Loss Tomography from Tree Topologies to General Topologies
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Abstract—Loss tomography has received considerable attention in recent years and a number of estimators based on maximum likelihood (ML) or Bayesian principles have been proposed. Almost all of the estimators are devoted to the tree topology despite the general topology is more common in practice. Among the proposed estimators, most of them use an iterative procedure to search for the maximum of a likelihood equation obtained from observations. Without the detail of the solution space, an iterative procedure can be computationally expensive and may even converge to a local maximum. To overcome those, the following three questions need to be addressed: 1) whether there is a closed form solution to estimate the loss rates of a link or a path for the tree topology; 2) whether there is a unique maximum likelihood estimate (MLE) for a link or a path of the general topology; and 3) if so, how to obtain the MLE by a method other than an iterative procedure. This paper is devoted to address the three questions and provide the results obtained recently that include a closed form solution to estimate the loss rates of a tree network, a direct expression of the MLE for the general topology, a divide-and-conquer strategy to decompose a general network into a number of independent trees, and the method to estimate the loss rates of the decomposed trees.

Index Terms—Decomposition, General topology, Loss tomography, Top down algorithm, Tree topology, Transformation.

I. INTRODUCTION

Network characteristics, such as loss rate, delay, available bandwidth, and their distributions, are critical to various network operations and also important to network research, e.g. modeling. Because of these, considerable attention has been given to network measurement, in particular to large networks. However, due to various reasons, e.g. security, commercial interests and administrative boundary, some of the characteristics cannot be obtained directly from a large network. To overcome the limitations, network tomography is proposed in [1], where the author suggests the use of end-to-end measurement and statistical inference to estimate the characteristics of a large network. In an active approach, a number of sources are attached to the network of interest to send probes to the receivers attached to the other side of the network, where the paths from the sources to the receivers cover the links of interest. The probes used in network tomography are sent in a coordinated manner that aims to create more information for inference. The arrivals, arrival orders and arrival time of probes carry the information of the network that deliver the probes to receivers. By properly organizing the paths of probing, we are able to collect information for various characteristics. The characteristics that have been estimated in this manner include link-level loss rates [2], delay distribution [3], [4], [5], [6], [7], and loss pattern [8]. In this paper, our attention is focused on the loss rate inference, which can be easily extended to estimate loss pattern.

The topologies connecting sources to receivers can be divided into two classes: tree and general. The tree topology as named has a single source attached to the root of a multicast tree to send probes to receivers attached to the leaf nodes of the multicast tree. In contrast, a network of the general topology requires a number of trees to cover all links of the network. Each of the trees has a source sending probes to receivers. Because of the use of multiple sources to send probes, the receivers, nodes, and links located in an intersection of multiple trees can receive probes sent from multiple sources. Then, estimation must consider the impacts of all probes sent by the sources. Because of the differences between the two topologies, estimators or algorithms have been divided into two classes, one for a topology. At this moment, almost all of the estimators proposed so far are devoted to the tree topology because the correlation embedded in the observation of a tree topology is much simpler than that of the general one. Although a large number of works have been devoted to the tree topology, there have been only a few of them providing analytical solutions as [9] [10] while the majority of them rely on iterative procedures to search for the maximum of a likelihood equation built on observations. In contrast to the tree topology, there has been little research for the general topology although the majority of networks in practice fall into this category. This is partially due to the lack of understanding the correlations embedded in the probes sent by multiple sources. To tackle the problems, we, in this paper, propose a closed from solution to obtain MLEs for the tree topology, including all proofs. We then propose a divide-and-conquer strategy to decompose a general network into two types of independent trees. Finally, we propose two estimators, one for a type of the independent trees.

To estimate link-level loss rates from end-to-end measurements, a loss model is needed to describe the loss behavior of a link with some unknown parameters that need to be determined by statistical inference. If the probes sent in an experiment are far apart and the traffic is stable, the observations obtained at receivers can be considered independent identical distributed (i.i.d.) and the corresponding likelihood function of the observations takes the product form of the individual ones. Statistical inference is then applied on the likelihood function to estimate those unknown parameters, where ML or Bayesian principles are often used in estimation. No matter which principle is used in estimation, an estimator

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must try to use all available information in observations, where sufficient statistics are crucial to an effective and efficient estimator. Unfortunately this issue has been overlooked in the past that leads to the wide use of iterative procedures, such as the expectation and maximization (EM), to approximate the maximum of the likelihood function. Nevertheless, an iterative approach, as the EM, may be computational expensive [11].

A. Contribution and Paper Organization

As stated, there has been a lack of scalable estimators in loss tomography for the tree topology and there has been a lack of both analytical results and scalable estimators for the general topology. To overcome the problems, we in this paper present the latest findings that partially solve the problems, which, comparing to [9], [12], have two-fold contributions to general topology. For the tree topology, there are 3 contributions:

1) A set of the minimal sufficient statistics is introduced and used to rewrite the likelihood function. From the likelihood function, a set of likelihood equations is derived that shows the loss rate of a link depends on the loss rates of its ancestors and descendants and can be expressed in a polynomial as [9].

2) An statistic equivalent replacement is proposed to use a tree to replace a multi-descendant node. Subsequently, there is a closed form solution to loss rate estimation. Further, a top down algorithm is proposed to estimate the loss rates of a network from top to bottom.

3) The solution obtained after replacement is the MLE and the solution space is proved to be concave if a Bernoulli model is used to describe the loss behavior of a link. The finding ensures that the estimates obtained by an iterative procedure, e.g. EM, are MLEs.

There are also 3 contributions to the general topology:

1) The analytical results obtained for the tree topology have been extended to the general topology, and the direct expression of the MLE is derived for the loss rate of a link. The direct expression has a similar structure as that of the tree topology. Then, the key results obtained from the tree topology hold for the general topology as well.

2) By proving the direct expression leading to an MLE, we also prove the solution space is concave and the estimate obtained by an iterative algorithm, such as the EM, the fixed point and MCMC, is an MLE.

3) A divide-and-conquer strategy is proposed to decompose a general network into two type of independent trees; one is those rooted at the original roots, the other is those rooted at the decomposed points. Further, two estimators are proposed, one for a type of the trees.

The rest of the paper is organized as follows. In Section 2, we present the related works and the notations used in this paper. In addition, the set of sufficient statistics is introduced in this section. Section 3 consists of three parts: a) writing the likelihood function of an experiment with the set of sufficient statistics; b) deriving the likelihood equation from the likelihood function; c) comparing the likelihood equation with that presented in [9]. Section 4 is devoted to prove the statistics introduced in this paper is the complete minimal sufficient ones. The statistical equivalent replacement and the top down algorithm are detailed in Section 5. Section 6 extends the results obtained from the tree topology to the general one.

II. RELATED WORKS AND PROBLEM FORMULATION

A. Related Works

Multicast Inference of Network Characters (MINC) is the pioneer of using multicast probes to create correlated observations, where a Bernoulli model is used to model the loss behaviors of a link. Using this model, the authors of [9] derive an MLE for the pass rate of a path connecting the source to a node. The MLE is expressed in a polynomial that is one degree less than the number of descendants of the node [9], [13], [14]. To ease the concern of using numeric method to solve a higher degree polynomial (≥5), the authors of [10] propose an explicit estimator. Although the estimator has the asymptotic variance as the MLE to first order, it is not an MLE. The two estimators are dedicated to the tree topology. Later, Bu et. al. attempted to extend the strategy to the general topology [15]. Using the same model as [9], Bu et. al. failed to derive a direct expression as the one presented in [9] for the general topology. The authors then resorted on iterative procedures, i.e. the EM, to approximate the maximum of the likelihood function. In addition, a minimum variance weighted average (MVWA) method is used for comparison with the EM one. The problem of MVWA is the bias of its estimate when the sample size is small [16]. The experimental results presented in [15] confirm this problem, where the EM outperforms the MVWA when the sample size is small. Nevertheless, the solution identified by the EM algorithm could not be verified to be the MLE since there was no proof that the search space is concave. In addition, iterative procedures, e.g. EM algorithm, have their interionic weakness as previously stated in [17]. Considering the unavailability of multicast in some networks, Harfoush et al. and Coates et al. independently proposed the use of the unicast-based multicast to send probes to receivers [18], [19], where Coates et al. also suggested the use of the EM algorithm to estimate link-level loss rates. Apart from those, Rabbat et al. in [20] consider network tomography on general networks and found a general network comprised of multiple sources and multiple receivers can be decomposed into a number of 2 by 2 components, and further proposed the use of the generalized likelihood ratio test to identify network topology. To improve the scalability of an estimator, Zhu and Geng propose a bottom up estimator for the tree topology in [21]. The estimator actually is topology independent and can be applied to the general topology as well [12]. The estimator adopts a step by step approach to estimate the loss rate of a link, one at a time and from bottom up. At each step the estimator uses a formula to compute the loss rate of a link. Despite the effectiveness, scalability, and extensibility to the general topology, the estimate obtained by the estimator is
not the MLE as that obtained by the explicit estimator of \[10\] because the statistics used by the two estimators are not sufficient ones.

**B. Notation**

The multicast tree used to send deliver probes to their receivers is slightly different from an ordinary one at its root that has only a single descendant. Let \( T = (V, E, \theta) \) donate the multicast tree, where \( V = \{v_0, v_1, \ldots, v_m\} \) is the set of nodes representing routers and switches of a network; \( E = \{e_1, \ldots, e_m\} \) is the set of directed links connecting the nodes of \( V \), the two nodes connected by a link are called the parent node and the child node of the link, where the parent forwards received probes to the child; and \( \theta = \{\theta_1, \ldots, \theta_m\} \) is the set of parameters to be estimated, one for a link to describe the loss rate of the link. Figure 1 is an example of the multicast tree discussed above. To distinguish a link from another, each link is assigned a unique number from 1 to \( m \); similarly, each node also has a unique number from 0 to \( m \), where link \( i \) is used to connect node \( i \)'s parent node to node \( i \). The numbers are assigned to the nodes from small to big along the tree structure from top to bottom as shown in Figure 1. The source is attached to node 0 to send probes to the receivers attached to the leaf nodes of \( T \). \( R \) is used to denote all receivers. In contrast to \[9\] and \[10\] that use node \( i \) for every possible \( x \), but also needs to count the number of occurrences of each \( x \). However, after those efforts, it can be long and tedious, if not impossible, to derive the likelihood equation from it. Thus, the log-likelihood function needs to be rewritten since the current one is more symbolic than practical.

**D. Internal State and Internal View**

Instead of using the log-likelihood function as \( \frac{\partial}{\partial \theta} \), we consider to rewrite it in a different form. Let \( P_{i3}(\theta) \) denote the likelihood function. Under the i.i.d. assumption of probes, we have

\[
P_{i3}(\theta) = \prod_{i=1}^{n} P(X^{(i)}, \theta)
\]

and under the Bernoulli assumption of losses at a link, we have

\[
P(X^{(i)}, \theta) = \prod_{j \in E_1} (1 - \beta_j) \prod_{k \in E_2} (\theta_k + (1 - \theta_k)(1 - \beta_k))
\]

for \( X^{(i)} \), where \( E_1 \) is the set of links that observe the probe corresponding to \( X^{(i)} \), i.e. the observation is confirmed from \( X^{(i)} \); \( E_2 \) is the set of links whose observation of the probe cannot be confirmed but their parents’ are confirmed, and \( \beta_j = P(\forall r \in R(i) X_r = 1 | X_i = 1; \theta) \) is the pass rate of subtree \( i \). If we do not distinguish the difference between \( E_1 \) and \( E_2 \) and let \( P_{i3}(\theta_j) \) denote the probability of link \( j \) for \( X^{(i)} \), we have

\[
P_{i3}(\theta) = \prod_{i=1}^{n} \prod_{j \in E} P_{i3}(\theta_j)
\]

With finite \( n \) and \( |E| \), the order of the products is commutable and different orders can lead to different outcomes. In order to rewrite the log-likelihood function, we change the order of the products and have

\[
P_{i3}(\theta) = \prod_{j \in E} \prod_{i=1}^{n} P_{i3}(\theta_j).
\]
Based on the likelihood function, we need to find the state of each link from $X^{(i)}, i \in \{1, n\}$ and accumulate them that lead to two concepts called internal state and internal view. The internal state of a link is obtained from $X^{(i)}$ to confirm whether node $i$ passes link $i$ while the internal view of a link sums the internal states of the link from $X^{(1)}$ to $X^{(m)}$. The formal definitions of them are as follows.

1) **Internal State:** Given the observations of probe $j$ at $R(i)$ and $R(f(i))$, we are able to partially confirm whether the probe passing link $i$. For observation $X^{(j)}_k, k \in R(i)$, we have

$$Y^j_i = \max_{k \in R(i)} x^j_k, \quad j \in \{1, \ldots, n\}$$

as the internal state of link $i$ for probe $j$. If $Y^j_i = 1$ probe $j$ reaches at least one receiver attached to $T(i)$, that also implies the probe passes link $i$. Further, considering the values of $Y^j_i$ and $Y^j_{f(i)}, i, f(i) \in E$, we have following scenarios:

1) $Y^j_i = Y^j_{f(i)} = 1$, this means that probe $j$ passes link $i$; or
2) $Y^j_i = 0$ and $Y^j_{f(i)} = 1$, this means that probe $j$ reaches node $f(i)$ and then become uncertain in $T(i)$, i.e. it is uncertain whether the probe is lost on link $i$ or lost in the subtrees rooted at node $i$; or
3) $Y^j_i = Y^j_{f(i)} = 0$, this means the probe becomes uncertain at one of a(a(i) and the uncertainty is transferred from the ancestor to node $i$.

For link $i$, if the first scenario occurs, we need to have $(1 - \theta_i)$ in the likelihood function for this probe; if the second one occurs, we need to have $(\theta_i + (1 - \theta_i)(1 - \beta_i))$ in the likelihood function; and if the last one occurs, we need to have 1 in the likelihood function for this probe since $P(Y^j_i = 0)Y^j_{f(i)} = 0 = 1$. Because the likelihood function is in a product form, we only need to consider the first two scenarios in the process.

2) **Internal View:** Accumulating the states of each link in an experiment, and let

$$n_i(1) = \sum_{j=1}^n Y^j_i,$$

We have the number of probes passing link $i$ confirmed from observations, i.e. at least $n_i(1)$ probes pass link $i$ in the experiment. In addition, let

$$n_i(0) = n_{f(i)}(1) - n_i(1)$$

denote the number of probes that become uncertain in $T(i)$. $n_i(1)$ and $n_i(0)$ are called the internal view of node $i$ that are the statistics collected from observations. We will prove that $n_i(1), i \in E$ is a set of sufficient statistics later in the paper. Using $n_i(1), i \in E$, we are able to rewrite the likelihood function of $\Omega$ in a different format.

III. NEW LIKELIHOOD FUNCTION AND SOLUTION

A. New Likelihood Function

Given $n_i(1)$ and $n_i(0), i \in E$ as defined, we then have the following theorem to write the log-likelihood function.

**Theorem 1.** Given $n_i(1), n_i(0), \theta_i$, and $(1 - \beta_i), i \in \{1, \ldots, m\}$ as previously defined, we can write the log-likelihood function as follows:

$$L(P_{\Theta}(\theta)) = \sum_{i \in E} \left[ n_i(1) \log(1 - \theta_i) + n_i(0) \log(\theta_i + (1 - \theta_i)(1 - \beta_i)) \right] \tag{2}$$

**Proof:** Based on the definition of $n_i(1)$ and $n_i(0)$, we have

$$P(X^j_i; \theta) = \prod_{i \in E} \left[ (1 - \theta_i)Y^j_i (\theta_i + (1 - \theta_i)(1 - \beta_i))^{Y^j_i - Y^j_i} \right].$$

Thus,

$$L(P_{\Theta}(\theta)) = \sum_{k=1}^n \log(P(X^k_i; \theta))$$

$$= \sum_{k=1}^n \sum_{i \in E} \left[ Y^j_i \log(1 - \theta_i) + (Y^j_{f(i)} - Y^j_i) \log(\theta_i + (1 - \theta_i)(1 - \beta_i)) \right]$$

$$= \sum_{i \in E} \left[ n_i(1) \log(1 - \theta_i) + n_i(0) \log(\theta_i + (1 - \theta_i)(1 - \beta_i)) \right] \tag{3}$$

It is easy to prove that (3) is equal to (1) by expanding the complex logarithm functions embedded in (1) to the simplest forms, then grouping similar logarithm functions to have the likelihood function with $n_i(x), i \in E, x \in \{0, 1\}$ as coefficients. (2) is much simpler than (1) and we are able to calculate the derivative of (2) and further have the likelihood equation.

B. Likelihood Equations and Solution

Differentiating (2) with respect to (wrt) each parameter and letting the derivatives be 0, we have a set of likelihood equations as:

$$\frac{\partial L(P_{\Theta}(\theta))}{\partial \theta_i} = - \frac{n_i(1)}{1 - \theta_i} + \frac{n_i(0) \beta_i}{\theta_i + (1 - \theta_i)(1 - \beta_i)} + \sum_{k \in a(i)} \frac{n_k(0) \prod_{i \in E} [(1 - \theta_i) - (1 - \theta_{f(k)})]}{\theta_k + (1 - \theta_k)(1 - \beta_k)} = 0, \quad i = 1, \ldots, m.$$

Reorganizing it, we have

$$\theta_i = \begin{cases} 
\frac{n_i(1)}{1 - \frac{n}{\beta_i}}, & i = 1 \\
\frac{n_i(1)}{n_{f(i)}(1 + \text{imp}(f(i)))}, & i \in E \setminus (L \cup 1) \\
\frac{n_i(0) + \text{imp}(f(i))}{n_{f(i)}(1 + \text{imp}(f(i)))}, & i \in L 
\end{cases} \tag{4}$$
where $L$ denotes the set of links connecting $R$. Now, (4) is a polynomial in the form of $\theta_i = f(\theta_i)$, where

$$imp(f(i)) = \sum_{k \in a(i)} \frac{n_k(0) * pa_i(k) * \xi_i * \prod_{l \in a(i) \setminus \{i\}} \xi_l}{\xi_k} * 1 - \beta(f(i))$$

is the estimated number of $n_j(0), j \in a(i)$ that reaches node $f(i)$ before being lost in $T(i)$, where

$$\xi_i = \theta_i + (1 - \theta_i)(1 - \beta_i)$$

$$\beta_i = 1 - \prod_{j \in d_i} \xi_j$$

$$pa_i(k) = \prod_{l \in a(i) \setminus \{i\}} (1 - \theta_l).$$

Each term in the summation of (5) is for an ancestor $j, j \in a(i)$ and represents the impact of $n_j(0)$ on $\theta_i$, where $pa_i(k)$ is the pass rate of the path from node $f_k(i)$ to node $f(i)$; and $\frac{n_k(0)}{\xi_k}$ is the estimate of the number of probes reaching node $f_k(i)$.

(6)

Each term in the summation of (5) is for an ancestor $j, j \in a(i)$ and represents the impact of $n_j(0)$ on $\theta_i$, where $pa_i(k)$ is the pass rate of the path from node $f_k(i)$ to node $f(i)$; and $\frac{n_k(0)}{\xi_k}$ is the estimate of the number of probes reaching node $f_k(i)$. (5) can be expressed recursively as

$$imp(f(i)) = \frac{(1 - \theta_f(i))(1 - \beta_f(i))(n_{f(0)}(0) + imp(f_2(i))))}{\xi_f(i)}$$

$$= (1 - \theta_f(i))(1 - \beta_f(i)) \cdot \hat{n}_{f(0)}(1)$$

$$= \hat{n}_{f(0)}(1)[\theta_f(i) + (1 - \theta_f(i))(1 - \beta_f(i))] - \hat{n}_{f(0)}(1) \cdot \theta_f(i)$$

$$= \hat{n}_{f(0)}(1) - \hat{n}_{f(0)}(1) \cdot \theta_f(i)$$

where $\hat{n}_i(1)$ is the MLE of the probes reaching node $i$ and $\hat{n}_{i(0)}$ is the MLE of the probes lost in subtree $i$. Note that since $imp(0) = 0$ and $\hat{n}_i(0) = n_i(1), imp(1) = \hat{n}_i(0) - n_i$.

C. Comparison to $H_k(A_k, \gamma)$

Let $\beta_i = 1, \forall i, i \in L$, the three formulæ of (4) become one as:

$$\theta_i = 1 - \frac{\gamma(a(i))}{\beta_i},$$

where $\gamma(a(i))$ is the pass rate of $T(i)$, its empirical pass rate is equal to $\hat{\gamma}(a(i)) = \frac{n_{f(i)}(1)}{n_{f(i)}(1) + imp(f(i))}$. Then we have,

$$\beta_i = 1 - \prod_{j \in d_i} (1 - \frac{\hat{\gamma}(a(j))}{1 - \theta_i})$$

If knowing $\hat{\gamma}(a(i))$, we have $\hat{\beta_i} = \frac{\hat{\gamma}(a(i))}{1 - \theta_i}$, and then (9) turns to a polynomial as that derived in (9), i.e.

$$H_k(A_k, \gamma) = 1 - \frac{\gamma_k}{A_k} - \prod_{j \in d_k} (1 - \frac{\hat{\gamma_j}}{A_k}) = 0$$

where $\gamma_k$ is the pass rate from the source to $R(k)$ and $A_k$ is the pass rate from the source to node $k$. Despite the similarity between (9) and (10), there are three subtle differences between them:

1) (10) is obtained from probabilistic equivalence before being proved that the $A_k$ obtained from the formula is the MLE (9), (22). (9) is obtained directly from the log-likelihood function. If the solution space of the likelihood equation in $[0,1]^m$ is strictly concave, (9) is the MLE.

2) (10) is path-oriented that aims to estimate the pass rate of the path connecting the root to node $k$. $A_k$ is link-oriented, which, based on internal views, estimates the loss rate of link $i$.

3) (9), plus (7), shows the evolution of the internal view from top to bottom and step by step, which also unveils that the uncertainty is built from top to bottom and can be resolved in the same manner in estimation.

IV. PROOF OF SUFFICIENT STATISTICS

There are a number of approaches to prove (9) is the MLE. One of them is to prove the equivalence between (9) and (10) that has been achieved in the previous section. To prove the internal view proposed in this paper is the complete minimal sufficient statistics, we need to prove the (9) belongs to the exponential families.

Given the loss rate of $T(i)$ as

$$\xi_i = \theta_i + (1 - \theta_i)(1 - \beta_i),$$

there is a bijection $\Gamma$ from $\Theta$ to $\xi$, which is invertible. The inverse transform $\Gamma^{-1}$ is

$$\theta_i = \frac{\xi_i - \sum_{k \in C_i} \xi_k}{1 - \prod_{k \in C_i} \xi_k} i \in E.$$  (12)

Using $\xi$ to replace $\theta$ in (2), we have:

$$L(\xi) = \sum_{i \in V} \left[ n_{i(1)} \log\left(\frac{1 - \xi_i}{1 - \prod_{k \in C_i} \xi_k} \right) + \frac{n_i(0) \log \xi_i} \right]$$  (13)

Differentiating (13) wrt. $\xi_i$ and letting the derivatives be 0, we have a set of likelihood equations, one for a link. Solving them, we have:

$$\left\{ \begin{array}{l} 
\xi_1 = \frac{n_{1(0)}}{n_{0(1)}} = \frac{n_1(0)}{n} \\
\xi_j = \frac{n_j(0)}{n_{f(j)}(1)} + \frac{n_j(1)}{n_{f(j)}(1)} \cdot (1 - \beta_{f(j)}) \quad j \neq 1 
\end{array} \right.$$  (14)

Since $1 - \beta_i = \prod_{j \in d_i} \xi_j$, we have

$$1 - \beta_i = \prod_{j \in d_i} \left[ \frac{n_j(0)}{n_{i(1)}} + \frac{n_j(1)}{n_{i(1)}} \cdot (1 - \beta_i) \right]$$

$$= \prod_{j \in d_i} \left[ (1 - \frac{n_j(1)}{n_{i(1)}}) + \frac{n_j(1)}{n_{i(1)}} \cdot (1 - \beta_i) \right].$$  (15)
One should notice the similarity between (15) and (10). (15) implies that with appropriate conditions, there is a unique solution to (15), and subsequently a unique solution to (4).

To prove this, we have:

Lemma 1. Assume $c_i \in (0, 1)$, if $\sum_i c_i > 1$, $x = \prod_i \left[ (1 - c_i ) + c_i x \right]$ has a unique solution in $(0, 1)$. Otherwise, if $\sum_i c_i < 1$, there is either no solution or have multiple solutions in $(0, 1)$ for the equation.

Proof: See appendix

To complete the proof, we use $\psi_i$ to replace $\xi_i$ in the likelihood functions, where:

$$
\psi_i = \begin{cases} 
\log \frac{\xi_i - \theta}{\xi_i} & i \in E \setminus R \\
\log \frac{\xi_i}{\xi_i} & i \in R 
\end{cases}
$$

Then, the log-likelihood function presented in (9) turns to

$$
L(\xi) = n \log \xi_1 + \sum_{i \in E} n_i (1) \psi_i .
$$

It is easy to prove that (16) belongs to the standard exponential family by changing it from log-likelihood back to the likelihood one. Then, it is easy to prove $\psi$ is the MLE of $\xi_i$ since there is a unique MLE for the standard exponential family in a specific parameter space. Further, applying functional invariance, we are able to prove the estimate obtained by (9) is the MLE for $\theta_i$. What we need to do here is to prove there is a bijection between $\psi$ and $\xi$. The following proposition confirms this:

Proposition 1. Let $\Theta = (0, 1)^m$, $\Xi = \Gamma(\Theta)$ and $\Psi = \Lambda(\Xi) = \Lambda \circ \Gamma(\Theta)$, $\Gamma$ is a bijection from $\Theta$ to $\Xi$, $\Lambda$ is a bijection from $\Xi$ to $\Psi$, and $\Lambda \circ \Gamma$ is a bijection from $\Theta$ to $\Psi$. Thus, the likelihood inference based on the three parameter systems are equivalent, i.e.

$$
\arg \max_{\theta \in \Theta} L(\theta) = \Gamma^{-1}(\arg \max_{\xi \in \Xi} L(\xi)) = (\Lambda \circ \Gamma)^{-1}(\arg \max_{\psi \in \Psi} L(\psi));
$$

and the likelihood equations from three likelihood functions have same solution, i.e. let $\theta^* = \Gamma^{-1}(\xi^*) = (\Lambda \circ \Gamma)^{-1}(\psi^*)$, we have

$$
\frac{\partial L(\theta)}{\partial \theta}\bigg|_{\theta=\theta^*} = 0 \iff \frac{\partial L(\xi)}{\partial \xi}\bigg|_{\xi=\xi^*} = 0 \iff \frac{\partial L(\psi)}{\partial \psi}\bigg|_{\psi=\psi^*} = 0
$$

Proposition 1 shows that we can transform the statistical inference problem from the original parameter space, $\theta$, to alternative parameter spaces $\Xi$ or $\Psi$. We then have the following theorem:

Theorem 2. Because likelihood function (16) belongs to the standard exponential family with $\psi$ as the natural parameters, we have the following results:

1. statistics $\{n_1(1), \ldots, n_m(1)\}$ are complete minimal sufficient statistics;
2. the likelihood equation $\frac{\partial L(\psi)}{\partial \psi} = 0$ has at most one solution $\psi^* \in \Psi$;
3. if $\psi^*$ exists, $\psi^* \ (or \ \theta^* = (\Lambda \circ \Gamma)^{-1}(\psi^*))$ is the MLE.

Proof: See appendix

The proof of MLE ensures the solution of (4) is asymptotically optimal, that implies asymptotically efficient, asymptotically unbiased and asymptotically normal.

Finally, applying Lemma 1 on (15) and replacing $c_i$ by $\frac{n_f(j)}{n_{f(j)}(1)}$ we have the following theorem:

Theorem 3. With the internal view, we have $0 \leq n_i(1) \leq n_{f(j)}(1), \forall i \in E$. If $\forall i \in E, n_i(1) < \sum j \in E_n, n_j(1)$, (12) has an unique solution $\xi^* \in (0, 1)^m$. If $\xi^* \in \Gamma(\theta_i), \theta^* = \Gamma^{-1}(\xi^*)$ is the MLE.

Proof: Based on Theorem 2 and Lemma 1 this theorem can be deduced directly.

V. TOP DOWN ALGORITHM

The top down dependence expressed in (7) indicates the possibility of using a top down procedure to complete the estimation in a step by step manner. Then, the obstacle laying in front of finding a closed form solution to (9) is whether there is an analytic solution to a high degree polynomial since the degree of the likelihood equation of a link or a path is one less than the number of descendants ending at the link or path. It is known that there is no analytic solution to a polynomial of degree 5 or greater according to Galois theory. Then, the task to have a closed form solution rests on whether we are able to replace a multi-descendant node with a two level tree that has its root connecting the node and has its leave connecting the descendants. Before proving the statistical equivalence of the replacement, a concept called statistical equivalent replacement is introduced here.

Proposition 2. Assume $B'$ is obtained from $B$ by replacing subgraph $A$ of $B$ with another graph $C$. If the sufficient statistics of the common part of $B$ and $B'$ are always identical for all possible observations, $C$ is called a statistical equivalent replacement of $A$.

It is clear that a number of serially connected links is a statistical equivalent replacement of a link. This can be considered an axiom. Such a replacement can not only maintain the statistic of the incoming link of the serial links and the statistic of the outgoing link of the serial links, but also provide the freedom to set the statistics of the newly introduced links as far as the statistic of a upstream link is larger than that a downstream one. Apart from link, using a tree to replace a multi-descendant node, including the egress links of the node, is also a statistical equivalent replacement. The following lemma confirms this.

Lemma 2. Using a tree to replace a multi-descendant node is a statistical equivalent replacement.

Proof: Assume node $i$ is a multi-descendant node. Recall the definition of internal state of link $i$.

$$
Y_i = \max_{k \in R(i)} x_{jk}^i, \quad j \in \{1, \ldots, n\}.
$$
and internal view,

\[ n_i(1) = \sum_{j=1}^{n} Y_i^j. \]

The internal view of link \( i \) is independent from the topology of \( T(i) \) but the receivers attached to \( T(i) \), so do the statistics of the links connecting to \( j, j \in a_i \). Then, the lemma follows since replacing an non-leaf multi-descendant node does not affect the receivers attached to the subtree rooted at the multi-descendant node.

Using a tree to replace a multi-descendant node is equal to create a number of serially connected links to replace the egress links of the node. The difference between the two replacements, a number of serial links for a link and a tree for a multi-descendant node, is that the sufficient statistics of the links introduced by the tree replacement are determined according to the definition of internal view. To have a common parent link in the tree replacement requires there are intersections in terms of observations between the descendants rooted from the parent link. Given the statistics, a closed form solution to loss tomograph follows.

**Theorem 4.** An non-leaf multi-descendant node can be replaced by a tree. The estimates obtained after the replacement are consistent with that obtained before the replacement.

**Proof:** See appendix.

Given a multi-descendant node, there are a number of ways to construct a tree to replace the node, each of them is statistically equal to one another although the estimates may be slightly different from each other when \( n \) is small just as the estimate obtained from (26) may be different from that obtained from (27). Among the estimators, the one that has its observed correlation equals or nearly equals to the modeled correlation is the estimator that resembles the estimator before replacement. As \( n \to \infty \), the estimates obtained from the estimators approximate the true parameter. Note that the maximum likelihood principle is built on large sample theory and the accuracy of an estimate measured by the properties, such as unbiased, minimum variance, is assured only if the samples used in estimation is large. Among the trees that can be used to replace a multi-descendant node, which one is the most likely one to the original one is an issue that will be studied in future.

Given Theorems 4, we can use a tree with two descendants on the top layer to replace a multi-descendant node. Then, we only need to solve a linear equation to estimate the loss rate of a link (or path), then the formula presented in (12), as follows, can be used to obtain the MLE of \( \theta_i \):

\[ \hat{\theta}_i = \frac{n_i(0)n - n_i(0)(n_i(0) + n_i(1))}{(n - [n_i(0) + n_i(1)] + n_i(0))n} \]

\[ = \frac{n_i(0)[n_i(1) - n_i(2)(0)] - n_i(0)n_i(2)(0)}{n \times (n_i(1) - n_i(2)(0))} \]

\[ = \frac{n_i(0)[n_i(1) - n_i(2)(0)] - n_i(0)n_i(2)(0)}{n \times (n_i(2)(1) - n_i(2)(0)),} \quad (17) \]

With (17), the top down algorithm is emerged and presented in Algorithm 1 where \( n_{f(i)}(1) = n_0(1) = n \). The algorithm starts from the root of \( T \) and uses (17) to obtain the loss rate of the root link. Once having the loss rate, it updates the statistics of the descendants of the link that is equal to adding \( imp(f(i)) \) to \( n_{f(i)}(1) \) and \( n_i(0) \). Then, the algorithm goes to the descendants of the root node and performs the same operation as above. This process continues until it explores all links.

The computation complexity of the top down algorithm is of \( O(n) \), \( n \) is the number of links of a network, which is significantly better than an iterative algorithm. For instance, the complexity of an EM is of \( O((2 + m + x)) \), where \( 2m \) is the complexity of a single iteration (19), (23) and \( x \) is the number of iterations that is a random variable related to the shape of search space.

To measure the gain of the closed form solution presented in this paper against an EM one, a series of simulations are conducted on four four-layer trees. The difference between the trees is the number of descendants attached to an internal node; they are 2, 3, 4, and 5, respectively. The simulation results are presented in Table I where the loss rates given to the descendants are presented in the first row and in a sequence, e.g. 1.5 is for a binary tree that sets the loss rates of the two links connecting to their descendants to 1% and 5%; 1,1,5 is for a triple tree that sets the loss rates of the three descendants 1%, 1%, and 5%, respectively. With the increase of the number of descendants, the number of iterations is reduced and the variance approaches to 0. This indicates there is a significant change of the steepness of the search space when the number of descendants increases from 2 to 3, and then the change becomes slow. To measure the impact of number of probes on the iteration number, two sets of simulations are performed with 500 and 1000 probes, respectively. We find there is little change in terms of the number of iterations that indicates the concaveness of the search space is insensitive to the number of probes. In contrast, the number of iterations is related to the stop criteria selected. For instance, if we change the stop criteria from \( 10^{-7} \) to \( 10^{-10} \), the average number of iterations for the binary tree is increased to 12.1, and the number for the 3ry tree to 7. This indicates there is an area around the maximum of the search space that have elegant difference. Based on the results presented here, the proposed estimator is at least 10 times faster than an EM algorithm.

Compared with the explicit estimator presented in (10), the top down algorithm presented here is the ML estimator with a similar computation complexity as the explicit one. The difference between them is the one proposed here considers all available information, while the other only considers a part of the information.

**VI. LOSS RATE ANALYSIS FOR GENERAL NETWORKS**

**A. Goals and Background**

As stated, using multiple trees and multiple sources in a general network makes the estimation harder if \( n < \infty \). Unfortunately, there has been no result to express the correlation of probes sent by the sources and there is no information about the shape of the search space formed by the likelihood equations of a general network. In this section, we prove the
TABLE I
THE NUMBER OF ITERATIONS

<table>
<thead>
<tr>
<th>Loss pattern</th>
<th>1.5</th>
<th>1.5</th>
<th>1.1,1,5</th>
<th>1.1,1,5</th>
</tr>
</thead>
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<tr>
<td>samples</td>
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<td>1000</td>
<td>500</td>
<td>1000</td>
</tr>
<tr>
<td>statistics</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean</td>
<td>var</td>
<td>mean</td>
<td>var</td>
<td>mean</td>
</tr>
<tr>
<td>round 1</td>
<td>9.92</td>
<td>0.38</td>
<td>9.67</td>
<td>0.22</td>
</tr>
<tr>
<td>round 2</td>
<td>9.60</td>
<td>0.24</td>
<td>9.60</td>
<td>0.24</td>
</tr>
<tr>
<td>round 3</td>
<td>9.86</td>
<td>0.12</td>
<td>9.73</td>
<td>0.12</td>
</tr>
<tr>
<td>round 4</td>
<td>9.90</td>
<td>0.16</td>
<td>9.87</td>
<td>0.20</td>
</tr>
<tr>
<td>round 5</td>
<td>9.67</td>
<td>0.22</td>
<td>9.87</td>
<td>0.12</td>
</tr>
<tr>
<td>round 6</td>
<td>9.67</td>
<td>0.22</td>
<td>9.87</td>
<td>0.12</td>
</tr>
<tr>
<td>round 7</td>
<td>9.86</td>
<td>0.12</td>
<td>9.40</td>
<td>0.24</td>
</tr>
<tr>
<td>round 8</td>
<td>9.67</td>
<td>0.22</td>
<td>9.60</td>
<td>0.16</td>
</tr>
<tr>
<td>round 9</td>
<td>9.73</td>
<td>0.20</td>
<td>9.80</td>
<td>0.16</td>
</tr>
<tr>
<td>round 10</td>
<td>9.55</td>
<td>0.25</td>
<td>9.67</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Algorithm 1 Top down algorithm
1: Given tree T and observations, compute \( n_i(1) \) and \( n_i(0) \) for each node;
2: TOP DOWN(1);
3: procedure TOP DOWN(i) \( \triangleright i \) is the node number
4: if \( i = leaf \) then
5: \( \hat{\theta}_i \leftarrow \frac{n_i(0)}{n_f(i)(1)} \);
6: Return;
7: else
8: use (17) to compute \( \hat{\theta}_i \);
9: \( n_i(1) \leftarrow n_i(1) + n_f(i)(0) - n_f(i)(1) \cdot \hat{\theta}_i \);
10: for \( j \in d_i \) do
11: \( n_j(0) \leftarrow n_j(0) + n_f(i)(0) - n_f(i)(1) \cdot \hat{\theta}_i \);
12: TOP DOWN(j)
13: end for
14: end if
15: end procedure

shape of the search space is strictly concave. This assures that under the similar conditions as those stated in Theorem 3 there is a unique MLE for the loss rates of a general network and one can use an iterative procedure to approximate it. We further derive a direct expression of the MLE that has a similar and one can use an iterative procedure to approximate it. We cover a general network. Let \( N \) denote the receivers attached to \( T_i \) is intersected with \( T^j \), intersection is commutative. The largest intersection between two trees is called the shared link, or simply segment. A segment is an ordinary tree and its root is called the joint node of the segment. Let \( J \) denote all joint nodes and let \( p(i), i \in J \) denote the links that end at node \( i \). As the tree topology, each node in the general topology is assigned a unique number, so is a link. However, a node, say \( i \), in a general network may have multiple parents that receive probes from their own sources. \( S(i) \) is used to denote the sources sending probes to link \( i \). More, \( f^j(i) \) is used to denote the parent link of link \( i \) on the path to \( S^j \) and \( a^j(i) \) is used to denote the ancestor links of link \( i \) toward \( S^j \). Further, \( f^j_k(i) \) denotes the ancestor link of link \( i \) located \( k \) hops away from link \( i \) to \( S^j \). In a shared segment, the links and nodes are numbered after the parent that has the smallest number. Figure 2 shows an example of this assignment. Applying internal view to the general topology, we need to distinguish the sources of probes, where \( n^i_1(1) \) and \( n^i_0(0) \) denote the the statistics created by source \( j, j \in S \), on link \( i \). Then, the internal view of link \( i \) is equal to

\[
n_i(1) = \sum_{r=1}^{k} n^r_1(1), \quad n_i(0) = \sum_{r=1}^{k} n^r_0(0).
\]

Apparently, if \( i \notin T^j \), \( n^i_1(1) = n^i_0(0) = 0 \). Apart from the above, the notations and definitions used in the tree topology, such as the parameter systems, \( \theta, \zeta \) and \( \psi \), and their relationships remains.

C. MLE Formula

With the extended notations, the log-likelihood function of a general network can be written as:

\[
L(P_\theta) = \sum_{r=1}^{k} \sum_{i \in E} \left[ n^r_1(1) \log(1 - \theta_i) + n^r_0(0) \log \xi_i \right]
\]

\[
= \sum_{i \in E} \left[ \sum_{r=1}^{k} n^r_1(1) \cdot \log(1 - \theta_i) + \sum_{r=1}^{k} n^r_0(0) \cdot \log \xi_i \right]
\]

\[
= \sum_{i \in E} \left[ n_i(1) \cdot \log(1 - \theta_i) + n_i(0) \cdot \log \xi_i \right]
\] 18

Note that in the above derivation, we use \( n_i(1) \) to replace \( \sum_{r=1}^{k} n^r_1(1) \) and \( n_i(0) \) to replace \( \sum_{r=1}^{k} n^r_0(0) \) in the last step. Then, (18) has the same form as (2). In contrast to the tree
obtained previously, we are interested in the applicability
of the top down algorithm to general networks due to its
simplicity and efficiency. To answer this question, we com-
pare the top down algorithm to general networks due to its
definition above.

\[
\theta_i = \begin{cases}
\frac{n_i(1)}{\beta_i}, & i \in RL, \\
1 - \frac{n_i(1)}{\beta_i} - n_f(i)(1) + \text{imp}(f(i)), & i \in SBRL, \\
1 - \frac{n_f(i)(1) + \text{imp}(f(i))}{\beta_i}, & i \in SSNL, \\
\frac{\sum_{j \in S(i)} n_j^q(1) + \text{imp}(f(i))}{\beta_i}, & i \in AOI, \\
\frac{\sum_{j \in S(i)} [n_j^q(0) + \text{imp}(f(i))]}{\sum_{j \in S(i)} [n_j^q(1) + \text{imp}(f(i))]}, & i \in RL,
\end{cases}
\]

where \(n^i\) is the number of probes sent by \(S^i\), and \(\text{imp}(i)\)
is the impact of \(n_k^i(0), k \in a^i(j), j \in S(i)\) on the loss rate
of link \(i\). The similarity between \(19\) and \(4\) is noticeable and
the statistical implication of \(19\) is the same as that of
\(4\). Then, the immediate question is whether the theorems
and lemmas presented previously for the tree topology hold
here, in particular for Proposition 1 and theorem 2–3 The
following corollary provides the answer.

**Corollary 1.** In a general network, the relationships
defined for \(\theta, \xi\) and \(\psi\) remain as that in a tree
network:

1. \(\Gamma\) still leads to a bijection from \(\Theta_\Gamma \to \mathcal{P}_\Gamma\)
   thereby
   \[
   \Gamma^{-1}\left(\arg \max_{\xi \in \mathcal{P}_\Gamma} L(\xi)\right) = \arg \max_{\theta \in \Theta_\Gamma} L(\theta).
   \]
2. \(\Phi\) still leads to a bijection from \(\Theta_\Phi\) to \(\psi\), thereby
   \[
   \Phi\left(\arg \max_{\xi \in \mathcal{P}_\Phi} L(\xi)\right) = \arg \max_{\psi \in \psi} L(\psi)
   \]
   and the two likelihood equations \(\frac{\partial L(\xi)}{\partial \xi} = 0\) on \(\mathcal{P}_\Psi\) and
   \(\frac{\partial L(\psi)}{\partial \psi} = 0\) on \(\psi\) are equivalent, i.e.

\[
\forall \xi^* \in \xi_\Phi, \frac{\partial L(\xi)}{\partial \xi} \bigg|_{\xi = \xi^*} = 0 \iff \frac{\partial L(\psi)}{\partial \psi} \bigg|_{\psi = \Phi(\xi^*)} = 0.
\]

Apart from the validity of the theorems and proposition
obtained previously, we are interested in the applicability
of the top down algorithm to general networks due to its
simplicity and efficiency. To answer this question, we compare
the formula presented in \(19\) with those presented in \(4\). \(19\)
uses

\[
\frac{\sum_{j \in S(i)} n_j^q(1)}{\sum_{j \in S(i)} [n_j^q(1) + \text{imp}(f(i))]} (20)
\]
to estimate the loss rate of a link in a shared segment. That
requires information from a number of sources, however,
they are related to each other. Then, the top down algorithm
is incapable of obtaining the MLE directly for a general
topology since there is no method yet to compute \(\text{imp}(i)\)
and \(\text{imp}(k), j, k \in S(i)\). To regain the validity of
the top down algorithm for a general network, we propose a
divide-and-conquer strategy to decompose the network into
a number of independent trees. The most suitable decomposing
points are the joint points. If \(i\) is a joint point and we have
\(n_j^i(1) = \sum_{r \in S(i)} n_j^i(1), i \in J\), the estimated number
of probes reaching node \(i\), the subtrees connected to the node
can be decomposed into two classes: one is those having their
roots in \(S(i)\); the other is those that have a common root, i.e. node \(i\).
We call the former truncated trees, and the latter independent
trees. Given \(n_j^i(1)\), each of the independent trees can be
estimated independently by using the top down algorithm.
However, for the truncated trees, given \(n_j^i(1), r \in S(i)\), we
must recalculate the internal view of each link, and then
use a new method to complete the estimation. The next two
subsections are devoted to discuss the recalculation and the
new method.

**D. Estimate Loss Rate**

To start the the divide-and-conquer strategy stated above,
we need to have a method to obtain \(n_j^i(1), i \in J\). Considering
a number of options, we decide to use an iterative procedure
based on fixed point theory to accomplish this since \(13\) is
a transcendental equation that expresses the loss rate of
the ordinary tree rooted at node \(i\), i.e., \((1 - \beta_i)\). The iterative
procedure uses:

\[
(1 - \beta_i)^{(q+1)} = \prod_{j \in S_i} \left[ \frac{\sum_{k \in S(j)} n_k^i(0)}{\sum_{k \in S(j)} n_k^i(1)} + \frac{\sum_{k \in S(j)} n_k^i(1)}{\sum_{k \in S(j)} n_k^i(1)} \cdot (1 - \beta_j)^{(q)} \right]
\]

(21)
to compute \((1 - \beta_i)^{(q+1)}\). According to Lemma \(1\) there is only
one \((1 - \beta_i)^{(q)}\) in \((0, 1)\) satisfying \(21\). The iterative procedure
is certain to converge at \((1 - \beta_i)^{(q)}\) and may only need a few
iterations to converge at \((1 - \beta_i)^{(q)}\) if the initial value of \((1 -
\beta_i)^{(0)}\), is located close to the fixed point. There
are a number of methods to select \((1 - \beta_i)^{(0)}\), one of them
is a weighted average the estimates obtained by the top down
algorithm on the trees involved in the shared segment; another is to use the bottom-up algorithm proposed in [12]. Both have the complexity of $O(m)$, $m$ is the number of links of the tree involved. In our experience for a network as Figure 2 the procedure needs 5.25 iterations on average to converge to the fixed point when $\epsilon$ is set to $10^{-10}$. Alternatively, Newton Raphson algorithm can be applied on (21) to find the $(1-\beta)^*$. Given $(1-\beta)^*$, we can use the following formula to estimate $\hat{n}^*_j(1)$:

$$\hat{n}^*_j(1) = \frac{n^*_j(1)}{\beta^*_i}, j \in p(i).$$

(22)

where $\beta^*_i = 1-(1-\beta)^*$. Given $\hat{n}^*_j(1), \forall j, j \in p(i)$, segment $i$ can be decomposed from the rest of the network and node $i$ is replicated as many as the number of links directly connected to node $i$ regardless the links are ingress or egress. However, if a replica is connected to an ingress link, i.e. a leaf node of a truncated subtree after the decomposition, the internal views of link $k, \forall k \in a^*(i)$ need to be updated accordingly since $\hat{n}^*_i(1) \geq n^*_i(1)$ and $\hat{n}^*_i(1) - n^*_i(1) = imp^*(i)$.

E. Path-based Estimator for Truncated Subtree

Given $\hat{n}^*_i(1)$, it is hard, if not impossible, to update the statistics of $k, k \in a^*(i)$ since it depends on the observations of the siblings of $k$. In this situation, (10) is used to estimate the pass rate of the path from $r$ to $f^*(i)$. Once having the pass rate, $A_{f^*(i)}$,

$$1 - \theta_i = \frac{\hat{n}^*_i(1)}{A_{f^*(i)}}.$$

This process continues until it reaches source $r$. During this process, those subtrees rooted at $k, k \in a^*(i)$ are decomposed into independent trees and the top down algorithm can be applied to estimate the link-level loss rates of the subtrees.

**Theorem 5.** The estimates obtained by the divide-and-conquer is the MLE.

**Proof:** Because Theorem 3 holds here, there is a unique MLE, which implies there is a unique most likely $\hat{n}^*_j(1)$ for link $j$. Note that $\hat{n}^*_j(0) = \hat{n}^*_j(1) - \hat{n}^*_j(1)$. Given the number of probes reaching node $j, j \in J$, the subtrees rooted at node $j$ can be decomposed from its parents according to the separation [25]. Then, the loss rates estimated by the top down algorithm is the maximum likely ones. The estimates obtained by the estimator of (10) are also MLEs.

VII. CONCLUSION

In this paper, the findings obtained recently on loss tomography are presented that include theoretical findings and practical algorithms. In theory, the internal view introduced in the paper leads to a set of complete minimal sufficient statistics that contain all the information needed to compute the loss rates of a network. Based on the statistics, the frequently used likelihood function is rewritten, and subsequently a set of likelihood equations is obtained. Solving the likelihood equations, a direct expression of the MLE is obtained for the link-level loss rates of the tree topology. The direct expression indicates the feasibility to decompose the parameter space formed by the unknown parameters, one at a time and from top to bottom. Then, a top down algorithm is proposed to estimate the loss rate of a tree network in a step by step manner. Apart from the algorithm, an statistic equivalent replacement is introduced that allows us to replace a multi-descendant node with a tree. The replacement only affects the estimate of the link connecting the replaced node and the estimate obtained after the replacement is consistent with that obtained before the replacement. Then, an established formula is used to estimate the loss rate of the link. The complexity of the algorithm is $O(m)$, where $m$ is the number of links in the network. The proposed method is at least 10s time better than an iterative algorithms.

With the success achieved on the tree topology, a direct expression of the MLE is derived for the loss rate of a link in a general network. The direct expression has a similar structure as its tree counterpart, which ensures most of the theorems obtained for the tree topology hold for the general topology. To gain the applicability of the top down algorithm, a divide-and-conquer strategy is proposed to decompose a general network into a number of trees and a fixed-point procedure is proposed to estimate the number of probes reaching the root of a shared segment. With the estimated number, the subtrees rooted at a joint node become independent from each other. Then, the top down algorithm can be applied to the subtrees located in the shared segment. For the truncated subtrees, the estimator proposed in [9] is used to gain the MLEs of the links, which also has an $O(m)$ complexity. $m$ is the number of links in a truncated subtree.

APPENDIX

**Lemma 1**

**Proof:** Let $h_1(x) = x$ and $h_2(x) = \prod_i \left[ (1-c_i) + c_i x \right]$. We have $h'_1(1) = 1$ and $h'_2(x) = h_2(x) \sum_i \frac{c_i}{(1-c_i)+c_i x}$. Let $q_i = \sum_i (1-c_i) \sum_j \frac{c_j}{1-c_j}$, we have $h'_1(x) = 0$ and $h'_2(x) = h_2(x) \left( \sum_i q_i x^2 - \sum_i q_i x \right) > 0$, if $x \in [0,1]$. Let $h(x) = h_1(x) - h_2(x)$, that is strictly concave on $[0,1]$. In addition, $h'(0) = 1 - \prod (1-c_i) \sum_i \frac{c_i}{1-c_i} > 0$ and $h'(1) = 1 - \sum_i c_i$. If $\sum_i c_i > 1, h'(1) < 0$, there is a unique solution to $h(x) = 0$ for $x \in [0,1]$ since $h(x)$ is continuously differentiable on $[0,1]$, and $h(0) = - \prod (1-c_i) < 0$ and $h(1) = 0$. Otherwise, if $\sum_i c_i < 1, h'(1) > 0$, there is no solution to $h(x) = 0$ for $(0,1)$.

**Theorem 2**

**Proof:** To prove theorem 2 we need to use the following lemmas:

**Lemma 3.** If random variable $X$ has a distribution of the standard exponential family with $k$ parameters, i.e. its p.d.f can be written in the form

$$f(x; \theta) = C(\theta) \cdot h(x) \cdot \exp \sum_{i=1}^k T_i(x) \theta_i$$

(23)

where $\theta_1, \cdots, \theta_k$ are called the natural parameters, $C(\theta)$ is a function in $\theta$ alone, and $T_1(x), \cdots, T_k(x)$, $h(x)$ are well-
be proved by directly applying lemma 3. To prove the other
2) and 3) of the theorem. If using empirical probabilities \( \hat{\gamma}_i = \frac{n_i(1)}{n(1)} \), \( i \in \{1, 2, 3, 4\} \) to replace \( \gamma_i \), the left hand side of (25) is the observed
loss rate of subtree \( k \) and the right hand side (RHS) is the
modeled loss rate of subtree \( k \). The difference between (24)
and (25) is that the former uses observed loss rate to replace
the modeled one. Expanding the RHSs of (24) and (25),
one can find that (24) uses observed correlation \( \frac{n_{ij}(1)n_{kl}(1)}{n_A} \)
i, j \in k, j = \{1, 2\} to replace the modeled correlation
\( \frac{n_3(1)n_4(1)}{n^2A^2} \) and \( \frac{n_3(1)n_4(1)}{n^2A^2} \) used by (25). The observed
loss rate or correlation is statistically equal to the modeled
one, and as \( n \) increases, the difference between them becomes
negligible. This can be viewed as using accumulative statistics
to replace the product of individual ones, which has been
widely used in statistical inference. For instance, the estimator
of (10) is also built on the equality, which as (24) can be
written as

\[
1 - \frac{\gamma_k}{A_k} = \prod_{j \in d_k} (1 - \frac{\gamma_j}{A_k}) \quad (26)
\]
or

\[
1 - \frac{\gamma_k}{A_k} = \prod_{j \in d_k} \left( \prod_{l \in d_j} (1 - \frac{\gamma_l}{A_k}) \right), \quad (27)
\]

where the degree of (27) is much higher than that of (26).
When \( n \) is small, the estimates obtained from the two may
differ from each other.

To be general, let us consider link \( i \) has \( d_i, |d_i| > 5 \)
descendants. It is known that \( n_i(1) = \sum_{l=1}^{(|d_i|)} Y_{ij}^l \). If we
divide \( d_i \) into \( d_{i1} \) and \( d_{i2} \) and let \( Y_{ij}^l \) be the internal state of \( d_{ik} \), \( n_i(1) = \sum_{l=1}^{n_{ij}} Y_{ik}^l \) is the internal view of \( d_i \). The internal view of link \( ik \) can be obtained by

\[
n_{ik}(1) = \sum_{l=1}^{n_{ij}} Y_{ij}^l \]

\[
= \sum_{j \in d_{ik}} n_j(1) - \sum_{i < j \in d_{ik}} n_{ij}(1) + \sum_{i, j, k \in d_{ik}} n_{ijk}(1) - \cdots - (-1)^{|d_{ik}|-1} n_{d_{ik}}(1)
\]

Using the internal views of \( i1 \) and \( i2 \), we have

\[
(1 - \frac{\gamma_i}{A_i}) = \left( 1 - \frac{\gamma_{i1}}{A_{i1}} \right) (1 - \frac{\gamma_{i2}}{A_{i2}})
\]
as the likelihood equation, where the origin is

\[
(1 - \frac{\gamma_i}{A_i}) = \prod_{j \in d_{i1}} (1 - \frac{\gamma_j}{A_j}) \prod_{j \in d_{i2}} (1 - \frac{\gamma_j}{A_j}).
\]

\( n_{ij}(1) \) is a statistic as \( n_i(1) \) counts the number of probes passing link \( i \)
and link \( j \) confirmed by observation.
Expanding the first term on the RHS of (28) and the first product of the RHS of (29), we have:

\[
\left(1 - \frac{\gamma_{ij}}{A_i}ight) = 1 - \frac{1}{A_i} \times \\
\left(\sum_{j \in d_{i1}} n_j(1) - \sum_{j < k \in d_{i1}} n_{jk}(1) + \sum_{j < k < l \in d_{i1}} n_{jkl}(1) - \ldots + (-1)^{|d_{i1}|-1} n_{d_{i1}}(1)\right)
\]

and

\[
\prod_{j \in d_{i1}} \left(1 - \frac{\gamma_j}{A_i}\right) = 1 - \sum_{j \in d_{i1}} \frac{\gamma_j}{A_i} + \sum_{j < k \in d_{i1}} \frac{\gamma_{jk}}{(A_i)^2} - \ldots + (-1)^{|d_{i1}|} \prod_{k \in d_{i1}} \frac{\gamma_k}{(A_i)^{|d_{i1}|}}.
\]

Statistically, (30) consists of the observed correlations and (31) consists of modeled correlations. A statistical equivalent replacement actually uses observed correlation to replace the modeled correlation that are consistent with each other. So does the estimate. If they were not equal to each other, the likelihood equation of (10) would not hold.

**REFERENCES**


