On Zero-Sum Subsequences of Restricted Size

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Communicated by Alan C. Woods

Received March 1, 1995; revised March 13, 1996

Let $G$ be a finite abelian group with exponent $e$, let $r(G)$ be the minimal integer $t$ with the property that any sequence of $t$ elements in $G$ contains an $e$-term subsequence with sum zero. In this paper we show that if $r(C_{2^n})=4n\leq 3$ and if $n\geq (3m-4)(m-1)m^2-3)/4m$, then $r(C_{2nm})=4mn\leq 3$ provided that $n=2^{3^k}5^l$, $m\geq \cdots \geq m_1$, and $n\geq (3m-4)(m_1-1)m_1^2-3)/4m_1$. © 1996 Academic Press, Inc.

1

Let $C_n$ be the cyclic group of order $n$, and $C_n^k$ the product of $k$ copies of $C_n$. Erdős, Ginzburg and Ziv [5] showed that if $a_1, \ldots, a_{2n-1}$ is a sequence of $2n-1$ elements in a finite abelian group of order $n$ (written additively) then 0 can be written in the form $0 = a_{i_1} + \cdots + a_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq 2n-1$. This result has been generalized in several directions ([1–3], [7, 11]).

In [10], Harborth showed that if $n = 2^3b$ then any sequence of $4n-3$ elements in $C_2^n$ contains an $n$-term subsequence with sum zero. In [11], Kemnitz obtained the same result for $n = 2^35^l7^d$ and conjectured that it is true for any positive integer $n$. Recently, Alon and Dubiner [1] proved that any sequence of $6n-5$ elements in $C_2^n$ contains an $n$-term subsequence with sum zero. But the $4n-3$ conjecture is still open. In Section 2 of this paper we show that if the conjecture is true for $n$ and $n\geq (3m-4)(m-1)m^2+3)/4m$ then the conjecture is true also for $nm$, in particular, this result implies that the conjecture is true for $n = 2^{3^k}5^l7^d$ and $nm\geq \cdots \geq m_1$. The final Section 3 contains some concluding remarks.

Let $S = (a_1, \ldots, a_k)$ be a sequence of elements in a finite abelian group $G$ (written additively). By $\sum S$ we denote the sum $\sum_{i=1}^{k} a_i$, and we say $S$ with sum zero if $\sum S = 0$. By $\lambda$ we denote the empty sequence and adopt the convention that $\sum \lambda = 0$. By a subsequence of $S$ we mean a sequence $T = (a_{i_1}, \ldots, a_{i_l})$ with $1 \leq i_1 < \cdots < i_l \leq k$, the index set $\{i_1, \ldots, i_l\}$ is denoted
by $I_T$. We say two subsequences $T$, $W$ of $S$ disjoint if $I_T \cap I_W = \emptyset$. We say more than two subsequences disjoint they are pairwise disjoint. For disjoint subsequences $S_1, ..., S_t$, we define $S_1 + ... + S_t$ to be the subsequence $X$ of $S$ with $I_X = I_{S_1} \cup ... \cup I_{S_t}$, and we define $S = S_1 - ... - S_t$ to be the subsequence $Y$ of $S$ with $I_Y = \{1, ..., k\} - I_{S_1} - ... - I_{S_t}$. For any two subsequences $A, B$ of $S$ we define $A \cap B$ to be the subsequence $C$ with $I_C = I_A \cap I_B$.

Let $G$ be a finite abelian group and let $e(G)$ be the exponent of $G$, i.e. the maximal order of an element of $G$. Put $e = e(G)$, we define $r(G)$ to be the minimal integer $r$ with the property that any sequence of $r$ elements in $G$ contains an $e$-term subsequence with sum zero. It is well known that $G$ can be written in the form $G = G_{n_1} \oplus ... \oplus G_{n_t}$ with $n_1 | ... | n_t$ and $n_1 \geq 2$, we call $t$ the rank of $G$.

2

In this section we consider finite abelian groups with rank two.

**Theorem 2.1.** If $r(C_{2m}^d) = 4n - 3$ and if $n \geq ((3m - 4)(m - 1)m^2 + 3)/4m$ with $m \geq 2$, then $r(C_{2nm}^d) = 4nm - 3$.

To prove Theorem 2.1 we need several preliminary results.

**Lemma 2.1 ([1]).** $r(C_{2m}^d) \leq 6m - 5$.

**Lemma 2.2 ([4], [12]).** If $a_1, ..., a_{3d - 2}$ is a sequence of $3d - 2$ elements in $C_{2d}^d$, then one can find a nonempty subsequence $T$ of the sequence with sum zero and $|T| \leq d$.

**Remark.** The conclusion of Lemma 2.2 was first shown by Olson [12] for the case $d$ is a prime, but one can easily extend it to the general case by induction on $d$, and such a proof is contained in [4].

The next lemma is crucial in our proof.

**Lemma 2.3.** Let $a_1, ..., a_{4d - 3}$ be a sequence of $4d - 3$ elements in $C_{2d}^d$. Suppose that there exists an element in $C_{2d}^d$ which occurs at least $d - 1$ times in the sequence. Then the sequence contains a $d$-term subsequence with sum zero.

**Proof.** Without loss of generality, we may assume that $a_1 = ... = a_{d - 1} = a$ (say), consider the $(3d - 2)$-term sequence $a_d - a, ..., a_{4d - 3} - a$, it
follows from Lemma 2.2 that there exist \( i_1, \ldots, i_c \) with \( d \leq i_1 < \cdots < i_c \leq 4d - 3 \), such that \( \sum_{j=1}^c (a_{i_j} - a) = 0 \) and such that \( 1 \leq c \leq d \), hence
\[
a_0 + \cdots + a_c + a + \cdots + a = 0.
\]

This completes the proof.

Proof of Theorem 2.1. We first prove \( r(C_{nm}^2) \leq 4nm - 3 \), to do this we consider any sequence \( S = (a_1, \ldots, a_{4nm-3}) \) of \( 4nm - 3 \) elements in \( C_{nm}^2 \) and want to show that \( S \) contains an \( nm \)-term subsequence with sum zero.

Let \( \phi \) be a homomorphism from \( C_{nm}^2 \) onto \( C_n^2 \) with \( \ker \phi = C_n^2 \) (up to isomorphism). For any \( g \in C_{nm}^2 \), by \( S_g \) we denote the subsequence of \( S \) consisting of all terms \( a_i \) with \( \phi(a_i) = g \). Choose \( h \in C_m^2 \) so that \( |S_h| = \max \{ |S_g|, g \in C_{nm}^2 \} \). We show next that there exist at least \( 4n - 3 \) disjoint \( m \)-term subsequences of \( S \) such that each of them has sum in \( \ker \phi = C_n^2 \).

We now split the proof into steps.

Step 1. Clearly, one can find some disjoint \( m \)-term subsequences \( R_1, \ldots, R_k \) of \( S \) such that \( R_i \) has sum in \( \ker \phi = C_n^2 \) and if \( |S_i| \leq 6m - 6 \) follows from Lemma 2.1.

Step 2. Put \( W = S - S_h - R_1 - \cdots - R_k \). In this step we show that there exist \( u \geq 0 \) disjoint nonempty subsequences \( A_1, \ldots, A_u \) of \( S_h \) and \( u \) disjoint nonempty subsequences \( B_1, \ldots, B_u \) of \( W \) satisfying the following conditions (i) and (ii).

(i) Either \( u = 0 \) or for every \( v = 1, \ldots, u \), \( A_v + B_v \) has sum in \( \ker \phi = C_n^2 \).

(ii) \(|W - B_1 - \cdots - B_u| \leq 3m - 3\).

If \(|W| \leq 3m - 3\), then \( u = 0 \) meets with the conditions.

If \(|W| > 3m - 2\), take an arbitrary subsequence \( C_1 \) of \( S_h \) with \(|C_1| = \max \{|D|, D \in C_{m-1}^2 \} = m - 1\). By using Lemma 2.3 one can find an \( m \)-term subsequence \( D_1 \) of \( C_1 + W \) with \( \sum D_1 \in \ker \phi = C_n^2 \). Since \( W \) contains no \( m \)-term subsequence having sum in \( \ker \phi = C_n^2 \), so we have \( D_1 \cap C_1 \neq \emptyset \), but \(|C_1| = m - 1\), so we also have \( D_1 \cap W \neq \emptyset \), put \( A_1 = D_1 \cap C_1, B_1 = D_1 \cap W, S_1 = S_h \), and put \( S_2 = S_1 + C_1 - A_1 \).

If \(|W - B_1| \leq 3m - 3\), then putting \( u = 1 \) meets with the conditions (i) and (ii). Otherwise, \(|W - B_1| > 3m - 2\), take an arbitrary subsequence \( C_2 \) of \( S_h \) with \(|C_2| = m - 1\), they by using Lemma 2.3 one can find an \( m \)-term subsequence \( D_2 \) of \( C_2 + (W - B_1) \) with \( \sum D_2 \in \ker \phi = C_n^2 \), similarly as above one can show that \( D_2 \cap C_2 \neq \emptyset \) and \( D_2 \cap (W - B_1) \neq \emptyset \), put
$A_2 = D_2 \cap C_2$, $B_3 = D_2 \cap (W - B_1)$, and put $S_3 = S_2 + C_2 - A_2$. Continue the same process and notice that
\[
\begin{align*}
&u \leq 6m - 6 - (3m - 2) = 3m - 4 \\
&\leq \frac{4nm - 3}{m'(m - 1)} \quad \text{(since } n \geq (3m - 4)(m - 1)m^2 + 3) \\
&\leq \frac{|S_u|}{m - 1},
\end{align*}
\]

we must get an integer $u \geq 0$ meeting with conditions (i) and (ii).

Step 3. Write $|S_h - A_1 - \cdots - A_u| = mq + r$ with $0 \leq r \leq m - 1$, take an arbitrary subsequence $A$ of $S_h - A_1 - \cdots - A_u$ with $|A| = r$, and divide $S_h - A_1 - \cdots - A_u - A$ into $q$ disjoint $m$-term subsequences, obviously, each of them has sum in ker $\phi = C_n^2$.

Step 4. In the three steps above we get altogether $t = (4nm - 3 - |A| - W - B_1 - \cdots - B_u)/m$ disjoint $m$-term subsequences $E_1, \ldots, E_t$ (say) with $\sum E_i \in \ker \phi = C_n^2$ for $i = 1, \ldots, t$. Since $|A| = r \leq m - 1$ and $|W - B_1 - \cdots - B_u| \leq 3m - 3$, we have
\[
t \geq \frac{4nm - 3 - (m - 1 + 3m - 3)}{m} = 4n - 4 + 1/m,
\]

hence $t \geq 4n - 3 = r(C_n^2)$, therefore one can find an $n$-subset $I$ of $\{1, \ldots, t\}$ with
\[
\sum_{i \in I} E_i = 0.
\]

This proves that $r(C_{nm}^2) \leq 4nm - 3$.

To prove the lower bound we consider the following example:

Let $T$ be a sequence of $4nm - 4$ elements in which each of the four elements $(0,0), (0, 1), (1, 0), (1, 1)$ occurs $nm - 1$ times, clearly, $T$ contains no $nm$-term subsequence with sum zero, hence, $r(C_{nm}^2) = 4nm - 3$. This completes the proof.

From the proof of Theorem 2.1 it is easy to see that the inequality $n \geq ((3m - 4)(m - 1)m^2 + 3)/4m$ can be replaced by $n \geq ((m - 1)(r(C_m^2) - 3m + 1)m^2 + 3)/4m$, so we have

**Corollary 2.1.** If $r(C_m^2) = 4n - 3$ and if $n \geq ((m - 1)(r(C_m^2) - 3m + 1)m^2 + 3)/4m$ with $m \geq 2$, then $r(C_{nm}^2) = 4nm - 3$. 


Corollary 2.2. Let \( n = 2^a 3^b 5^c 7^d \) with \( a, b, c, d \geq 0 \), and let \( m_1, \ldots, m_r \). Suppose that \( n \geq (3m_1 - 4)(m_1 - 1)m_1^2 + 3)/4m_1 \). Then, \( r(C_{m_1 \cdots m_r}) = 4n m_1 \cdots m_r - 3 \).

Proof. Since it is proved that \( r(C^2) = 4n - 3 \) in [11], the corollary follows from Theorem 2.1 by using induction on \( r \).

Corollary 2.3. Let \( n = 2^a 3^b 5^c 7^d \) with \( a, b, c, d \geq 0 \), and let \( m_1, \ldots, m_r \). Suppose that \( n \geq (3m_1 - 4)(m_1 - 1)m_1^2 + 3)/4m_1 \). Then, for any positive integer \( m \), \( r(C_{m m_1 \cdots m_r}) = 4nm_1 \cdots m_r - 3 \).

Proof. Put \( 2^{a+b+c}+2^{2a+c+d} \) to prove \( r(C_{2^a 3^b 5^c 7^d}) = 4^{a+b+c}+2^{2a+c+d} \) we consider any sequence \( S \) of \( 2^{a+b+c}+2^{2a+c+d} \) elements in \( C_{2^a 3^b 5^c 7^d} \). It follows from Corollary 2.2 that \( r(C^2) = 4d - 3 \), notice that \( 2d + 2dm - 3 = (2m - 2)d + 4d - 3 \), one can find \( 2m - 1 \) disjoint \( d \)-term subsequences \( S_1, \ldots, S_{2m-1} \) of \( S \) such that \( \sum S_i \in C_{2^a 3^b 5^c 7^d} \), so by using the Erdős-Ginzburg-Ziv theorem, one can find an \( m \)-subset \( I \) of \( \{1, \ldots, 2m-1\} \) such that \( \sum_{i \in I} S_i = 0 \). This proves that \( r(C_{2^a 3^b 5^c 7^d}) \leq 2d + 2dm - 3 \).

To prove the lower bound we consider a sequence \( T \) of \( 2d + 2dm - 4 \) elements in \( C_{2^a 3^b 5^c 7^d} \) in which the four elements \((0, 0), (0, 1), (1, 0), (1, 1)\) occur \( dm \), \( dm \), \( d \), \( d \) times, respectively. It is easy to see that \( T \) contains no \( dm \)-term subsequence with sum zero. Hence \( r(C_{2^a 3^b 5^c 7^d}) \leq 2d + 2dm - 3 \). This completes the proof.

Let \( G = C_{n_1} \oplus \cdots \oplus C_{n_k} \) with \( 1 < n_1 | n_2 | \cdots | n_k \). By \( M(G) \) we denote the number \( n_1 + n_2 + \cdots + n_k - k + 1 \).

Proposition 3.1. Let \( p \) be a prime, \( H \) be a finite abelian \( p \)-group, and let \( G = H \oplus C_{pm} \). Suppose \( p^n \geq M(H) \) and suppose

\[
m \geq \frac{p^n |H| (p^n - 1)(p^n |H| - p^n - M(H) + 1) - p^n - M(H) + 3}{2p^n}.
\]

Then, \( r(G) \leq 2p^m + p^n + M(H) - 3 \).

Lemma 3.1 ([4], (3.9) Theorem). Let \( p \) be a prime, \( H \) a finite abelian \( p \)-group, and \( G = H \oplus C_{p^n} \). Suppose that \( p^n \geq M(H) \). Then, any sequence of \( 2p^n + M(H) - 2 \) elements in \( G \) contains a nonempty zero-sum subsequence of length not exceeding \( p^n \).
Lemma 3.2. Let $p$ be a prime, $H$ a finite abelian $p$-group, $G = H \oplus C_p^n$, and $S$ a sequence of $3p^n + M(H) - 3$ elements in $G$. Suppose that $p^n \geq M(H)$ and suppose that there exists an element in $G$ which occurs $p^n - 1$ times in $S$. Then, $S$ contains a $p^n$-term subsequence with sum zero.

Proof. The proof is similar to that of Lemma 2.3 and we omit it here.

Proof of Proposition 3.1. The proof is similar to that of Theorem 2.1 and we omit the details here.

When $m = p$ is a sufficiently large prime, one can use the result of $r(C_p^n) \leq 5p - 2$ proved in [1] to improve Theorem 2.1.

For arbitrary finite abelian groups $G$, recently, a considerable progress was made on $r(G)$ by Alon and Dubiner, they proved $r(G) \leq c e$, where $c$ is a constant depending only on $e = e(G)$. In [8], the author showed that $r(G) \leq |G| + e - 1$. But the problem of determining $r(G)$ for all finite abelian groups $G$ remains wide open and seems to be very difficult, even for $G = C_2^n$ the problem is still open in general.

ACKNOWLEDGMENT

The author is very grateful to the referee for helpful suggestions and comments.

REFERENCES

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