Closed-Form Expression for the Bit Error Probability of Rectangular QAM Subject to Rayleigh Fading

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Abstract—In this paper a closed-form expression for the bit error probability (BEP) of M-ary pulse amplitude modulation (M-PAM) subject to Rayleigh fading is derived. This expression is used to derive another general result: a closed-form expression for the BEP of rectangular quadrature amplitude modulation (R-QAM) subject to Rayleigh fading. In this work the fading channel is seen as an additive noise channel whose additive noise is modeled as the ratio between Gaussian and Rayleigh random variables.

Index Terms—Rectangular quadrature amplitude modulation, Rayleigh fading, bit error probability

I. INTRODUCTION

The requirements for multimedia traffic over wireless communications systems demands an ever-increasing need for bandwidth. Spectrally efficient digital modulation schemes, such as quadrature amplitude modulation (QAM), are attractive techniques to achieve high transmission rates without affecting the bandwidth of those systems.

The performance evaluation of QAM for noisy channels in terms of bit error probability (BEP) has been addressed previously (e.g. [1]–[9]). Although some approximate expressions give accurate error rates for high channel signal-to-noise ratio (SNR), the evaluation of the error rates with those expressions leads to a deviation from the corresponding exact values, when SNR is low. Therefore, the derivation of closed-form expressions for the BEP is a relevant problem.

In a recent paper [10], a convenient method for deriving closed-form expressions for the BEP of modulation schemes subject to Rayleigh fading channel was presented. The method assumes that the Rayleigh fading channel can be seen as an additive noise channel whose additive noise, denoted by \( m(t) \), is modeled as the ratio between a Gaussian random variable (r.v.) and a Rayleigh r.v. The method consists on using the cumulative density function (CDF) of this noise, given by [10]

\[
P_M(m) = \int_{-\infty}^{m} p_M(x)dx = \frac{1}{2} \left( \frac{m}{\sqrt{m^2 + N_0}} + 1 \right),
\]

for deriving the closed-form expressions for the BEP.

In the present paper, this method is firstly used to obtain a closed-form expression for the BEP of M-ary pulse amplitude modulation (M-PAM) for a Rayleigh fading channel. Then, this expression is used to obtain another general result: a closed-form expression for the BEP of rectangular quadrature amplitude modulation (R-QAM) subject to Rayleigh fading.

The remaining of the paper is organized as follows. Section II presents the derivation of the BEP expression of M-PAM and shows numerical and Monte Carlo simulation results. The BEP expression of R-QAM is derived in Section III, which presents numerical and simulation results for some examples of R-QAM schemes. The conclusion of the work is presented in Section IV. In order to maintain the paper self-contained, an appendix with the derivation of the CDF of \( m(t) \) is also provided.

II. BIT ERROR PROBABILITY OF M-PAM

The signal waveforms of M-ary pulse amplitude modulation can be expressed as

\[
s(t) = A_I \cos(2\pi f_c t), \quad 0 \leq t < T,
\]

where \( A_I \) is the signal amplitude of the in-phase components, \( f_c \) is the carrier frequency and \( T \) is the symbol interval. In an M-PAM scheme, a serial data sequence is converted to \( \log_2 M \) bits. In (2), the amplitude \( A_I \) is selected from the set \( \{ \pm d, \pm 3d, \ldots, \pm (M-1)d \} \), where \( 2d \) is the minimum distance between signal points, given by

\[
d = \sqrt{\frac{3 \log_2 M \cdot E_b}{(M^2 - 1)}},
\]

where \( E_b \) is the bit energy. The received PAM signal can be demodulated coherently.

Recently, in [11], a closed-form expression for the BEP of M-PAM under additive white Gaussian noise (AWGN) has been derived. In the following, results presented by Cho and Yoon in [11] are used to obtain a closed-form expression for the BEP of M-PAM subject to Rayleigh fading.
Based on the consistency of the bit mapping of a Gray coded signal constellation [12], Cho and Yoon have derived in [11] an expression for the BEP of square $M$-PAM for an AWGN channel. It is given by

$$P_b = \frac{1}{\log_2 M} \sum_{k=1}^{\log_2 M} P_b(k),$$  \hspace{1cm} (4)

with

$$P_b(k) = \frac{1}{M} \sum_{i=0}^{(1-2^{-k})M-1} \left\{ w(i, k, M) \cdot \text{erfc}\left((2i+1)\sqrt{\frac{3\log_2 M \cdot \gamma}{M^2-1}}\right) \right\},$$  \hspace{1cm} (5)

where

$$w(i, k, M) = (-1)^{\left\lfloor \frac{i-1}{2^k-1} \right\rfloor} \left( 2^{k-1} - \left\lfloor \frac{i \cdot 2^{k-1}}{2^k+1} + \frac{1}{2} \right\rfloor \right),$$  \hspace{1cm} (6)

$\gamma = E_b/N_0$ denotes the signal-to-noise ratio (SNR) per bit, $\lfloor x \rfloor$ denotes the largest integer smaller than $x$, and erfc$(\cdot)$ denotes the complementary error function, given by

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt.$$  \hspace{1cm} (7)

Notice that the BEP of $M$-PAM subject to AWGN is expressed in terms of a weighted sum of complementary error functions. The term erfc$(\cdot)$ in (5) corresponds to twice the probability of the additive Gaussian noise exceeding $(2i+1)\sqrt{\frac{3\log_2 M \cdot E_b}{(M^2-1)}}$. For non-Gaussian additive channels, the weights in (6) (which incorporates the effect on the BEP of the bit positions in a symbol with $\log_2 M$ bits) can be used in conjunction with the cumulative density function (CDF) of the corresponding additive noise for determining the BEP of an M-PAM scheme.

Considering the Rayleigh fading channel, the CDF of the r.v. which models the corresponding additive noise is given by (1). Thus, twice the probability that the one-dimensional additive noise $m(t)$ exceeds $(2i+1)\sqrt{\frac{3\log_2 M \cdot E_b}{(M^2-1)}}$ is given by

$$2 \times P \left( m \geq (2i+1)\sqrt{\frac{3\log_2 M \cdot E_b}{(M^2-1)}} \right) = 2 \times \int_{(2i+1)^2 \log_2 M \cdot E_b}^{\infty} \frac{1}{(M^2-1)^\frac{1}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{m^2}{2(M^2-1)^\frac{1}{2}}} \, dm = \frac{1}{(M^2-1)^\frac{1}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{((2i+1)^2 \log_2 M \cdot E_b)}{(M^2-1)^2 N_0}} \left( 1 - \frac{\sqrt{\frac{3(2i+1)^2 \log_2 M \cdot E_b \cdot N_0}{(M^2-1)^2 N_0}}}{\sqrt{\frac{3(2i+1)^2 \log_2 M \cdot E_b}{(M^2-1)^2 N_0}} + 1} \right).$$  \hspace{1cm} (8)

Using (8) and the weights in (6), the expression for the BEP of $M$-PAM for a channel subject to Rayleigh fading, $P_{M\text{-PAM, Ray}}$, is finally obtained as

$$P_{M\text{-PAM, Ray}} = \frac{1}{\log_2 M} \sum_{k=1}^{\log_2 M} P_{M\text{-QAM, Ray}}(k),$$  \hspace{1cm} (9)

with

$$P_{M\text{-PAM, Ray}}(k) = \frac{1}{M} \sum_{i=0}^{(1-2^{-k})M-1} \left\{ w(i, k, M) \cdot \left( 1 - \frac{\sqrt{\frac{3(2i+1)^2 \log_2 M \cdot \gamma}{M^2-1}}}{\sqrt{\frac{3(2i+1)^2 \log_2 M \cdot \gamma}{M^2-1}} + 1} \right) \right\}.$$  \hspace{1cm} (10)

To the best of authors’ knowledge, this is a new expression for the BEP of $M$-PAM subject to Rayleigh fading.

Some numerical examples obtained from the closed-form expression for the BEP of $M$-PAM are shown in Figure 1, which shows the BEP performance of the $M$-PAM scheme as a function of the signal-to-noise ratio per bit for $M = 2, 4, 8, 16, 32, 64$ and $128$ for a channel subject to to Rayleigh fading. As shown in Figure 1, the Monte Carlo simulation results strongly agree with the results obtained from (9), (10) and (6). It can be seen in Figure 1, for instance, that 3–4 dB of SNR per bit has to be invested to transmit an extra bit, in order to maintain the average bit error probability of $2 \times 10^{-2}$.

III. Bit Error Probability of R-QAM

In an arbitrary $I \times J$ R-QAM scheme, the signal waveforms consist of two independently amplitude-modulated carriers in quadrature, which can be expressed as

$$s(t) = A_I \cos(2\pi f_c t) - A_J \sin(2\pi f_c t), \hspace{1cm} 0 \leq t < T,$$  \hspace{1cm} (11)

where $A_I$ and $A_J$ are the signal amplitudes of in-phase and quadrature components, respectively, $f_c$ is the carrier frequency and $T$ is the symbol interval. In an arbitrary $I \times J$ R-QAM scheme, $\log_2(I \cdot J)$ bits of serial information stream are mapped onto a two-dimensional signal constellation using Gray coding. Among the $\log_2(I \cdot J)$ bits, $\log_2 I$ bits are mapped onto the in-phase channel, the amplitude $A_I$ of which is selected from the set $\{ \pm d_1, \pm 3d_1, \ldots, \pm (I-1)d_1 \}$, where $2d_1$ is the minimum distance between two signals considering their projections in the in-phase axis. Similarly, $\log_2 J$ bits are mapped onto the quadrature channel, the amplitude $A_J$ of which is selected from the set $\{ \pm d_J, \pm 3d_J, \ldots, \pm (J-1)d_J \}$, where $2d_J$ is the minimum distance between two signals considering their projections in the quadrature axis. The demodulation of the received QAM signal is achieved by performing two parallel PAM demodulations.

In this section, results presented in the previous sections for the BEP of $M$-PAM are extended to obtain a new, exact and closed-form expression for the BEP of an arbitrary $I \times J$ R-QAM scheme subject to Rayleigh fading.

From the results presented in Section II, it follows that the bit error rate for the $k$-th bit, $P_{I\text{-PAM, Ray}}(k)$, with $k \in \{ 1, 2, \ldots, \log_2 M \}$ (where $M$ is the number of symbols of the QAM constellation), is given by (10), with the weights given by (6).

Thus, considering an $I \times J$ rectangular QAM, the error probability for the $k$-th bit of the in-phase component and the
error probability for the $l$-th bit of the quadrature component are respectively given by

$$P_l(k) = \frac{1}{I} \sum_{i=0}^{(1-2^{-k})I-1} \left\{ w(i, k, I) \cdot \left( 1 - \frac{\sqrt{\frac{3(2+1)^2 \log_2(I \cdot J) \gamma}{I^2 + J^2 - 2} + 1}}{\sqrt{\frac{3(2+1)^2 \log_2(I \cdot J) \gamma}{I^2 + J^2 - 2} + 1}} \right) \right\}$$  \hspace{1cm} (12)

and

$$P_J(l) = \frac{1}{J} \sum_{j=0}^{(1-2^{-l})J-1} \left\{ w(j, l, J) \cdot \left( 1 - \frac{\sqrt{\frac{3(2+1)^2 \log_2(I \cdot J) \gamma}{I^2 + J^2 - 2} + 1}}{\sqrt{\frac{3(2+1)^2 \log_2(I \cdot J) \gamma}{I^2 + J^2 - 2} + 1}} \right) \right\}$$  \hspace{1cm} (13)

with

$$w(i, k, I) = (-1)^{\lfloor \frac{i \cdot 2^{k-1}}{I} \rfloor} \cdot \left( 2^{k-1} - \left\lfloor \frac{i \cdot 2^{k-1}}{I} + \frac{1}{2} \right\rfloor \right)$$  \hspace{1cm} (14)

and

$$w(j, l, J) = (-1)^{\lfloor \frac{j \cdot 2^{l-1}}{J} \rfloor} \cdot \left( 2^{l-1} - \left\lfloor \frac{j \cdot 2^{l-1}}{J} + \frac{1}{2} \right\rfloor \right).$$  \hspace{1cm} (15)

It is worth mentioning that in an $M$-ary amplitude modulation ($M$-PAM) and rectangular quadrature amplitude modulation (R-QAM) subject to Rayleigh fading. In this work the fading channel is seen as an additive noise channel whose additive noise is modeled as the ratio between Gaussian and Rayleigh random variables.

As future works, the authors will investigate the effects of channel estimation errors as well as multipath scenarios on the bit error probability of modulation schemes subject to Rayleigh fading.

IV. CONCLUSION

This paper presented new, exact and closed-form expressions for the bit error probability (BEP) of $M$-ary pulse amplitude modulation ($M$-PAM) and rectangular quadrature amplitude modulation (R-QAM) subject to Rayleigh fading. In the present paper is more compact than that one presented in [13], which involves hyper-geometric and gamma functions.

Figure 2 presents some numerical results obtained from the closed-form expression for the BEP of R-QAM subject to Rayleigh fading. The figure shows the BEP as a function of the signal-to-noise per bit for the schemes $8 \times 16$ R-QAM and $16 \times 32$ R-QAM. It is observed that the numerical results are corroborated by Monte Carlo simulation results.

APPENDIX

CDF OF THE ADDITIVE NOISE

Consider the wireless system depicted in Figure 3, where the transmitter uses $M$-ary modulation.
Assuming a frequency-nonselective slow fading channel, the received signal \( r_c(t) \) can be expressed as

\[
    r_c(t) = \alpha e^{-j\phi} s(t) + z(t), \quad 0 \leq t \leq T,
\]

where \( s(t) \) represents the transmitted signal, \( \alpha \) is the fading amplitude, \( \phi \) is the phase shift due to the channel, \( z(t) \) denotes the additive white Gaussian noise, and \( T \) is the signaling interval.

The fading amplitude \( \alpha \) is modeled as a Rayleigh r.v., whose probability density function (pdf) is expressed as

\[
    p_A(\alpha) = 2\alpha e^{-\alpha^2} u(\alpha),
\]

where \( u(\cdot) \) is the unit step function. The additive noise \( z(t) \) is modeled as a two-dimensional Gaussian r.v. having zero mean and variance \( N_0/2 \) per dimension. Without loss of generality, a normalized fading power is considered, that is, \( E[\alpha^2] = 1 \), where \( E[\cdot] \) is the expected value operator.

Assuming that the fading is sufficiently slow so that the phase shift \( \phi \) can be estimated from the received signal without error, the receiver can perform the phase compensation (multiplication of \( r_c(t) \) by \( e^{j\phi} \)). Then, the resulting received signal \( r(t) \) can be expressed as

\[
    r(t) = r_c(t) \cdot e^{j\phi} = \alpha s(t) + z(t) \cdot e^{j\phi} = \alpha s(t) + \eta(t).
\]

It is important to note that the additive noise \( \eta(t) = z(t) \cdot e^{j\phi} \) is also a two-dimensional Gaussian r.v. having zero mean and variance \( N_0/2 \) per dimension. This follows from the fact that the error probability is unaffected by a rotation, since \( p_N(\eta) \) is spherically symmetric [14, pp. 247].

The maximum a posteriori criterion [15] establishes that the optimum detector, on observing \( r(t) \), sets \( \hat{s}(t) = s_k(t) \) as the received symbol whenever the decision function

\[
    P(s_i(t)) p_r(r(t) | s_i(t) = s_k(t)), \quad i = 0, 1, \ldots, M - 1,
\]

is maximum for \( i = k \).

Based on the maximum a posteriori criterion and considering equiprobable constellation symbols, the detector can use the following strategy for determining the most probable transmitted symbol from the noisy observation \( r(t) \): compare \( r(t)/\alpha \) with all the constellation symbols and choose as the received symbol the closest one to \( r(t)/\alpha \), that is, choose as the received symbol the one that minimizes the metric \( |r(t)/\alpha - s_k(t)| \).

The decision rule for the detector is

\[
    \hat{s}(t) = \arg \min_{s_i(t)} \left| \frac{r(t)}{\alpha} - s_i(t) \right|.
\]

In this scheme, after providing fading compensation (division of \( r(t) \) by \( \alpha \)), the channel works as an additive noise channel.
because
\[ s(t) = \arg \min_{s_i(t)} \left| \frac{\alpha s(t) + \eta(t)}{\alpha} - s_i(t) \right| \]
\[ = \arg \min_{s_i(t)} |s(t) + m(t) - s_i(t)|, \tag{23} \]
where \( m(t) = \eta(t)/\alpha \) is the additive noise obtained from the ratio between a Gaussian r.v. and a Rayleigh r.v.

Now the pdf and CDF of a random variable obtained from the ratio between a Gaussian r.v. and a Rayleigh r.v. is given by (19).

Under these assumptions, the pdf of \( M \) is given by [16]
\[ p_M(m) = \int_{-\infty}^{\infty} |\alpha|p_{N\alpha}(m\alpha, \alpha) d\alpha, \tag{25} \]
where \( p_{N\alpha}(\eta, \alpha) \) is the joint probability of \( N \) and \( A \), given by
\[ p_{N\alpha}(\eta, \alpha) = \frac{2}{\sqrt{\pi N_0}} \alpha e^{-(\alpha^2 + \eta^2)/N_0} u(\alpha). \tag{26} \]
Thus, the pdf of \( M \) is given by
\[ p_M(m) = \frac{2}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} \alpha^2 e^{-\alpha^2(1+m^2/N_0)} d\alpha. \tag{27} \]
Using the fact that \([17, \text{pp. 1030}]\)
\[ \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \sqrt{\pi}/4, \tag{28} \]
(27) can be expressed as
\[ p_M(m) = \frac{1}{2} \cdot \frac{N_0}{(m^2 + N_0)^{3/2}}. \tag{29} \]
Finally, the CDF of \( M \) can be obtained by integrating the previous expression (using [18, Eq. 2.264]). Thus, the CDF of \( M \) is given by
\[ P_M(m) = \int_{-\infty}^{m} p_M(x) dx = \frac{1}{2} \left( \frac{m}{\sqrt{m^2 + N_0}} + 1 \right). \tag{30} \]

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