Dynamics of a family of two-dimensional difference systems

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Abstract

The boundedness and global asymptotic behavior of positive solutions of the nonlinear two-dimensional difference systems

\[ x_{n+1} = A + \frac{x_n - m}{x_{n-k}}, \quad n \in \mathbb{N}_0, \]

where \( A, r > 0, \ p \geq 0 \) and \( k, m \in \mathbb{N}_0 \), and its numerous special cases had been studied in a large number of papers, e.g., see [1–4,7–9,11,12,14,26–34] and the references therein. Note that in all these papers the equations treated therein are of the form \( x_{n+1} = f(x_{n-k}, x_{n-1}) \), for some \( k, l \in \mathbb{N} \) where the function \( f(x,y) \) is monotone in both variables \( x \) and \( y \). For related max-type difference equations see, e.g. [25,29], and the references therein.

Usually, many specific difference equations are studied before an abstract one is investigated. For example, Camouzis and Papaschinopoulos [6] investigated a specific system which consists of the following two rational difference equations

\[ x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n \in \mathbb{N}_0, \]

and suggested to study the generalized system

\[ x_{n+1} = A + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-m}}, \quad n \in \mathbb{N}_0. \]

For some related papers on systems of difference equations, see, e.g. [5,15–19,21–24]. The following abstract nonlinear difference equation

\[ x_{n+1} = f(x_{n-3}, x_{n-1}), \]

1. Introduction

There has been a long history of interest in difference equations and difference systems, e.g., see [1–37] and the references therein. In particular, a wide variety of nonlinear difference systems have been studied because they model numerous real life problems in biology, physics, population dynamics, economics and so on.

One of the difference equations that has attracted some attention is

\[ x_{n+1} = A + \frac{x_n - m}{x_{n-k}}, \quad n \in \mathbb{N}_0, \]

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where \( s, t \in \mathbb{N}_0 \) satisfying \( s < t \), was studied by Sun and Xi [37], in which they gave some sufficient conditions such that every positive solution of this equation converges to the unique positive equilibrium point.

Furthermore, Sun and Xi [35] investigated the global asymptotic behavior of positive solutions to the following difference system

\[
    x_{n+1} = f(x_n, y_{n-k}), \quad y_{n+1} = f(y_n, x_{n-k}), \quad n \in \mathbb{N}_0,
\]

with positive initial conditions and \( k \in \mathbb{N} \). Certain appropriate assumptions about the function \( f \) in system (2) were given to guarantee that each positive solution to this system converges to a positive equilibrium point.

Recently, Sun and Xi [36] have further studied the generic difference equation system

\[
    x_{n+1} = f(y_{n-p} x_{n-k}), \quad y_{n+1} = g(x_{n-t}, y_{n-p}), \quad n \in \mathbb{N}_0,
\]

where \( p, q, s, t \) are all nonnegative integers satisfying \( s > t \) and \( p > q \), and the initial values are positive, by posing some sufficient conditions such that the unique positive equilibrium is a global attractor.

Inspired by the above works, we study in this paper the dynamics of the following class of difference systems

\[
    \text{(A1)}: \quad \text{Let } I = (a, +\infty) \text{ or } (a, +\infty), \text{ where } a \geq 0, \text{ and } \Phi, \Psi : I^2 \to I \text{ are continuous with } \gamma = \inf_{x, y \in I^2} \Phi(x, y) = \inf_{x, y \in I^2} \Psi(x, y). \text{ Furthermore, all the initial values of system (3) } \begin{array}{c}
        x_0 = \Phi(x_{n-1}, y_{n-1}), \\
        y_0 = \Psi(y_{n-2}, x_{n-2}) \end{array} \text{, } n \in \mathbb{N}_0, \text{ where } \begin{array}{c}
        x_1, s_2, t_2 \text{ are all positive integers and both } \Phi(x, y) \text{ and } \Psi(x, y) \text{ are decreasing in } x \text{ and increasing in } y. \text{ For } s_1, s_2, t_1, t_2 \text{ in system (3), let } t = \max(t_1, t_2) \text{ and } s = \max(s_1, s_2) \text{ in the sequel.}
\end{array}
\]

As far as we know, system (3) has not yet been studied. Through careful analysis we have found that all positive solutions of system (3) converge to the unique equilibrium under some assumptions. These conditions are different from those given in [36].

2. Main results

In this section, we first present six basic assumptions about the mappings \( \Phi \) and \( \Psi \) in system (3), which are necessary for the main result at the end of this section. For the two mappings in system (3), we assume

\[ \text{(A1): Let } I = (a, +\infty) \text{ or } (a, +\infty), \text{ where } a \geq 0, \text{ and } \Phi, \Psi : I^2 \to I \text{ are continuous with } \gamma = \inf_{x, y \in I^2} \Phi(x, y) = \inf_{x, y \in I^2} \Psi(x, y). \]

\[ \text{(A2): There exists exactly one fixed point } \lambda \in (\gamma, +\infty) \text{ such that } \Phi(\gamma, \lambda) = \Psi(\lambda, \lambda) = \lambda; \]

\[ \text{(A3): The system } x = \Phi(x, y), \quad y = \Psi(y,x) \text{ has a unique solution } (\bar{x}, \bar{y}) \text{ with } \bar{x} \in [\Phi(\lambda, \gamma), \lambda], \quad \bar{y} \in [\Psi(\lambda, \gamma), \lambda]; \]

\[ \text{(A4): Both the mappings } \Phi(x, y) \text{ and } \Psi(x, y) \text{ are strictly decreasing in } x \text{ and strictly increasing in } y \text{ for each } x \in I; \]

\[ \text{(A5): Both } \Phi(\gamma, x)/x \text{ and } \Psi(\gamma, x)/x \text{ are nonincreasing on the interval } (\gamma, +\infty); \]

\[ \text{(A6): The system } M = \Phi(m,T), \quad m = \Phi(M,t), \quad T = \Psi(t,M), \quad t = \Psi(T,m) \text{ with } M, m, t \in [\Phi(\lambda, \gamma), \lambda], \quad T, m \in [\Psi(\lambda, \gamma), \lambda] \text{ has a unique solution with } M = m \text{ and } T = t. \]

**Lemma 2.1.** Assume that there exist two bounded sequences \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) such that \( a_n = \Phi(a_{n-1}, b_{n-1}), \quad b_n = \Psi(b_{n-2}, a_{n-2}), \quad n \in \mathbb{N}_0 \), where both \( \Phi \) and \( \Psi \) satisfy (A1), (A4). Then

\[
\begin{align*}
\limsup_{n \to \infty} a_n & \leq \Phi(\liminf_{n \to \infty} a_{n-1}, \limsup_{n \to \infty} b_{n-1}), \\
\liminf_{n \to \infty} a_n & \geq \Phi(\limsup_{n \to \infty} a_{n-1}, \liminf_{n \to \infty} b_{n-1}), \\
\limsup_{n \to \infty} b_n & \leq \Psi(\liminf_{n \to \infty} b_{n-2}, \limsup_{n \to \infty} a_{n-2}), \\
\liminf_{n \to \infty} b_n & \geq \Psi(\limsup_{n \to \infty} b_{n-2}, \liminf_{n \to \infty} a_{n-2}).
\end{align*}
\]

**Proof.** The proof is simple and known but is given here for the completeness. Since the sequences \( \{a_n\} \) and \( \{b_n\} \) are bounded, denote

\[
\begin{align*}
\alpha_{1,k} &= \inf_{n \geq k} \{a_n\} = \inf_{n \geq k} \{a_n, a_{k+1}, \ldots\}, \quad \alpha_{2,k} = \inf_{n \geq k} \{b_n\} = \inf_{n \geq k} \{b_n, b_{k+1}, \ldots\}, \\
\beta_{1,k} &= \sup_{n \geq k} \{a_n\} = \sup_{n \geq k} \{a_n, a_{k+1}, \ldots\}, \quad \beta_{2,k} = \sup_{n \geq k} \{b_n\} = \sup_{n \geq k} \{b_n, b_{k+1}, \ldots\}.
\end{align*}
\]

Then by (5) and the assumption (A4), we have that for \( k \geq \max(s_1, t_1), \)

\[
\beta_{1,k} = \sup_{n \geq k} \{\Phi(a_{n-t_1}, b_{n-s_1})\} \leq \Phi\left(\inf_{n \geq k} \{a_{n-t_1}\}, \sup_{n \geq k} \{b_{n-s_1}\}\right) = \Phi(\alpha_{1,k-t_1}, \beta_{2,k-s_1}),
\]

where \( s, t \in \mathbb{N}_0 \) satisfying \( s < t \), was studied by Sun and Xi [37], in which they gave some sufficient conditions such that every positive solution of this equation converges to the unique positive equilibrium point.
\[ x_{1,k} = \inf_{n,k} \{ \Phi(\alpha_{n-t_1}, b_{n-s_2}) \} \geq \Phi \left( \sup_{n,k} \{ \alpha_{n-t_1} \}, \inf_{n,k} \{ b_{n-s_2} \} \right) = \Phi(\beta_{1,k-t_1}, \alpha_{2,k-t_2}). \]

Therefore by taking limits on both sides of the above two inequalities and in light of the continuity of the function \( \Phi \), we have

\[ \limsup_{n \to \infty} \limsup_{k \to \infty} x_{1,k} \leq \Phi \left( \limsup_{k \to \infty} \alpha_{1,k-t_1}, \limsup_{k \to \infty} \alpha_{2,k-t_2} \right) = \Phi \left( \liminf_{n \to \infty} \alpha_{n-t_1}, \limsup_{n \to \infty} b_{n-s_2} \right). \]

\[ \liminf_{n \to \infty} \liminf_{k \to \infty} x_{1,k} \geq \Phi \left( \liminf_{k \to \infty} \alpha_{1,k-t_1}, \liminf_{k \to \infty} \alpha_{2,k-t_2} \right) = \Phi \left( \limsup_{n \to \infty} \alpha_{n-t_1}, \liminf_{n \to \infty} b_{n-s_2} \right). \]

The proof of the last two inequalities of (4) is similar and thus omitted. \( \square \)

**Lemma 2.2.** Under the assumptions (A1), (A2), (A4) and (A5), the following two statements hold:

1. \( x_n \leq \hat{\lambda} \) if \( y_{n-s_2} \leq \lambda \), while \( x_n \leq y_{n-s_2} \) if \( y_{n-s_2} > \lambda \), \( n \geq \max\{s_1, t_1\} \);
2. \( y_n \leq \hat{\lambda} \) if \( x_{n-t_2} \leq \lambda \), while \( y_n \leq x_{n-t_2} \) if \( x_{n-t_2} > \lambda \), \( n \geq \max\{s_2, t_2\} \).

**Proof.** Observe that \( x_n = \Phi(x_{n-t_1}, y_{n-s_2}) \leq \Phi(\gamma, x_{n-t_2}) \), \( n \geq \max\{s_1, t_1\} \). By the assumptions (A2), (A4) and (A5), we have that if \( y_{n-s_2} \leq \lambda \), then

\[ x_n \leq \Phi(\gamma, y_{n-s_2}) \leq \Phi(\gamma, \lambda) = \hat{\lambda}, \]

while if \( y_{n-s_2} > \lambda \) then

\[ \frac{\Phi(\gamma, y_{n-s_2})}{y_{n-s_2}} \leq \frac{\Phi(\gamma, \lambda)}{\lambda} = 1, \]

which implies \( x_n \leq \Phi(\gamma, y_{n-s_2}) \leq y_{n-s_2} \). Likewise, we get

\[ y_n = \Psi(y_{n-s_2}, x_{n-t_2}) \leq \Psi(\gamma, x_{n-t_2}) \], \( n \geq \max\{s_2, t_2\} \).

From this inequality, we know that if \( x_{n-t_2} \leq \lambda \), then

\[ y_n = \Psi(y_{n-s_2}, x_{n-t_2}) \leq \Psi(\gamma, \lambda) = \hat{\lambda}, \]

while if \( x_{n-t_2} > \lambda \) then

\[ \frac{\Psi(\gamma, x_{n-t_2})}{x_{n-t_2}} \leq \frac{\Psi(\gamma, \lambda)}{\lambda} = 1, \]

which indicates that \( y_n \leq \Psi(\gamma, x_{n-t_2}) \leq x_{n-t_2} \). \( \square \)

**Lemma 2.3.** For each \( j \in \{0, 1, \ldots, s_1 + t_2 - 1\} \), there exists \( N_j \in \mathbb{N} \) such that \( x_{n(t_2+s_1)+j} \leq \hat{\lambda} \) and \( y_{n(t_2+s_1)+t_2+j} \leq \lambda \) for all \( n \geq N_j \).

**Proof.** Suppose that for some \( j \in \{0, 1, \ldots, s_1 + t_2 - 1\} \) there exists \( w_j \in \mathbb{N}_0 \) such that \( x_{w_j(t_2+s_1)+j} \leq \hat{\lambda} \). By using both statements in **Lemma 2.2** we have that

\[ x_{w_j(t_2+s_1)+j} \leq \hat{\lambda} \text{ for all } n \geq w_j \]

and

\[ y_{n(t_2+s_1)+t_2+j} \leq \hat{\lambda} \text{ for all } n \geq w_j. \]

Likewise, if for some \( j \in \{0, 1, \ldots, t_2 + s_1 - 1\} \) there exists \( z_j \in \mathbb{N}_0 \) such that \( y_{z_j(t_2+s_1)+j} \leq \lambda \). Then recursively we get that for all \( n \geq z_j \) there hold

\[ y_{n(t_2+s_1)+j} \leq \lambda \text{ and } x_{n(t_2+s_1)+s_2+j} \leq \lambda. \]

Therefore, on the contrary assume that there exists some \( j \in \{0, 1, \ldots, s_1 + t_2 - 1\} \) such that **Lemma 2.3** does not hold. Then applying the above two conclusions, we can simply get that

\[ x_{n(t_2+s_1)+j} > \hat{\lambda} \text{ and } y_{n(t_2+s_1)+t_2+j} > \lambda \text{ for all } n \in \mathbb{N}_0. \]

Using **Lemma 2.2**, we have that for all \( n \in \mathbb{N}_0 \),

\[ \lambda < x_{(n+1)(t_2+s_1)+j} \leq x_{n(t_2+s_1)+j} \text{ and } \lambda < y_{(n+1)(t_2+s_1)+t_2+j} \leq y_{n(t_2+s_1)+t_2+j}. \]
Denote
\[ \lim_{n \to \infty} x_{n(t_2 + s_1) + j} = \lambda j \quad \text{and} \quad \lim_{n \to \infty} y_{n(t_2 + s_1) + t_2 + j} = \gamma j, \]
then \( \lambda j \geq \lambda \) and \( \gamma j \geq \lambda \). By Lemma 2.2 we know \( \{x_n\}, \{y_n\} \) are bounded. Let
\[ \phi_j = \liminf_{n \to \infty} x_{n(t_2 + s_1) + t_2 + j}, \quad \mu_j = \liminf_{n \to \infty} y_{n(t_2 + s_1) + t_2 + j}, \]
onobviously we have \( \phi_j \geq \gamma, \mu_j \geq \gamma \). By system (3), we have that
\[ x_{n(t_2 + s_1) + j} = \Phi(x_{n(t_2 + s_1) + t_2 + j}, y_{n(t_2 + s_1) + t_2 + j}), \]
\[ y_{n(t_2 + s_1) + t_2 + j} = \Psi(y_{n(t_2 + s_1) + t_2 + j}, x_{n(t_2 + s_1) + j}). \]
Employing Lemma 2.1, it follows from (6) that
\[ \lambda j \leq \Phi(\phi_j, \gamma) \leq \Phi(\gamma, \gamma) = \gamma j \Phi(\gamma, \gamma) / \lambda \leq \gamma j, \]
\[ \gamma j \leq \Psi(\lambda, \gamma) \leq \Psi(\gamma, \gamma) / \lambda \leq \gamma j. \]
Thus \( \lambda j = \gamma j \), which plus the assumptions (A2) and (A4) leads to \( \lambda j = \gamma j = \lambda \), \( \phi_j = \mu_j = \gamma \).
Let \( \lambda_j = \limsup_{n \to \infty} n(x_{n(t_2 + s_1) + j} - t_2 + s_1) \) and \( B_j = \liminf_{n \to \infty} y_{n(t_2 + s_1) + j - t_2 - s_1} \), then \( \lambda J, B_j \geq \gamma \). Again by system (3) we get
\[ x_{n(t_2 + s_1) + j - t_2 - s_1} = \Phi(x_{n(t_2 + s_1) + j - 2t_2}, y_{n(t_2 + s_1) + j - t_2 - s_1}), \]
It follows by Lemma 2.1 and (7) that \( \gamma \geq \Phi(\lambda J, B_j) > \Phi(\lambda, B_j) \geq \gamma \), which is obviously a contradiction. The proof is complete. \( \square \)

The following lemma follows directly from Lemma 2.3 and shows the explicit bounds of positive solution to system (3) for sufficiently large \( n \).

Lemma 2.4. Suppose (A1)-(A5) hold for system (3), then there exists \( N \in \mathbb{N} \) such that \( \Phi(\lambda, \gamma) \leq x_n \leq \lambda \) and \( \Psi(\lambda, \gamma) \leq y_n \leq \lambda \), \( n > N \).

Proof. According to Lemma 2.3, let
\[ N = \max \{N_{\lambda j} : (s_1 + t_2) + \max_{i=1,2} |s_i| \}, \]
then \( x_{n-s_t}, y_{n-s_t} \leq \lambda, i = 1,2 \), for all \( n > N \). It follows by system (3) and (A4) that \( x_n = \Phi(x_{n-s_t}, y_{n-s_t}) \geq \Phi(\lambda, \gamma), y_n = \Psi(y_{n-s_t}, x_{n-s_t}) \geq \Psi(\lambda, \gamma) \), for all \( n > N \). The proof is complete. \( \square \)

Theorem 2.1. If (A1)-(A6) hold for system (3), then every positive solution \( \{x_n, y_n\} \) to system (3) converges to the unique equilibrium \( (\bar{x}, \bar{y}) \).

Proof. Let \( t = \max\{t_1, t_2\}, s = \max\{s_1, s_2\} \), and define four sequences \( \{n_j\}_{j=1}^{\infty}, \{M_i\}_{i=1}^{\infty}, \{k_i\}_{i=1}^{\infty}, \{T_i\}_{i=1}^{\infty} \) recursively in the following way:
\[ M_i = \Phi(M_{i-1}, T_{i-1}), \quad M_i = \Phi(M_{i-1}, T_{i-1}), \]
\[ T_i = \Psi(T_{i-1}, M_{i-1}), \quad t_i = \Psi(T_{i-1}, M_{i-1}), \]
where \( i \in \mathbb{N}_0 \) and the initial values satisfy
\[ m_i = \Phi(\lambda, \gamma), \quad M_j = \lambda, \quad j \in \{0, -1, -2, \ldots\}, \]
\[ t_j = \Psi(\lambda, \gamma), \quad T_j = \lambda, \quad j \in \{0, -1, -2, \ldots\}. \]

Obviously we have that \( m_{k-1} \leq m_k \leq M_j \leq M_{j+1} \), \( j = \{-t + 1, \ldots, -1\} \) and \( t_{j-1} \leq t_j \leq T_{j+1}, j \in \{-s + 1, \ldots, -1\} \). Assume that \( m_{k-1} \leq m_k \leq M_j \leq M_{j+1} \) hold for \( -t + 1 \leq j \leq k \), and \( t_{j-1} \leq t_j \leq T_{j+1}, \) \( j \leq M_j \), \( \Psi(\lambda, \gamma) \leq T_j \leq T_{j+1} \) hold for \( -s + 1 \leq j \leq k \). Then employing (A4), for \( k = j + 1 \) we obtain that
\[ m_{k+1} = \Phi(M_{k-1} + T_{k-1}), \quad M_{k+1} = \Phi(M_{k-1} + T_{k-1}), \]
\[ T_{k+1} = \Psi(T_{k-1} + M_{k-1}), \quad t_{k+1} = \Psi(T_{k-1} + M_{k-1}). \]
Working inductively, we have that \( \{m_i\}_{i=1}^{\infty} \) and \( \{t_i\}_{i=1}^{\infty} \) are nondecreasing and bounded from above by \( \lambda \), and \( \{M_i\}_{i=1}^{\infty} \) and \( \{T_i\}_{i=1}^{\infty} \) are nonincreasing and bounded from below by \( \Phi(\lambda, \gamma), \Psi(\lambda, \gamma) \), respectively.
Hence \( \{m_i\}, \{M_i\}, \{t_i\}, \{T_i\} \) are convergent, and denote

\[
m = \lim_{i \to \infty} m_i, \quad M = \lim_{i \to \infty} M_i, \quad t = \lim_{i \to \infty} t_i, \quad T = \lim_{i \to \infty} T_i.
\]

Then \( m, M \in [\Phi(\lambda, \gamma), \lambda] \) and \( t, T \in [\Psi(\lambda, \gamma), \lambda] \). By taking limits on both sides of (8) and in view of the continuity of \( \Phi \) and \( \Psi \), we have that

\[
M = \Phi(m, T), \quad m = \Phi(M, t), \quad T = \Psi(t, M), \quad t = \Psi(T, m).
\]

It follows from (A3) and (A6) that \( m = M = x, t = T = y \).

Let \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) be a solution to system (3), then by Lemma 2.4 there exists \( K \in \mathbb{N} \) such that

\[
\Phi(\lambda, \gamma) \leq x_j \leq \lambda \quad \text{for all } j \geq K - t
\]

and

\[
\Psi(\lambda, \gamma) \leq y_j \leq \lambda \quad \text{for all } j \geq K - s.
\]

By (8) and (9), we have

\[
m_j - K \leq x_j \leq M_j - K, \quad K - t \leq j \leq K - 1
\]

and

\[
t_j - K \leq y_j \leq T_j - K, \quad K - s \leq j \leq K - 1.
\]

If \( m_j - K \leq x_j \leq M_j - K \) holds for \( K - t \leq j \leq n - 1 \) and \( t_j - K \leq y_j \leq T_j - K \) holds for \( K - s \leq j \leq n - 1, n \geq K \), then observe that

\[
x_n = \Phi(x_{n-1}, y_{n-1}) \geq \Phi(M_{n-K-1}, T_{n-K-1}) = M_{n-K},
\]

\[
x_n = \Phi(x_{n-1}, y_{n-1}) \leq \Phi(m_{n-K-1}, T_{n-K-1}) = M_{n-K},
\]

\[
y_n = \Psi(y_{n-1}, x_{n-1}) \leq \Psi(T_{n-K-2}, M_{n-K-2}) = T_{n-K},
\]

\[
y_n = \Psi(y_{n-1}, x_{n-1}) \leq \Psi(T_{n-K-2}, M_{n-K-2}) = T_{n-K}.
\]

By induction, we derive \( m_{n-K} \leq x_n \leq M_{n-K} \) and \( t_{n-K} \leq y_n \leq T_{n-K} \), for all \( n \geq K \). This plus \( m = M = x, t = T = y \) yields \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \). The proof is complete. \( \Box \)

3. Examples

In this section, we will discuss several specific examples to verify the correctness of the main result given in Section 2.

**Example 3.1.** Consider the following difference equation system

\[
x_n = p + \frac{y_{n-s}}{x_{n-t}}, \quad y_n = p + \frac{x_{n-t}}{y_{n-t}}, \quad n \in \mathbb{N}_0,
\]

where \( s_1, t_1, s_2, t_2 \in \mathbb{N}, p > 1 \) and the initial values \( x_i, y_j > 0 \), for \( i = -t, \ldots, -1, j = -s, \ldots, -1 \), where \( t = \max\{t_1, t_2\}, s = \max\{s_1, s_2\} \).

**Proof.** Define \( \Phi(x, y) = \Psi(x, y) = p + y/x \) with \( x, y \in \mathbb{R}^+ \). It can be seen that \( \Phi, \Psi \) satisfy assumptions (A1)–(A5) with \( \gamma = p, \lambda = p^2/(p - 1), x = y = p + 1 \).

Hence it suffices to verify the following system

\[
M = p + T/m, \quad m = p + t/M, \quad T = p + M/t, \quad t = p + m/T
\]

has a unique solution with \( m = M \). \( t = T \). It follows from (11) that

\[
p(M - m) = T - t \quad \text{and} \quad p(T - t) = M - m,
\]

which implies \( T = t, M = m \) for \( p > 1 \). Thus (A6) holds for \( \Phi, \Psi \).

In view of **Theorem 2.1**, for every positive solution \( (x_n, y_n) \) to system (10) we conclude \( \lim_{n \to \infty} x_n = x = p + 1, \lim_{n \to \infty} y_n = y = p + 1 \). \( \Box \)

**Example 3.2.** Consider the following difference equation system

\[
x_n = \frac{py_{n-s} + x_{n-t}}{q + x_{n-t}}, \quad y_n = \frac{px_{n-t} + y_{n-s}}{q + y_{n-s}},
\]

where \( s_1, s_2, t_1, t_2 \in \mathbb{N}, 0 < q < p < q + 1 \) with \( p + q > 1 \) and initial values \( x_i, y_j \in (q/p, +\infty), \) for \( i = -t, \ldots, -1, j = -s, \ldots, -1 \), where \( t = \max\{t_1, t_2\}, s = \max\{s_1, s_2\} \).
Example 3.3. Consider the system
\[ x_n = p + qy_{n-1} + \frac{A}{x_{n-1}}, \quad y_n = p + qx_{n-2} + \frac{A}{y_{n-2}}, \quad n \in \mathbb{N}_0, \]
where all the initial values are positive, and \( p, A > 0, 0 < q < 1 \) satisfy the inequality
\[ p(1 + q + A(1 - q)/(A + p^2)) > \sqrt{A/(1 - q)}. \] (14)

Proof. The equation \( \Phi(x, y) = \Psi(x, y) = (py + x)/(q + x) \) with \( x, y \in (q/p, +\infty) \). It is straightforward to verify that (A1)–(A5) hold for \( \Phi, \Psi \) with \( \gamma = 1, \lambda = 1/(q - p + 1) \). Let \( x = p - q + 1 \). Easily, it follows from the system
\[
M = \frac{pT + m}{q + m}, \quad m = \frac{pt + M}{q + M}, \quad T = \frac{pM + t}{q + M}, \quad t = \frac{pm + T}{q + M},
\]
that \( (q + 1)(M - m) = p(T - t) \) and \( (q + 1)(T - t) = p(M - m) \).

So we get \((q + 1)^2(T - t) = p^2(M - m)\) which implies \( T = t \), and then \( M = m \). Hence (A6) holds for \( \Phi, \Psi \). In light of Theorem 2.1, for every positive solution \((x_n, y_n)\) of system (12) we have \( \lim_{n \to \infty} x_n = x = p - q + 1 \), \( \lim_{n \to \infty} y_n = y = p - q + 1 \). \( \square \)

Example 3.4. Consider the following system (from Riccati equation)
\[ x_n = \frac{p_1 + y_{n-1}}{q_1 + x_{n-1}}, \quad y_n = \frac{p_2 + x_{n-2}}{q_2 + y_{n-2}}, \quad n \in \mathbb{N}_0, \]
with nonnegative initial values, and \( p_1, p_2 > 0, q_1, q_2 > 1 \) satisfying
\[
p_1q_2 + p_2 = p_2q_1 + p_1, \quad q_1^2 - 2q_2 + q_1 \geq 0,
\]
\[
q_1q_2 - 2p_2q_2 - 1 \geq 0, \quad p_2^2 - q_1p_2 - p_1 < 0.
\] (17)

Proof. Define two mappings \( \Phi(x, y) = (p_1 + y)/(q_1 + x) \) and \( \Psi(x, y) = (p_2 + y)/(q_2 + x), x, y \in [0, +\infty) \). It is easy to verify that (A1),(A2) and (A4)–(A6) hold for \( \Phi, \Psi \) with \( \gamma = 0, \lambda = p_1/(q_1 - 1) = p_2/(q_2 - 1) \). Hence it suffices to show that the system
\[ x^2 + q_1x = p_1 + y, \quad y^2 + q_2y = p_2 + x \]
(18)
has a unique positive solution \((x, y)\). It follows from system (18) that
\[ y^4 + 2q_1y^3 + (q_1 + q_2 - 2p_2)y^2 + (q_1q_2 - 2p_2q_2 - 1)y = q_1p_2 + p_1 - p_2^2. \]
Let \( H(t) = t^4 + 2q_1t^3 + (q_1 + q_2^2 - 2p_2)t^2 + (q_1q_2 - 2p_2q_2 - 1)t + p_2^2 - q_1p_2 - p_1 \), then
\[ H(t) = 4t^4 + 6q_1t^3 + 2(q_1 + q_2^2 - 2p_2)t^2 + q_1q_2 - 2p_2q_2 - 1, \text{ and } H'(t) = 12t^3 + 12q_1t^2 + 2(q_1 + q_2^2 - 2p_2). \]

We conclude by (17) that \((x, y)\) has only one positive root, denoted by \( \bar{y} \). Then by substituting \( y = \bar{y} \) into (18) we get \( \bar{x} = \left(-q_1 + \sqrt{q_1^2 + 4p_1 + 4\bar{y}}\right)/2 \). It follows from Theorem 2.1 that every positive solution to system (16) converges to the unique positive equilibrium \((\bar{x}, \bar{y})\). \( \square \)

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