Energy Detection Limits under Log-Normal Approximated Noise Uncertainty

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Abstract

We revisit, in this letter, the impact of noise uncertainty on the performance of the well known energy detector. Mainly, we reconsider the case of a Log-Normal approximated noise uncertainty suggested in the work of Alexander Sonnenschein and Philip M. Fishman. We show that under a Log-Normal noise uncertainty, closed form expressions of the detector’s performances and limits can be provided. Thus we show that, relying on mild approximations, we can design a detector with a fixed probability of false alarm function of the uncertainty, and present a new expression of the SNR-wall that depends on the desired performances of the detector as well as the introduced uncertainty parameter.

I. INTRODUCTION

The Neyman-Pearson Energy Detector (NP-ED, also known as Energy Detector or radiometric detector) is a commonly used spectrum sensor. It has been extensively analyzed [1] [2] for its properties as a semi-blind low complexity spectrum sensor, since it ignores the characteristics of the received signals and only relies on the perceived energy of the signal. The main detection process relies on the comparison of the perceived energy to a fixed threshold that depends on the desired performances of the detector as well as the noise power level. However, despite its general assets, the NP-ED’s performances decrease quickly in case of imperfect knowledge on the noise level [3], [4].

As a matter of fact, in their seminal paper, Alexander Sonnenschein and Philip M. Fishman [3] performed a worst case analysis on the performances of the energy detector in the case of
imperfect knowledge on the noise level, referred to as noise uncertainty. As a matter of fact, depending on the information held by the decision maker on the noise level and its uncertainty, the analysis suggested in [3] relies on an upper-bound of the probability of false alarm and on a lower bound of the probability of detection. The main results showed that if the noise power level is only known through a confidence bound, there exists an SNR-wall, value of the Signal-to-Noise Ratio (SNR) beyond which detection is theoretically impossible.

In this letter, we reconsider the case of a Log-Normal approximated noise uncertainty suggested in [3]. However, rather than reducing the analysis of the noise uncertainty to a bounded distribution, we suggest to redefine the uncertainty based on the estimated noise distribution’s variance. We then propose a new expression of the SNR-wall function of the introduced uncertainty parameter. Moreover, unlike the previously introduce SNR-wall expression [3], the herein introduced SNR-wall formula takes into account the desired performances of the detector. Consequently, it enables an accurate evaluation of the detection limits.

The rest of this letter is organized as follows: first we start by presenting the general system model in Section II. Then we analyze the performances of the ED in case of Log-Normal noise uncertainty in Section III. Finally Section IV concludes this letter.

II. SYSTEM MODEL

A. Network Assumption

Let $\mathbf{r}_t = [r_{t,0}, r_{t,1}, \cdots, r_{t,M-1}]$ be $M$ independent and identically distributed (i.i.d.) samples gathered by the receiver at the current slot $t \in \mathbb{N}$. The outcome of the sensing process can be modeled as a binary hypothesis\(^1\) test described as follows:

$$
\mathbf{r}_t = \begin{cases} 
\mathbf{n}_t; & H_0 \\
\mathbf{x}_t + \mathbf{n}_t; & H_1 
\end{cases}
$$

where hypotheses $H_0$ and $H_1$ refer respectively to the case of an absent or a present signal on the sensed channel. On the one hand, $\mathbf{x}_t = [x_{t,0}, x_{t,1}, \cdots, x_{t,M-1}]$ refers to the source signal where every sample $x_{t,k}$ is perceived as an i.i.d. realization of a Gaussian stochastic distribution $\mathcal{N}(0, \sigma^2_{x,t})$. On the other hand, $\mathbf{n}_t = [n_{t,0}, n_{t,1}, \cdots, n_{t,M-1}]$ refers to i.i.d. additive white Gaussian

\(^1\)In this letter, we associate the numerical value 1 to the presence of a signal on the channel and 0 otherwise.
noise (AWGN) samples $N(0, \sigma^2_{n,t})$. Moreover, $x_t$ and $n_t$ are assumed to be independent. Thus, we consider the following Gaussian received signals under either hypothesis $\forall r_{t,i} \in \{0, \cdots, M-1\}$:

$$
\begin{align*}
\mathbb{H}_0 : r_{t,i} &\sim N(0, \sigma^2_{n,t}) \\
\mathbb{H}_1 : r_{t,i} &\sim N(0, \sigma^2_{x,t} + \sigma^2_{n,t})
\end{align*}
$$

Within this context, the detection outcome can be modeled as the output of a decision making policy $\pi$ that maps the current samples $r_t$ into a binary value $d_t = \pi(r_t)$, $d_t \in \{0, 1\}$.

**B. Performance Evaluation of a Detection Policy $\pi$**

Under the previously considered binary hypothesis test, one can define two probabilities that characterize the performance of the detection policy $\pi$ at the slot number $t$: The probability of false alarm ($P_{fa,t}$) and the probability of detection ($P_{d,t}$):

$$
\begin{align*}
P_{fa,t} &= P(d_t = 1|\mathbb{H}_0) \\
P_{d,t} &= P(d_t = 1|\mathbb{H}_1)
\end{align*}
$$

Usually, constraints impose to fix the $P_{fa,t}$ under a given level $\alpha$, such that $P_{fa,t} \leq \alpha$. The most powerful decision policy is then defined as the one having the largest $P_{d,t}$ value for a given $P_{fa,t} = \alpha$.

**C. Neyman-Pearson Energy Detector**

NP-ED assumes that the values of the noise level $\sigma^2_{n,t}$, at every slot number $t$, are perfectly known. For the sake of simplicity and without loss of generality, we consider in the rest of the letter a constant noise level for all $t$, $\sigma^2_{n,t} = \sigma^2_{n}$. Under these assumptions, the NP-ED is proven to be the most powerful test.

NP-ED relies on the computation of the received energy statistic $T_t$ at the slot number $t$ defined such as:

$$
T_t = \sum_{i=0}^{M-1} |r_{t,i}|^2
$$

The decision policy $\pi_{NP-ED}$ is a simple threshold policy that only depends on the comparison of the statistic $T_t$, at the current slot $t$, with a given threshold $\xi_t(\alpha)$. This latter is selected to guaranty $P_{fa} \leq \alpha$. Such policies are usually described using the following notation:

$$
T_t \leq_{\mathcal{H}_0} \xi_t(\alpha)
$$
Evolution of the maximum absolute error between Chi2 distributions and LogNormal/Normal distributions

Residual function: Comparison between Chi2, LogNormal and Normal distributions

Fig. 1. Residual functions. In the right figure, the curves describe the differences between, Chi-square, Log-Normal and Normal probability density functions relative to $M = 250$ degrees of freedom. The relative order of the curves still holds for very large degrees of freedom while the amplitude decreases as $1/M$ as described in the left figure.

The following equations remind us of the expressions of $P_{f,t}$ and $P_{d,t}$ (where $T_t \sim \chi^2_M$) as well as their approximations for large $M$ (where $T_t$ is assumed to follow a Gaussian distribution):

$$
\begin{align*}
\mathbb{P}_{f,t} &= 1 - F_{\chi^2_M} \left( \frac{\xi_t(\alpha)}{\sigma_n^2} \right) \\
\approx Q \left( \sqrt{\frac{M}{2}} \left( \frac{\xi_t(\alpha)}{\sigma_n^2} - 1 \right) \right)
\end{align*}
$$

$$
\begin{align*}
\mathbb{P}_{d,t} &= 1 - F_{\chi^2_M} \left( \frac{\xi_t(\alpha)}{\sigma_n^2 + \sigma_{x,t}^2} \right) \\
\approx Q \left( \sqrt{\frac{M}{2}} \left( \frac{\xi_t(\alpha)}{\sigma_n^2 + \sigma_{x,t}^2} - 1 \right) \right)
\end{align*}
$$

where $F_{\chi^2_M} (\cdot)$ refers to the cumulative distribution function of a $\chi^2$-distribution with $M$ degrees of freedom, and $Q(\cdot)$ is the complementary cumulative distribution function of Gaussian random variable (also known as Marcum function) [5].

NP-ED provides satisfactory behavior when $\sigma_n^2$ is known. Unfortunately, when such knowledge is unavailable, its performances, through a worst-case analysis, is shown to significantly degrade [3], [4].
III. ENERGY DETECTOR UNDER LOG-NORMAL NOISE UNCERTAINTY

A. Noise Uncertainty and Energy Statistic’s Approximation

Let $\hat{\sigma}^2_n$ denote the estimated noise power level. We assume that $\hat{\sigma}^2_n$ follows an unbiased Log-Normal distribution as suggested in [3]. Assuming that the expectation and the variance of the estimated noise level, respectively, verify $E[\hat{\sigma}^2_n] = \sigma^2_n$ and $V[\hat{\sigma}^2_n] = u\sigma^4_n$, with $u$ the defined uncertainty parameter, also defined through a (non conventional) decibel value $u = 10^{U_{dB}/10} - 1$, then:

$$\hat{\sigma}^2_n \sim LogN(\mu_u, V_u), \text{ s.t. :} \begin{cases} V_u = \log (1 + u) \\ \mu_u = 2 \log (\sigma_n) - \frac{V_u}{2} \end{cases}$$

where $\mu_u$ and $V_u$ respectively refer to the mean and variance parameters of the uncertainty noise distribution.

Moreover, for mathematical reasons, we introduce a Log-Normal approximation of the $\chi^2$-distribution. Approximation empirically validated in Figure 1 for $M$ usually considered large enough, i.e., $M > 50$. As a matter of fact, we can observe in Figure 1, right curves, that the Log-Normal approximation leads to an estimation error on the probability density function. However, the amplitude of this error function, called residual function, is uniformly bounded by a function that decreases as $1/M$.

Thus, let us assume that the power statistic $T_t/M$ can be accurately approximated by a Log-Normal distribution such that $E[T_t/M] = \sigma^2_T$ and $V[T_t/M] = 2\sigma^4_T/M$ then:

$$\frac{T_t}{M} \sim LogN(\mu_T, V_T), \text{ s.t. :} \begin{cases} V_T = \log \left(1 + \frac{2}{M}\right) \\ \mu_T = 2 \log (\sigma_T) - \frac{V_T}{2} \end{cases}$$

where $\sigma^2_T$ is the value of the power level of the collected samples depending on the current state of the channel at the slot $t$.

Finally, we introduce the following statistic $W_t$ defined as:

$$W_t = \log \left(\frac{T_t}{M\hat{\sigma}^2_n}\right) \quad (1)$$

---

2 Other distributions can be considered to model noise uncertainty [4], [6]. However, the evaluation and comparison of their pros and cons is still a matter of debate and is out of the scope of this letters.

3 We use, in this letter, the natural logarithm function $\log(\cdot)$

4 Stronger mathematical arguments are currently under investigation.
In the next subsection, we analyze the performance of the Energy Detector based on the statistic $W_t$. The following Lemma and Theorems rely on two main assumptions. On the one hand, a Log-Normal approximated noise uncertainty and, on the other hand a Log-Normal approximation of $\chi^2$ distributions. The former was suggested in [3] and the latter is supported by empirical simulations (Cf. Figure 1). These results allow determining the threshold of the detection policy given a desired false alarm. Moreover, they provide a closed form expression of the SNR-wall that depends on the desired performances of the detector as well as the uncertainty parameter $u$ and the estimation parameter $M$.

**B. Energy Detector’s Performances and Limits**

Under the previously introduced assumptions, we present the main results of the paper. For that purpose, anticipating the next results, we use the following notations: the Signal-to-Noise Ratio $\gamma_t = \sigma^2_{x,t}/\sigma^2_n$ and:

$$
\begin{align*}
E[W_t|\mathcal{H}_0] &= \frac{1}{2} \left( V_u - V_T \right) = \frac{1}{2} \log \left( \frac{1+u}{1+2/M} \right) \\
E[W_t|\mathcal{H}_1] &= \log (1 + \gamma_t) + E[W_t|\mathcal{H}_0] \\
V[W_t] &= V_u + V_T = \log \left( (1 + \frac{2}{M}) (1 + u) \right)
\end{align*}
$$

**Lemma 1 (Distribution of $W_t$):** Let $W_t$ be the random variable defined in Equation 1. We assume that the previously introduced assumptions hold, then:

$$
\frac{W_t - E[W_t|\mathcal{H}_0]}{\sqrt{V[W_t]}} \sim \begin{cases} 
\mathcal{H}_0 & : N(0,1) \\
\mathcal{H}_1 & : N \left( \frac{\log (1+\gamma_t)}{\sqrt{V[W_t]}}, 1 \right)
\end{cases}
$$

**Sketch of the proof:** Notice that we can write:

$W_t = \log \left( \frac{T^2}{M} \right) - \log (\hat{s}_n^2)$. Thus, $W_t$ is a linear combination of two independent Gaussian random variables with, respectively, parameters: \{${\mu}_T, V_T$\} and \{${\mu}_u, V_u$\}. Using the previously introduced notations the end of the proof is obvious.

We next present the performances of the herein analyzed detector.

**Theorem 1 (Detector’s performances):** Let $\xi_t(\alpha)$ be a real value such that the threshold based policy is: $W_t \geq_{\mathcal{H}_0}^{\mathcal{H}_1} \log (\xi_t(\alpha))$, then the probabilities of false alarm and detection have the...
following forms:

\[
\begin{align*}
\mathbb{P}_{fa,t} &= Q \left( \frac{\log(\xi_t^{(\alpha)}\sqrt{\frac{1+2/M}{1+u}})}{\sqrt{\log((1+2/M)(1+u))}} \right) \\
\mathbb{P}_{d,t} &= Q \left( \frac{\log(\xi_t^{(\alpha)}\sqrt{\frac{1+2/M}{1+\gamma_t}})}{\sqrt{\log((1+2/M)(1+u))}} \right)
\end{align*}
\]

**Sketch of the proof:**

Since $W_t$ follows a Gaussian distribution under either hypotheses, we can write the probabilities of false alarm $\mathbb{P}_{fa,t}$ and detection $\mathbb{P}_{d,t}$ of this energy detector as follows:

\[
\begin{align*}
\mathbb{P}_{fa,t} &= Q \left( \frac{\log(\xi_t^{(\alpha)} - \mathbb{E}[W_t | H_0])}{\sqrt{\mathbb{V}[W_t]}} \right) \\
\mathbb{P}_{d,t} &= Q \left( \frac{\log(\xi_t^{(\alpha)} - \mathbb{E}[W_t | H_1])}{\sqrt{\mathbb{V}[W_t]}} \right)
\end{align*}
\]

Using the previously introduced notations, we obtain the stated results.

The contributions of Theorem 1 are twofold: On the one hand, Theorem 1 provides closed form expressions of $\mathbb{P}_{fa,t}$ and $\mathbb{P}_{d,t}$. On the other hand, as a corollary, it provides an explicit expression of the threshold for a given probability of false alarm:

\[
\log (\xi_t^{(\alpha)}) = Q^{-1}(\mathbb{P}_{fa}) \sqrt{\mathbb{V}[W_t] + \mathbb{E}[W_t | H_0]}
\]

The following result provides a general form of the SNR-wall as a function of the desired performances of the detector and the uncertainties \( \{2/M; u\} \).

**Theorem 2 (SNR-wall):** Let $W_t$ be the random variable defined in Equation 1 and let $\Delta = Q^{-1}(\mathbb{P}_{fa}) - Q^{-1}(\mathbb{P}_{d})$, then the SNR-wall of the ED under a Log-Normal approximated noise level is equal to:

\[
\gamma_{wall,t} = e^{\Delta \sqrt{\mathbb{V}[W_t]}} - 1
\]

**Sketch of the proof:**

Notice that by inversing the equations of the probabilities of false alarm and detection, we can write:

\[
\log (\xi_t^{(\alpha)}) = \begin{cases} 
Q^{-1}(\mathbb{P}_{fa}) \sqrt{\mathbb{V}[W_t]} - \log \left( \frac{1+2/M}{1+u} \right) \\
Q^{-1}(\mathbb{P}_{d,t}) \sqrt{\mathbb{V}[W_t]} - \log \left( \frac{1+2/M}{(1+u)(1+\gamma_t)^2} \right)
\end{cases}
\]

which lead to the following expression:

\[
\log (1 + \gamma_t) = \Delta \sqrt{\mathbb{V}[W_t]}
\]
which concludes the proof.

The impact of noise uncertainty on the detection performances, as described in the equations of Theorem 1, is illustrated in Figure 2 (left figure). On the left figure seven curves of probability of detection are drawn for a false alarm equal to 0.1. The first curve entitled “NP-Energy Detector” represents the results obtained by the NP-ED with no approximation and with no uncertainty on the noise level as described in Section II. The second curve, entitled “Energy Detector $U = 0$ dB”, that matches the first curve, illustrates the results as described in Section III with $U_{dB} = 0$. The other five curves show the impact of the uncertainty on the herein introduced detector under a Log-Normal model for the noise uncertainty. As expected, we can notice that the detection abilities of the suggested detector degrade as the uncertainty increases. However, the loss of performances can be accurately predicted. As a matter of fact, the right figure of Figure 2 illustrates the result of Theorem 2, where we can observe the evolution of the $SNR_{wall}$ as a function of the uncertainty (in this case to achieve a probability of false alarm equal to 0.1 and a probability of detection equal to 0.9). We can notice that the existing gap between the second and third curves (left figure), i.e. the case where the uncertainties are respectively equal to 0$dB$ and 0.5$dB$, and referred to as $\Delta SNR_1$, in both figures, matches the result of Theorem 2 (right figure). Similar comments apply for the gap $\Delta SNR_2$. Finally, this results show that, indeed, Theorem 1 and 2 enable to evaluate the detection limits of the system and conclude this letter.
IV. CONCLUSION

We suggested, in this letter, a different approach to define the noise level uncertainty and we evaluated its consequences in the case of an energy detector. For that purpose, we revisited the Log-Normal approximation of the noise power estimation, already suggested in the literature, and analyzed the energy detectors limits within this context. We mainly showed that there exists an SNR-wall which value depends on the desired performances of the detector as well as the uncertainty parameter introduced in this letter.

Nevertheless, these results rely on a Log-Normal approximation of $\chi^2$ distributions. Although empirical simulations support this approximation, stronger mathematical arguments are currently under investigation.

REFERENCES


