Note

A simple proof of Graham and Pollak’s theorem

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Abstract

Graham and Pollak [Bell System Tech. J. 50 (1971) 2495–2519] obtained a beautiful formula on the determinant of distance matrices of trees, which is independent of the structure of the trees. In this paper we give a simple proof of Graham and Pollak’s result.

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Suppose $T$ is a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. Let $D = (d_{ij})_{n \times n}$ be the distance matrix of $T$, where $d_{ij}$ equals the distance between vertices $v_i$ and $v_j$. Graham and Pollak [5] obtained a beautiful formula as follows:

\[
\det(D) = -(n-1)(-2)^{n-2},
\]

which is independent of the structure of $T$. Other proofs of (1) can be found in [2–4].

Zeilberger [6] gave an elegant combinatorial proof of the Dodgson’s determinant-evaluation rule [1] as follows:

\[
\det(A) \det(A_2) = \det(A_{11}) \det(A_{nn}) - \det(A_{1n}) \det(A_{n1}),
\]

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where $A$ is a matrix of order $n > 2$, $A_{ij}$ is the minor of $A$ by deleting the $i$th row and $j$th column and $A_2$ is the minor of $A$ by deleting two rows and columns 1 and $n$.

Now we prove (1) by induction on $n$. It is trivial to show that if $n \leq 3$ then (1) holds. Hence we may suppose $T$ is a tree with $n \geq 4$ vertices. Note that $T$ has at least two pendant vertices. Without loss of generality, we assume both $v_1$ and $v_n$ are two pendant vertices of $T$. The unique neighbor of $v_1$ (resp. $v_n$) is denoted by $v_p$ (resp. $v_q$). Let $d_i$ denote the $i$th column of $D$. By the definition of $v_1, v_p, v_q$ and $v_n$, $(d_1 - d_p)^T = (-1, 1, 1, \ldots, 1)$ and $(d_n - d_q)^T = (1, 1, \ldots, 1, -1)$. Hence we have the following:

$$\det(D) = \det(d_1 - d_p + d_q - d_n, d_2, \ldots, d_{n-1}, d_n).$$

Note that

$$(d_1 - d_p + d_q - d_n)^T = (-2, 0, 0, \ldots, 0, 2).$$

Hence we have

$$\det(D) = -2 \det(D_{11}) + 2(-1)^{n+1} \det(D_{n1}). \quad (3)$$

On the other hand, by Dodgson’s determinant-evaluation rule (2), we have

$$\det(D) \det(D_2) = \det(D_{11}) \det(D_{nn}) - \det(D_{1n}) \det(D_{n1}). \quad (4)$$

By the definition of the distance matrix of $T$, $\det(D_1n) = \det(D_{n1})$. Particularly, $D_2, D_{11}$ and $D_{nn}$ denote the distance matrices of trees $T - v_1 - v_n$, $T - v_1$ and $T - v_n$, respectively. Hence, by induction, we have the following:

$$\left\{ \begin{array}{l}
\det(D) = -2[-(n - 2)(-2)^{n-3}] + 2(-1)^{n+1} \det(D_{11}), \\
\det(D)[--(n - 3)(-2)^{n-4}] = [-(-n - 2)(-2)^{n-3}]^2 - [\det(D_{11})]^2,
\end{array} \right.$$

which implies (1) immediately.

The proof above also implies $\det(D_{1n}) = 2^{n-2}$. By a similar discussion, we have $\det(D_{ij}) = 2^{n-2}$ if $i$ and $j$ are two pendant vertices of $T$, which is a special case of Lemma 1 in [4].

References