Exact travelling wave solutions of a generalized Camassa–Holm equation using the integral bifurcation method

He Bin *, Rui Weiguo, Chen Can, Li Shaolin

Department of Mathematics, Honghe University, Mengzi, Yunnan 661100, PR China

A R T I C L E   I N F O

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A B S T R A C T

In this paper, a generalized Camassa–Holm equation is studied by using the integral bifurcation method. Many travelling waves such as peaked compacton, compacton, peaked solitary wave, solitary wave and kink-like wave are found. In some parameter conditions, exact parametric representations of these travelling waves in explicit form and implicit form are obtained.

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1. Introduction

Camassa and Holm [1] derived a shallow water wave equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{1}$$

which is called Camassa–Holm equation (CH equation in short). For $k = 0$, Camassa and Holm showed that Eq. (1) has solitary waves of the form $c e^{-|x-ct|}$, which were called peakons (or called peaked solitary wave) due to the discontinuity of the first derivative at the wave peak. Boyd [2] derived a perturbation series which converges even at the peakon limit, and gave three analytical representation for the spatially periodic generalization of the peakon, called “coshoidal wave”. Cooper and Shepard [3] derived approximate solitary wave solution by using some variational functions. Constantin [4] gave a mathematical description of the existence of interacting solitary waves. Recently, CH equation and some its generalized forms have been studied by many authors (see Refs. [5–17] and the references cited therein).

In Ref. [14], Liu and Qian discussed the peakons and their bifurcations when the integral constants taken as zero of the following generalized Camassa–Holm equation (GCH equation in short):

$$u_t + 2ku_x - u_{xxt} + au^m u_x = 2u_xu_{xx} + uu_{xxx}, \tag{2}$$

where $a > 0$, $k \in \mathbb{R}$, $m \in \mathbb{N}$, and they introduced Eq. (2) from the mathematical point of view. For $m = 1, 2, 3$, they obtained the explicit expressions for the peakons. In Ref. [15], Tian and Song derived some new exact peaked solitary wave solutions. In Ref. [16], Shen and Xu analyzed the dynamical behavior of travelling wave solutions of Eq. (2) by using the bifurcation theory and the method of phase portraits analysis.

In this paper, we aim to extend the previous works in Refs. [14–16] to obtain more new exact travelling wave solutions of Eq. (2) using the integral bifurcation method [13]. In some parameter conditions, many new exact parametric representations of the travelling waves such as peaked compacton, compacton, peaked solitary wave, solitary wave and kink-like wave in explicit form and implicit form are obtained.

* Corresponding author.
E-mail address: hbhhu@yahoo.com.cn (B. He).

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We look for travelling wave solutions to Eq. (2) in the form

\[ u(x, t) = \phi(x - ct) = \phi(\xi), \]  

(3)

where \( c \neq 0 \) is the wave speed and \( \xi = x - ct \). Substituting (3) into Eq. (2), we obtain

\[ (2k - c)\phi' + c\phi'' + a\phi^m \phi' - 2\phi'\phi'' - \phi\phi''' = 0, \]

(4)

where \( \phi' \) is the derivative with respect to \( \xi \).

Integrating Eq. (4) once with respect to \( \xi \) and the integral constant taken as zero, we have

\[ (c - \phi)\phi'' + \frac{a}{m + 1}\phi^{m+1} + (2k - c)\phi - \frac{1}{2}(\phi')^2 = 0. \]

(5)

Letting \( \phi' = y \), we obtain a planar integrable system

\[ \frac{d\phi}{dx} = y, \quad \frac{dy}{dx} = \frac{a}{m + 1}\phi^{m+1} + (2k - c)\phi - \frac{1}{2}y^2. \]

(6)

Transformed by \( d\xi = (\phi - c)d\xi \), system (6) becomes a Hamiltonian system

\[ \frac{d\phi}{d\xi} = (\phi - c)y, \quad \frac{dy}{d\xi} = \frac{a}{m + 1}\phi^{m+1} + (2k - c)\phi - \frac{1}{2}y^2 \]

(7)

with first integral

\[ H(\phi, y) = (\phi - c)y^2 - \frac{2a}{(m + 1)(m + 2)}\phi^{m+2} + (c - 2k)\phi^2 = h, \]

(8)

i.e.

\[ y^2 = \frac{2a}{(m + 1)(m + 2)}\phi^{m+2} - (c - 2k)\phi^2 + h. \]

(9)

Denote \( A_1 = c/(m+1)(c-2k) \), \( \phi_{1,2} = \pm(A_1)^{\frac{1}{3}} \), \( A_2 = 2c(m-2k)/(m+2) \), \( Y_1 = \sqrt{A_2} \). Obviously, systems (7) have only one equilibrium point \( O(0, 0) \) on the \( \phi \)-axis when \( c = 2k \); systems (7) have three equilibrium points \( O(0, 0), A_{1,2}(\phi_{1,2}, 0) \) on the \( \phi \)-axis when \( c > 2k \) and \( m \) is an even; systems (7) have two equilibrium points \( O(0, 0), A_1(\phi_1, 0) \) on the \( \phi \)-axis when \( c \neq 2k \) and \( m \) is an odd; systems (7) have two equilibrium points \( S_{\pm}(c, \pm Y_s) \) on the line \( \phi = c \) when \( A_2 > 0 \). From (8), we write

\[ h_0 = H(0, 0) = 0, \]

(10)

\[ h_1 = H(\phi_{1,2}, 0) = \frac{m(c - 2k)}{m + 2} \left( \frac{m + 1}(c - 2k) \right) \phi \]

(11)

\[ h_k = H(c, \pm Y_s) = -\frac{2a}{(m + 1)(m + 2)}c^{m+2} + (c - 2k)c^2. \]

(12)

The rest of this paper is organized as follows. In Section 2, we give some new exact travelling wave solutions of Eq. (2) in explicit form and implicit form when \( m = 1, 2, 3 \). A short conclusion will be given in Section 3.

2. Exact travelling wave solutions of the GCH equation

2.1. Exact travelling wave solutions of the GCH equation when \( m = 1 \)

When \( m = 1 \), we have

\[ \phi_1 = \frac{2(c - 2k)}{a}, \]

(13)

\[ y^2 = \frac{\frac{1}{2}a\phi^3 - (c - 2k)\phi^2 + h}{\phi - c}, \]

(14)

\[ h_0 = 0, \quad h_1 = \frac{4(c - 2k)^3}{3a^2}, \quad h_k = -\frac{1}{3}ac^3 + (c - 2k)c^2. \]

(15)

Case 2.1.1: Taking \( h = h_0 \) from (14) we get

\[ y^2 = \frac{\frac{1}{2}a(\phi - \frac{3}{2}(c - 2k))\phi^2}{\phi - c}. \]

(16)

2.1.1.1 When \( k = \frac{1}{4}c \), (16) can be reduced to

\[ y = \pm \sqrt{\frac{\frac{1}{2}a\phi\sqrt{\phi(\phi - c)}}{\phi - c}}. \]

(17)
Substituting (17) into the first expression of (6) and integrating it, we get

$$\int_{\phi(0)}^{\phi} \frac{(\sigma - c) \, d\sigma}{\sqrt{\frac{1}{2} a \sigma \sqrt{\sigma(\sigma - c)}}} = \pm \int_{0}^{y} \, d\sigma,$$

(18)

where $\phi(0)$ and $0$ are initial constants.

Choosing the $\phi(0) = c$ and completing (18), we obtain one implicit function expression for the compacton solution of Eq. (2)

$$2 \tanh^{-1} \sqrt{\frac{\phi - c}{\phi}} = \frac{2 \sqrt{\phi - c}}{\sqrt{\phi}} + \sqrt{\frac{1}{2} a \xi}.$$

(19)

Let $\tau = \frac{2 \sqrt{\phi - c}}{\sqrt{\phi}} + \sqrt{\frac{1}{2} a \xi}$, (19) can be rewritten as

$$\begin{align*}
\phi(\tau) &= c \cosh^2 \left( \frac{1}{2} \tau \right), \\
\xi(\tau) &= \frac{1}{\sqrt{c \tau}} \left( \tau - 2 \tanh \left( \frac{1}{2} \tau \right) \right),
\end{align*}$$

(20)

where $\tau$ is a parameter and $\omega_1 = \sqrt{\frac{1}{2} a}$.

2.1.1.2 When $k = \frac{1}{2} c - \frac{1}{6} ac$, (16) can be reduced to

$$y = \pm \sqrt{\frac{1}{3} a \phi}.$$

(21)

Substituting (21) into the first expression of (6) and integrating it, we get

$$\int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{\frac{1}{2} a \sigma}} = \pm \int_{0}^{y} \, d\sigma.$$

(22)

Choosing the $\phi(0) = c$ and completing (22), we obtain one explicit function expression for the peaked solitary wave solution of Eq. (2)

$$\phi(x - ct) = c e^{-\omega_2 |x - ct|},$$

(23)

where $\omega_2 = \sqrt{\frac{1}{2} a}$.

**Remark 1.** (23) has been obtained in Refs. [14,15].

Case 2.1.2: Taking $h = h_2$, from (14) we get

$$y^2 = \frac{1}{2} a (\phi - 1)(\phi - \frac{2k + c}{a}).$$

(24)

2.1.2.1 When $k = \frac{1}{2} c - \frac{1}{6} ac$, (24) can be reduced to

$$y = \pm \sqrt{\frac{1}{3} a \sqrt{(\phi - c)(\phi + \frac{1}{2} c)}}.$$

(25)

Substituting (25) into the first expression of (6) and integrating it, we get

$$\int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{\frac{1}{2} a \sqrt{(\sigma - c)(\sigma + \frac{1}{2} c)}}} = \pm \int_{0}^{y} \, d\sigma.$$

(26)

Choosing the $\phi(0) = c$ and completing (26), we obtain one explicit function expression for the compacton solution of Eq. (2)

$$\phi(x - ct) = c \left( \frac{1}{2} \sinh^2 (\omega_3 (x - ct)) + \cosh^2 (\omega_3 (x - ct)) \right),$$

(27)

where $\omega_3 = \frac{1}{2} \sqrt{\frac{1}{2} a}$. Choosing the $\phi(0) = -\frac{1}{2} c$ and completing (26), we obtain one explicit function expression for the compacton solution of Eq. (2)

$$\phi(x - ct) = -c (\sinh^2 (\omega_4 (x - ct)) + \frac{1}{2} \cosh^2 (\omega_4 (x - ct))),$$

(28)

where $\omega_4 = \frac{1}{2} \sqrt{\frac{1}{2} a}$.

2.1.2.2 When $k = \frac{1}{2} c + \frac{1}{6} ac$, (24) can be reduced to

$$y = \pm \sqrt{\frac{1}{3} a (\phi + 2c)}.$$

(29)
Substituting (29) into the first expression of (6) and integrating it, we get
\[ \int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{\phi}\left(\sigma + 2c\right)} = \pm \int_{0}^{c} d\sigma. \] (30)

Choosing the \( \phi(0) = c \) and completing (30), we obtain one explicit function expression for the peaked solitary wave solution of Eq. (2)
\[ \phi(x - ct) = c\left(3e^{-\omega_5|x| - ct} - 2\right), \] (31)
where \( \omega_5 = \sqrt{\frac{1}{2}a}. \)

**Remark 2.** (31) has been obtained in Ref. [14]

2.2. Exact travelling wave solutions of the GCH equation when \( m = 2 \)

When \( m = 2 \), we have
\[ \phi_{1.2} = \pm \sqrt{\frac{3(c - 2k)}{a}}. \] (32)
\[ y^2 = \frac{1}{6}a\phi^2 - (c - 2k)\phi + h, \] (33)
\[ h_0 = 0, \quad h_1 = \frac{3(c - 2k)^2}{2a}, \quad h_2 = -\frac{1}{6}ac^4 + (c - 2k)c^2. \] (34)

Case 2.2.1: Taking \( h = h_0 \), from (33) we get
\[ y^2 = \frac{1}{6}a\phi^2 - \frac{c}{3} \left(c - 2k\right)\phi. \] (35)

2.2.1.1 When \( k = \frac{1}{2}c \), (35) can be reduced to
\[ y = \pm \sqrt[6]{a}\phi^2 \sqrt{\phi - c}. \] (36)

Substituting (36) into the first expression of (6) and integrating it, we get
\[ \int_{\phi(0)}^{\phi} \frac{(\sigma - c)d\sigma}{\sqrt{\phi}\sigma^2 / \sqrt{\sigma - c}} = \pm \int_{0}^{c} d\sigma. \] (37)

Choosing the \( \phi(0) = c \) and completing (37), we obtain one implicit function expression for the peaked solitary wave solution of Eq. (2)
\[ \frac{1}{\sqrt{-c}} \tanh^{-1} \left( \frac{\phi - c}{-c} \right) = - \left( \frac{\sqrt{\phi - c}}{\phi} + \sqrt[6]{a}\xi \right), \quad c < 0 \] (38)
and one implicit function expression for the peaked compacton solution of Eq. (2)
\[ \frac{1}{\sqrt{c}} \tanh^{-1} \left( \frac{\phi - c}{c} \right) = \frac{\sqrt{\phi - c}}{\phi} + \sqrt[6]{a}\xi, \quad c > 0. \] (39)

Let \( \tau = -\left( \frac{\sqrt{\phi - c}}{c} + \sqrt[6]{a}\xi \right), \mu = \frac{\sqrt{\phi - c}}{\phi} + \sqrt[6]{a}\xi \). (38) and (39) can be rewritten respectively as
\[ \begin{align*}
\phi(\tau) &= c \text{sech}^2(\sqrt{-c}\tau), \\
\xi(\tau) &= \frac{1}{\omega_6} \left( \frac{1}{\sqrt{-c}} \sinh(2\sqrt{-c}\tau) - \tau \right), \quad c < 0
\end{align*} \] (40)
and
\[ \begin{align*}
\phi(\mu) &= c \sec^2(\sqrt{c}\mu), \\
\xi(\mu) &= \frac{1}{\omega_6} \left( -\frac{1}{2\sqrt{c}} \sin(2\sqrt{c}\mu) + \mu \right), \quad c > 0.
\end{align*} \] (41)
where \( \tau \) and \( \mu \) are two parameters and \( \omega_6 = \sqrt{\frac{1}{2}a} \). The profile of (41) is shown in Fig. 1(1-1).

2.2.1.2 When \( k = \frac{1}{2}c - \frac{a}{4}c^2 \), (35) can be reduced to
\[ y = \pm \sqrt[6]{a}\phi \sqrt{\phi + c}. \] (42)
Substituting (42) into the first expression of (6) and integrating it, we get
\[
\int_{\theta}^{\phi} \frac{d\sigma}{\sqrt{a\sigma^2 + c}} = \pm \int_{0}^{\phi} d\sigma.
\]
(43)

Choosing the \( \phi(0) = -c \) and completing (43), we obtain one explicit function expression for the compacton solution of Eq. (2)
\[
\phi(x - ct) = -c \sec^2(\omega_7(x - ct)), \quad c < 0
\]
(44)
and one explicit function expression for the kink-like wave solution of Eq. (2)
\[
\phi(x - ct) = \frac{2c}{\cosh(\omega_8(x - ct)) - 1}, \quad c > 0,
\]
(45)
where \( \omega_7 = \frac{1}{2} \sqrt{-\frac{1}{b}ac}, \omega_8 = \sqrt{\frac{b}{ac}}. \)

**Remark 3.** The kink-like wave is defined in Ref. [18].

Choosing the \( \phi(0) = c \) and completing (43), we obtain two explicit function expressions for the kink-like wave solutions of Eq. (2)
\[
\phi(x - ct) = \frac{4c}{(3 - 2\sqrt{2})e^{\omega_9(x - ct)} + (3 + 2\sqrt{2})e^{-\omega_9(x - ct)} - 2}, \quad c > 0,
\]
(46)
where \( \omega_9 = \sqrt{\frac{b}{ac}}. \)
Case 2.2.2: Taking $h = h_1$, from (33) we get
\[ y^2 = \frac{1}{6}a\phi^3 + \frac{1}{6}ac\phi^2 + \left(\frac{1}{6}ac^2 + 2k - c\right)\phi + \left(\frac{1}{6}ac^2 + 2k - c\right)c. \] (47)

2.2.2.1 When $k = \frac{1}{4}c$, (47) can be reduced to
\[ y = \pm \frac{1}{\sqrt{6}}a\sqrt{(\phi + c)(\phi^2 + c^2)}. \] (48)

Substituting (48) into the first expression of (6) and integrating it, we get
\[ \int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{a}(\sigma + c)(\sigma^2 + c^2)} = \pm \int_{0}^{\xi} d\sigma. \] (49)

Choosing the $\phi(0) = -c$ and completing (49), we obtain one explicit function expression of Jacobin elliptic function type for the compacton solution of Eq. (2)
\[ \phi(x - ct) = c(\sqrt{2} - 1)\text{cn}(\omega_{10}(x - ct), m_1) - (\sqrt{2} + 1), \quad c < 0 \] (50)

and one explicit function expression of Jacobin elliptic function type for the kink-like wave solution of Eq. (2)
\[ \phi(x - ct) = c(\sqrt{2} - 1) - (\sqrt{2} + 1)\text{cn}(\omega_{11}(x - ct), m_2)), \quad c > 0, \] (51)

where $\omega_{10} = \sqrt{-\frac{\sqrt{2}}{2}ac}$, $m_1 = \sqrt{\frac{2\sqrt{2}}{2v_1}}$, $\omega_{11} = \sqrt{\frac{\sqrt{2}}{2ac}}$, $m_2 = \sqrt{\frac{2\sqrt{2}}{2v_1}}$.

2.2.2.2 When $k = \frac{1}{4}c - \frac{1}{2}ac^2$, (47) can be reduced to
\[ y = \pm \frac{1}{\sqrt{6}}a(\phi + c)\sqrt{(\phi - c)}. \] (52)

Substituting (52) into the first expression of (6) and integrating it, we get
\[ \int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{a}(\sigma + c)\sqrt{(\sigma - c)}} = \pm \int_{0}^{\xi} d\sigma. \] (53)

Choosing the $\phi(0) = c$ and completing (53), we obtain one explicit function expression for the kink-like wave solution of Eq. (2)
\[ \phi(x - ct) = -c - \frac{4c}{\cosh(\omega_{12}(x - ct)) - 1}, \quad c < 0 \] (54)

and one explicit function expression for the compacton solution of Eq. (2)
\[ \phi(x - ct) = c(1 + 2\tan^2(\omega_{13}(x - ct))), \quad c > 0, \] (55)

where $\omega_{12} = \sqrt{-\frac{1}{2}ac}$, $\omega_{13} = \sqrt{\frac{1}{2ac}}$. The profile of (55) is shown in Fig. 1(1-2).

Case 2.2.3: When $c > 2k$, taking $h = h_1$, from (33) we get
\[ y = \pm \frac{\sqrt{2}}{\sqrt{6}}a(\phi - \phi_1)(\phi - \phi_2) \sqrt{\phi - c}, \] (56)

where $\phi_1$, $\phi_2$ are defined by (32).

Substituting (56) into the first expression of (6) and integrating it, we get
\[ \int_{\phi(0)}^{\phi} \frac{(\sigma - c)}{\sqrt{a}(\sigma - \phi_1)(\sigma - \phi_2)\sqrt{(\sigma - c)}} = \pm \int_{0}^{\xi} d\sigma. \] (57)

Choosing the $\phi(0) = c$ and completing (57), we obtain one implicit function expressions for the peaked solitary wave solution of Eq. (2)
\[ \frac{\sqrt{\phi_1 - c}}{\phi_1} \tanh^{-1}\left(\frac{\sqrt{\phi - c}}{\sqrt{\phi_2 - c}}\right) = \pm \frac{1}{\sqrt{6}}a\xi + \frac{\sqrt{\phi_1 - c}}{\phi_1} \tanh^{-1}\left(\frac{\sqrt{\phi - c}}{\sqrt{\phi_1 - c}}\right), \quad c < \phi_2 \] (58)

and one implicit function expressions for the solitary wave solution of Eq. (2)
\[ \sqrt{\phi_1 - c} \tanh^{-1}\left(\frac{\sqrt{\phi - c}}{\sqrt{\phi_1 - c}}\right) = \sqrt{c - \phi_2} \tanh^{-1}\left(\frac{\sqrt{\phi - c}}{\sqrt{c - \phi_2}}\right) - \frac{1}{\sqrt{6}}a\xi, \quad \phi_2 < c < \phi_1 \] (59)
and one implicit function expressions for the compacton solution of Eq. (2)

\[
\frac{\sqrt{c - \phi^2}}{\phi_1} \tan^{-1} \left( \frac{\sqrt{\phi - c}}{\phi - \phi_1} \right) = \sqrt{\frac{1}{c} a_1 \zeta + \frac{\sqrt{c - \phi_1}}{\phi_1} \tan^{-1} \left( \frac{\sqrt{\phi - c}}{\phi - \phi_1} \right)}, \quad \phi_1 < c.
\] (60)

Let \( X_1 = \sqrt{\frac{1}{c} a_1 \zeta + \frac{\sqrt{\phi_1}}{\phi_1} \tan^{-1} \left( \frac{\sqrt{\phi - c}}{\phi - \phi_1} \right) + \sqrt{\frac{1}{c} a_1 \zeta + \frac{\sqrt{\phi_1}}{\phi_1} \tan^{-1} \left( \frac{\sqrt{\phi - c}}{\phi - \phi_1} \right)} \).

\[(58)-(60)\] can be rewritten respectively as

\[
\begin{align*}
\phi(X_1) &= c + (\phi_2 - c) \tanh^2 \left( \frac{\phi_2}{\sqrt{c - \phi_2}} \right) X_1, \\
\xi(X_1) &= \frac{1}{\sqrt{c - \phi_2}} \left( X_1 - \frac{\sqrt{\phi_1 - \phi_2}}{\phi_1 - \phi_2} \tanh \left( \frac{\phi_2}{\sqrt{c - \phi_2}} X_1 \right) \right), \quad c < \phi_2
\end{align*}
\] (61)

and

\[
\begin{align*}
\phi(X_2) &= c + (\phi_1 - c) \tanh^2 \left( \frac{\phi_1}{\sqrt{c - \phi_1}} \right), \\
\xi(X_2) &= \frac{1}{\sqrt{c - \phi_1}} \left( X_2 - \frac{\sqrt{\phi_1 - \phi_2}}{\phi_1 - \phi_2} \tanh \left( \frac{\phi_1}{\sqrt{c - \phi_1}} X_2 \right) \right), \quad \phi_1 < c
\end{align*}
\] (62)

and

\[
\begin{align*}
\phi(X_3) &= c + (\phi_2 - c) \tanh^2 \left( \frac{\phi_2}{\sqrt{c - \phi_2}} \right) X_3, \\
\xi(X_3) &= \frac{1}{\sqrt{c - \phi_2}} \left( X_3 - \frac{\sqrt{\phi_1 - \phi_2}}{\phi_1 - \phi_2} \tanh \left( \frac{\phi_2}{\sqrt{c - \phi_2}} X_3 \right) \right), \quad \phi_1 < c
\end{align*}
\] (63)

where \( X_1, X_2, \) and \( X_3 \) are three parameters and \( \omega_{14} = \sqrt{\frac{1}{c} a} \). The profiles of (62) and (61) are shown in Fig. 1(1-3) and (1-4).

### 2.3. Exact travelling wave solutions of the GCH equation when \( m = 3 \)

When \( m = 3 \), we have

\[
\phi_1 = \left( \frac{4(c - 2k)}{a} \right)^{\frac{1}{3}},
\] (64)

\[
y^2 = \frac{1}{10} \frac{a \phi^5 - (c - 2k) \phi^3 + h}{\phi - c},
\] (65)

\[
h_0 = 0, \quad h_1 = \frac{6}{5} (c - 2k) \left( 2 \left( \frac{c - 2k}{a} \right)^{\frac{2}{3}} \right)^2, \quad h_s = -\frac{1}{10} a \phi^2 + (c - 2k)c^2.
\] (66)

Case 2.3.1: Taking \( h = h_0 \) from (65) we get

\[
y^2 = \frac{1}{10} \frac{a \phi^2 (\phi^3 - \frac{10}{a} (c - 2k))}{\phi - c}.
\] (67)

2.3.1.1 When \( k = \frac{1}{2} c \), (67) can be reduced to

\[
y = \pm \sqrt{\frac{1}{10} a \phi^2 \sqrt{\phi (\phi - c)}}
\] (68)

Substituting (68) into the first expression of (6) and integrating it, we get

\[
\int_0^\phi \frac{c - \sigma}{\sqrt{\sigma^2 + \sqrt{c} \sigma}} = \pm \int_0^z \sigma d\sigma.
\] (69)

Choosing the \( \phi(0) = c \) and completing (69), we obtain one explicit function expression for the travelling solutions of Eq. (2)

\[
\frac{18}{10} c \left( (18(c \omega_{15} (x - ct))^2) \right)^{\frac{1}{3}} + 9 (c \omega_{15} (x - ct))^2 + (218 (c \omega_{15} (x - ct))^2) \frac{1}{3} - 9 (9 (c \omega_{15} (x - ct))^2 - 4) ((c \omega_{15} (x - ct))^2)^{\frac{1}{3}},
\] (70)

where \( \omega_{15} = \sqrt{\frac{c}{10}} \).
2.3.1.2 When $k = \frac{1}{2}c - \frac{1}{2\omega}ac^3$, (67) can be reduced to
\[ y = \pm \sqrt{\frac{1}{10}a\phi \sqrt{c^2 + c\phi + \phi^2}}. \] (71)

Substituting (71) into the first expression of (6) and integrating it, we get
\[ \int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{10a\sigma\sqrt{c^2 + c\sigma + \sigma^2}}} = \pm \int_{0}^{\xi} d\sigma. \] (72)

Choosing the $\phi(0) = c$ and completing (72), we obtain two explicit function expressions for the kink-like wave solutions of Eq. (2)
\[ \phi(x - ct) = \frac{4c(3 + 2\sqrt{3})}{3(7 + 4\sqrt{3})e^{e^{\omega_{16}(x - ct)}} - 3e^{e^{\omega_{16}(x - ct)}} - 2(3 + 2\sqrt{3})}, \quad c > 0, \] (73)
where $\omega_{16} = c\sqrt{\frac{2}{10}}$. The profiles of (73) are shown in Fig. 1(1-5) and (1-6).

Choosing the $\phi(0) = -c$ and completing (72), we obtain two explicit function expressions for the kink-like wave solutions of Eq. (2)
\[ \phi(x - ct) = \frac{-4c}{3e^{e^{\omega_{17}(x - ct)}} - e^{e^{\omega_{17}(x - ct)}} + 2}, \quad c < 0, \] (74)
where $\omega_{17} = c\sqrt{\frac{2}{10}}$.

Case 2.3.2: Taking $h = h_n$, from (65) we get
\[ y = \pm \sqrt{\frac{1}{10}a\phi^4 + \frac{1}{10}ac\phi^3 + \frac{1}{10}ac^2\phi^2 + \left(\frac{1}{10}ac^3 + 2k - c\right)\phi + \left(\frac{1}{10}ac^3 + 2k - c\right)c}. \] (75)

When $k = \frac{1}{2}c - \frac{1}{2\omega}ac^3$, (75) can be reduced to
\[ y = \pm \sqrt{\frac{1}{10}a\phi^4 + \frac{1}{10}ac\phi^3 + \frac{1}{10}ac^2\phi^2}. \] (76)

Using the results of Ref. [17], we obtain the travelling wave solutions of Eq. (2) as follows:
\[ \phi(x - ct) = \frac{csech^2(\omega_{18}(x - ct))}{(1 \pm tanh(\omega_{18}(x - ct)))^2 - 1}, \] (77)
\[ \phi(x - ct) = \frac{ccsch^2(\omega_{18}(x - ct))}{1 - (1 \pm coth(\omega_{18}(x - ct)))^2}, \] (78)
\[ \phi(x - ct) = \frac{csech^2(\omega_{18}(x - ct))}{1 \pm 2 tanh(\omega_{18}(x - ct))}, \] (79)
\[ \phi(x - ct) = \frac{ccsch^2(\omega_{18}(x - ct))}{1 \pm 2 coth(\omega_{18}(x - ct))}, \] (80)
\[ \phi(x - ct) = \frac{2c csch(\omega_{18}(x - ct))}{\pm \sqrt{3} - csch(\omega_{19}(x - ct))} \] (81)
and
\[ \phi(x - ct) = \frac{c^2 e^{e^{\omega_{19}(x - ct)}}}{(e^{e^{\omega_{19}(x - ct)}} - \frac{1}{10}ac)^2}, \] (82)
where $\omega_{18} = \frac{1}{2}c\sqrt{\frac{10}{3}}$, $\omega_{19} = c\sqrt{\frac{10}{3}}$.

3. Conclusion

In this paper, some exact solutions of GCH equation are obtained by using the integral bifurcation method under some parameter conditions. Compare with Refs. [14–16], many new travelling waves such as peaked compacton, kink-like wave and compacton are found. It is shown that the integral bifurcation method is useful to many nonlinear partial equations.

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References