S_1^{-1}x^TA_{(1)}^{-1} \\ -S_1^{-1}A_{(1)}^{-1}y & A_{(1)}^{-1} + S_1^{-1}A_{(1)}^{-1}yx^TA_{(1)}^{-1} \end{pmatrix}, \quad (2.4)$$

where $S_1 = a_{11} - x^TA_{(1)}^{-1}y$.

Lemma 2.3. Let $(A)$ be a nonsingular d.d. $M$-matrix and $A^{-1} = (a'_{ij})$. Then

$$|a'_{11}| \leq \frac{s_j(A)}{|a_{ii}|}|a'_{11}| \leq \frac{A_j(A)}{|a_{ii}|}|a'_{11}|, \quad 2 \leq i \leq n \quad (2.5)$$

and

$$\frac{1}{|a_{11}| + s_1(A)} \leq |a'_{11}| \leq \frac{1}{|a_{11}| - s_1(A)}. \quad (2.6)$$

Proof. Let $A$ be partitioned into (2.3). Since $(A)$ is a nonsingular $M$-matrix, so is $(A_{(1)})$, which implies that $(A_{(1)})^{-1} \geq 0$ (e.g., see [1]). Let $(A_{(1)}) = D_{(1)} - L_{(1)} - U_{(1)}$, where $D_{(1)}$, $-L_{(1)}$, and $-U_{(1)}$ are nonsingular diagonal, strict lower and strict upper triangular parts of $(A_{(1)})$. Since $(A_{(1)}) \leq D_{(1)} - U_{(1)}$, we have $(A_{(1)})^{-1} \geq (D_{(1)} - U_{(1)})^{-1}$ (see [1]). Set $z = ([a_{22}] - A_2(A), \ldots, [a_{nn}] - A_n(A))^T$. By the assumption that $A$ is a d.d. matrix, we have $z \geq 0$, which implies that

$$(A_{(1)})^{-1}z \geq (D_{(1)} - U_{(1)})^{-1}z. \quad (2.7)$$

Notice that $(A_{(1)})e - |y| = z$. Then

$$(A_{(1)})^{-1}|y| = e - (A_{(1)})^{-1}z, \quad (2.8)$$

which together with (2.7) gives

$$(A_{(1)})^{-1}|y| \leq e - (D_{(1)} - U_{(1)})^{-1}z$$

$$= (D_{(1)} - U_{(1)})^{-1}[(D_{(1)} - U_{(1)})e - z]$$

$$= (D_{(1)} - U_{(1)})^{-1}[[A_2(A)] + L_{(1)})e - (A_{(1)})e + |y|]$$

$$= (D_{(1)} - U_{(1)})^{-1}[|y|, L_{(1)}]e. \quad (2.9)$$

It follows from Lemma 2.1 that

$$(D_{(1)} - U_{(1)})^{-1}[|y|, L_{(1)}]e = \left( \frac{s_2(A)}{|a_{22}|}, \ldots, \frac{s_n(A)}{|a_{nn}|} \right)^T,$$

which together with (2.9) gives

$$(A_{(1)})^{-1}|y| \leq \left( \frac{s_2(A)}{|a_{22}|}, \ldots, \frac{s_n(A)}{|a_{nn}|} \right)^T. \quad (2.10)$$

From Lemma 2.2 it follows that

$$|S_1^{-1}A_{(1)}^{-1}y| \leq |S_1^{-1}|(A_{(1)})^{-1}|y|. \quad (2.11)$$
Combining (2.4), (2.10) and (2.11) one may deduce that 
\[ |a'_{i1}| = |S^{-1}_1 (A^{-1}_i y)_i| \leq |S^{-1}_1| \frac{s_i(A)}{|a_{ii}|} = \frac{s_i(A)}{|a_{ii}|} |a'_{i1}|, \]
which proves the leftmost inequality of (2.5). By the assumption on \( A \), we have \( |a_{ii}| \geq \Lambda_i(A) \), and thus \( s_i(A) \leq \Lambda_i(A) \), \( i = 1, \ldots, n \), from which one may deduce the desired inequality (2.5).

From \( A^{-1}A = I \) we have
\[ a'_{11}a_{11} + \sum_{j=2}^{n} a_{1j}a'_{j1} = 1. \]
Hence
\[ |a'_{11}a_{11}| \leq 1 + \sum_{j=2}^{n} |a_{1j}a'_{j1}| \]
\[ \leq 1 + |a'_{11}| \sum_{j=2}^{n} |a_{1j}| \frac{s_j(A)}{|a_{jj}|} \]
\[ = 1 + |a'_{11}|s_1(A), \]
which implies that
\[ |a'_{11}|(|a_{11}| - s_1(A)) \leq 1, \]
from which the second inequality of (2.6) follows. The proof of the first inequality is analogous. ■

Recall the definitions of \( l_k \) and \( r_k \) in Section 1; we have the following main result in this note.

**Theorem 2.4.** Let \( A \in \mathbb{C}^{n \times n} \) be a w.c.d.d. matrix with \( |a_{kk}| + l_k > s_k(A), \ k = 1, \ldots, n \). Then
\[ \|A^{-1}\|_\infty \leq \sum_{i=1}^{n} \prod_{k=1}^{i} \frac{h_k}{a_{kk} + l_k - s_k(A)}, \tag{2.12} \]
where \( h_1 = 1, h_k = r_{k-1}(A) - s_{k-1}(A), \ k = 2, \ldots, n \).

**Proof.** By Theorem 3.3 of [3], \( A \) is a g.d.d. matrix, and hence \( \langle A \rangle \) is a nonsingular d.d. \( M \)-matrix. It follows from Lemma 2.2 that
\[ |A^{-1}| \leq \langle A \rangle^{-1}, \]
which implies that
\[ \|A^{-1}\|_\infty \leq \|\langle A \rangle^{-1}\|_\infty. \]
Hence without loss of generality we may assume that \( A \) is a nonsingular d.d. \( M \)-matrix.

By (2.4), the sum of row 1 of \( A^{-1} \) is
\[ (A^{-1}e)_1 = (-S_1^{-1}x^TA_{(1)}^{-1}e) + S_1^{-1} = S_1^{-1}(x^TA_{(1)}^1e + 1). \tag{2.13} \]
Since \( A \) is a nonsingular d.d. matrix, from (2.3) we have
\[ A_{(1)}e - |y| \geq 0. \tag{2.14} \]
Notice that a principal submatrix of a nonsingular \( M \)-matrix is also a nonsingular \( M \)-matrix (e.g. see [1]). Hence \( A_{(1)} \) is also a nonsingular d.d. \( M \)-matrix. Then \( A_{(1)}^{-1} \geq 0 \) (e.g., see [1]). By (2.14) we have
\[ A_{(1)}^{-1}|y| \leq e. \]
By (2.4), row \( i \) \((>1)\) of \( A^{-1} \) is

\[
(A^{-1} e)_i = [(A^{-1}_i)_1 e + S^{-1}_1 A^{-1}_i x^T A^{-1}_i e + S^{-1}_1 A^{-1}_i |y|]_i \\
= (A^{-1}_i)_1 e_i + S^{-1}_1 (A^{-1}_i)_i (|x^T A^{-1}_i e| + 1) \\
= (A^{-1}_i)_1 e_i + (A^{-1}_i |y|)_i (A^{-1} e)_1. \\
\leq (A^{-1}_i)_1 e_i + (A^{-1} e)_1.
\]

By (2.6),

\[
S^{-1}_1 \leq \frac{1}{a_{11} - s_1(A)}.
\]

From (2.13) and (2.16) one may deduce that

\[
(A^{-1} e)_1 = S^{-1}_1 (|x^T A^{-1}_i e| + 1) \leq \frac{1}{a_{11} - s_1(A)} (1 + u_1 \|A^{-1}_i\|_\infty).
\]

By (2.15) and (2.17), we have

\[
\|A^{-1}\|_\infty = \max_{1 \leq i \leq n} (A^{-1} e)_i \\
\leq (A^{-1}_i)_1 e_i + (A^{-1} e)_1 \\
\leq \|A^{-1}_i\|_\infty + \frac{1}{a_{11} - s_1(A)} (1 + u_1 \|A^{-1}_i\|_\infty) \\
\leq \frac{1}{a_{11} - s_1(A)} (1 + \frac{u_1}{a_{11} - s_1(A)}) \|A^{-1}_i\|_\infty \\
= \frac{1}{a_{11} - s_1(A)} + \left(1 + \frac{u_1}{a_{11} - s_1(A)}\right) \|A^{-1}_i\|_\infty \\
= \frac{h_1}{a_{11} - s_1(A)} + \left(\frac{h_1 h_2}{a_{11} - s_1(A)}\right) \|A^{-1}_i\|_\infty. 
\]

By the above proof, we have

\[
\|A^{-1}_i\|_\infty \leq \frac{1}{a_{22} - s_1(A)} + \left(1 + \frac{u_2}{a_{22} - s_1(A)}\right) \|A^{-1}_2\|_\infty.
\]

Since \( s_n(A_{(1)}) = \Lambda_n(A_{(1)}) = \Lambda_n(A) = a_{11} = s_1(A) \), it is easy to see that

\[
s_1(A_{(1)}) = \sum_{i=3}^n |a_{2i}| \frac{s_i(A_{(1)})}{|a_{ii}|} \leq \sum_{i=3}^n |a_{2i}| \frac{s_i(A)}{|a_{ii}|} = s_2(A) - l_2,
\]

which together with the assumption of this theorem gives

\[
a_{22} - s_1(A_{(1)}) \geq a_{22} + l_2 - s_2(A) > 0,
\]

and then from (2.19) it follows that

\[
\|A^{-1}_i\|_\infty \leq \frac{1}{a_{22} + l_2 - s_2(A)} + \left(1 + \frac{u_2}{a_{22} + l_2 - s_2(A)}\right) \|A^{-1}_2\|_\infty \\
= \frac{1}{a_{22} + l_2 - s_2(A)} + \frac{a_{22} - s_2(A) + l_2 + u_2}{a_{22} + l_2 - s_2(A)} \|A^{-1}_2\|_\infty \\
= \frac{1}{a_{22} + l_2 - s_2(A)} + \frac{r_2(A) - s_2(A)}{a_{22} + l_2 - s_2(A)} \|A^{-1}_2\|_\infty \\
= \frac{1}{a_{22} + l_2 - s_2(A)} + \frac{h_3}{a_{22} + l_2 - s_2(A)} \|A^{-1}_2\|_\infty. 
\]
From (2.18) and (2.20) it follows that
\[
\|A^{-1}\|_{\infty} \leq \frac{h_1}{a_{11} + l_1 - s_1(A)} + \frac{h_1 h_2}{(a_{11} + l_1 - s_1(A))(a_{22} + l_2 - s_2(A))} + \frac{h_1 h_2 h_3}{(a_{11} + l_1 - s_1(A))(a_{22} + l_2 - s_2(A))}\|A_{(2)}^{-1}\|_{\infty}.
\]

Going on in this way one may deduce the desired inequality (2.12). ■

**Remark 2.1.** Now we may compare (2.12) with (1.2).

- The condition in Theorem 2.4 is weaker than those in (1.2). In fact, since $A$ is a d.d. matrix, $|a_{ij}| \geq A_i(A)$, and hence by (2.1) we have $|a_{nn}| \geq A_n(A) = s_n(A)$, $|a_{n-1,n-1}| \geq A_{n-1}(A) \geq s_{n-1}(A)$, . . . , $|a_{11}| \geq A_1(A) \geq s_1(A)$, i.e., $s_i(A)/|a_{ii}| \leq 1$, and hence $\sum_{i=k+1}^{n} |a_{ki}| s_i(A)/|a_{ii}| \leq u_k$.

- The bound (2.12) is sharper than the bound (1.2) because the following inequalities hold:
  \[
  \frac{h_{k+1}}{a_{kk} + l_k - s_k(A)} = \frac{r_k(A) - s_k(A)}{a_{kk} + l_k - s_k(A)} = \frac{a_{kk} + l_k + u_k - s_k(A)}{a_{kk} + l_k - s_k(A)} = 1 + \frac{u_k}{a_{kk} - u_k}, \quad k = 1, \ldots, n - 1,
  \]
  \[
  \frac{1}{a_{jj} + l_j - s_j(A)} = \frac{1}{a_{jj} - u_j}, \quad j = 1, \ldots, n.
  \]

Hence our result in Theorem 2.4 always improves the corresponding one in [4].

**Remark 2.2.** It is noted from Remark 2.1 that $|a_{kk}| + l_k \geq s_k(A), \quad k = 1, \ldots, n$ for a d.d. matrix $A$. The condition in Theorem 2.4 that $|a_{kk}| + l_k > s_k(A), \quad k = 1, \ldots, n$ guarantees that the denominator in the bound (2.12) is nonzero, and hence Theorem 2.4 provides a finite bound for a w.c.d.d. matrix.

**Remark 2.3.** The sharper bound may be obtained if we replace $l_k - s_k(A)$ with $s_1(A_{(k-1)})$. But this bound seems complicated for the computation because for $k = 1, \ldots, n - 1$ one needs to compute $s_i(A_{(k-1)}), \ i = n, \ldots, k$.

**Acknowledgments**

The author would like to thank the referees for their valuable comments.

This work was supported in part by NNSF of China (10671077) and the Natural Science Foundation of Guangdong Province (06025061, 031496).

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