A NEW APPROACH IN THE SEARCH FOR PERIODIC ORBITS

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In this paper we present a new approach for finding periodic orbits in dynamical systems modeled by differential equations. It is based on the Homotopy Analysis Method (HAM) but it differs from the usual way it is applied. We apply the HAM to construct approximations of a formal series solution of the equation. These approximations exist for any value of the frequency. They can be obtained by choosing a suitable linear operator and allowing the initial conditions to vary freely. Then we study the behavior of the obtained expressions as a function of the frequency. This procedure allows us to find those values of the frequency for which the series converges and therefore to find the periodic orbits. We show several examples of application of the proposed method. Mathematica and MATCONT were applied for all the calculations.

Keywords: Homotopy Analysis Method – Hopf Bifurcation – Bautin Bifurcation – Ordinary Differential Equations

1. Introduction

The study of periodic orbits in dynamical systems has been of continuing importance since its beginning. This is true from theoretical and practical standpoint. Just by mentioning the index theory, the Poincaré-Bendixon theorem or famous sixteenth Hilbert problem to appreciate the enormous impact of this issue in the field of mathematics and its applications ([Guckenheimer & Holmes, 1983; Kuznetsov, 1995; Strogatz, 1994]).

The need to find explicit expressions for the orbits has motivated the development of several methods. Among them, the Homotopy Analysis Method of Liao [Liao, 2004a,b] has been applied to find periodic orbits in various situations. For example to approximate the limit cycle of the van der Pol equation or more general Liénard equations [Abbasbandy et al., 2011a; Chen & Liu, 2009], for wave solutions of mKdV equation [Wang et al., 2004] or cycles around a center [Bel et al., 2012; Turkyilmazoglu, 2011]. In all these cases, in the spirit of the Poincaré-Lindstedt method [Kevorkian & Cole, 1968], the successive terms of the solution are obtained by imposing conditions to ensure the vanishing of the secular terms in the solutions of the chosen linear operator. After obtaining the desired approximation, it is studied as a function of the
parameter $h$ of the method, this is the hallmark of the HAM. This procedure is summarized in [Bel et al., 2012].

This work is also based on the HAM, but we have developed a different approach, first we find a finite approximation of a formal series solution which depends on two parameters, $\omega$ and $h$. Then we study the obtained expression in order to find the values of these parameters, if any, for which the series converges. Then evaluating the expression at those parameter values we finally have the solution.

2. Description of the Method

This section describes the method as used in this work. An overview of the HAM can be found in the book of Liao [Liao, 2004a].

Consider the differential equation

$$x' = f(x, s),$$

with $x(s) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ a sufficiently differentiable nonlinear function. Suppose that the system has a periodic solution of frequency $\omega$. Then after replacing $t = \omega s$ and $y(t) = x(t/\omega)$ we obtain the equation

$$\omega y' = f(y, t/\omega),$$

for which we seek a solution with unit frequency. We write the previous equation as $N_\omega[y] = 0$, where the nonlinear operator $N_\omega$ depends algebraically on $\omega$.

Then consider the homotopy given by the family of operators $H_q$ depending on a deformation parameter $q \in [0, 1]$

$$H_q[\phi] = (1 - q) \mathcal{L}[\phi - y_0] - q h N_\omega[\phi],$$

where $h \neq 0$ is a real parameter, $y_0(t)$ is an initial approximation, and $\mathcal{L}$ is the linear operator

$$\mathcal{L}[\phi] = \frac{\partial \phi}{\partial t} + \phi.$$

The procedure is based on the search of a function $\phi(t, q, \omega, h)$, analytical in $q$ and periodic in $t$ such that $H_q[\phi] = 0$ for $q \in [0, 1]$. Then $y_P(t, \omega, h) = \phi(t, 1, \omega, h)$ is a solution of the system (2).

To find the function $\phi(t, q, \omega, h)$, we consider its series expansion

$$\phi(t, q, \omega, h) = \sum_{k=0}^{+\infty} y_k(t, \omega, h)q^k.$$

After replacing this series in $H_q[\phi] = 0$, and taking the $k$-th derivative with respect to $q$ evaluated at $q = 0$, it yields the following set of equations

$$\mathcal{L}[y_k - (1 - \delta_{1k})y_{k-1}] = \frac{h}{(k-1)!} \frac{\partial^{k-1} N_\omega[\phi]}{\partial q^{k-1}} \bigg|_{q=0}, \quad k = 1, \ldots,$$

where $\delta$ is the Kronecker delta.

The operator $\mathcal{L}$ has the following property. For any periodic function $p$ of unit frequency, the equation $\mathcal{L} \psi = p$ always has a periodic solution of the same frequency. Then for any $\omega$, applying the last equations, we have successive terms of an approximation of a periodic formal series solution of the system.

Here we want to note that the implemented method is aimed at finding periodic orbits around the origin. In general, we want to find them around any fixed equilibrium. In case the equilibrium is not the origin then we simply make a change of coordinates to center it.

Another point to emphasize is that any solution gives a continuum of solutions by varying the phase in a period (see for example [Baker et al., 2005]). Therefore, we must proceed to an arbitrary choice of the origin in the variable $t$. To make this choice we take a particular initial cycle. It seems logical to choose the most elementary cycle, i.e. a circle. For two-dimensional systems we take a circle that cut the positive axis of the first variable at $t = 0$. 

It remains to determine the amplitude of this initial cycle. This is a delicate point. The closer we put this amplitude to the amplitude of the periodic cycle, the faster the convergence of the method. Moreover, in the case of more than one cycle then different initial amplitudes lead to different cycles. Different cycles seem to have a different basin of attraction and depending on the initial amplitude the solution can fall into any of them.

In the case of the search of limit cycles we make a set of iterations in the initial condition in order to optimize the processing time and memory usage. It proceeds as follows, first we choose an initial amplitude, then we obtain the first terms of the solution. We apply the criteria explained in the following subsection, thus obtaining a preliminary result. If the amplitude of the obtained cycle differs significantly from the initial one, we make another run changing the initial amplitude. We proceed in this way until the difference between two iterations does not vary more than a predetermined amount. The order of HAM reported corresponds to the last iteration.

While we have not made a detailed study of this iterative process, we noticed that to get a result after a few cycles of iterations is more efficient than trying to run the method once with a very high order, which is not always possible without extreme memory usage.

We note that the finite approximation for the cycle that is obtained with the formula (6) is a truncated formal Fourier series. Its coefficients, in the autonomous case, depend polynomially on $h$ and $\omega$.

The following subsection describes how we can find, when available, an approximation to the true solution.

### 2.1. Heuristic search of solutions

As stated above we have obtained a finite approximation of $y_P(t, \omega, h)$. We call $\tilde{y}(\omega, h)$ to this expression evaluated at a fixed time.

Then we choose a rectangle in the $h$-$\omega$ plane. In the direction of $\omega$ we take an interval in which we suppose that the frequency is. In the $h$ direction we take the interval $[-1, 0]$.

If a periodic solution of (1) exists with frequency $\omega_0$ then replacing this value in (2) the HAM applied in the usual manner would proceed to find the solution and the convergence of the same can be adjusted with the parameter $h$.

The approximations of $y(t)$ and its derivatives for fixed $t$ are polynomials in $h$ that are called $h$-curves. As shown in Liao’s book [Liao, 2004a] the observation of the behavior of the $h$-curves allows us to select an appropriate value for $h$, see also [Abbasbandy et al., 2011b]. At values of $h$ for which the series converges, the $h$-curves tend, when the order goes to infinity, to a value independent of $h$. Thus, plotting these functions we can have a rough idea of where these convergence regions are, and to select an appropriate value for $h$. In Figures 1, 4 and 12 various $h$-curves are shown that permit us to distinguish these regions. The interval $[-1, 0]$ was chosen because that is where typically the $h$-curves have their approximately constant region.

We transformed these observations in a procedure that allows us to decide on the presence of periodic orbits.

We calculate the function $m(\omega, h)$. It gives the average, in an interval of length $2\Delta$, of the values of $\tilde{y}(\omega, h)$ in the direction of $h$. It is given by

$$m(\omega, h) = \frac{1}{2\Delta} \int_{h-\Delta}^{h+\Delta} \tilde{y}(\omega, \eta) \, d\eta.$$  \hspace{1cm} (7)

We have found that the value $\Delta = 0.1$ is suitable in most considered cases. Only for the anharmonic oscillator we change this value to $\Delta = 0.05$.

Then, we calculate the quadratic error by the following formula

$$\rho(\omega, h) = \frac{1}{2\Delta} \int_{h-\Delta}^{h+\Delta} (\tilde{y}(\omega, \eta) - m(\omega, \eta))^2 \, d\eta.$$  \hspace{1cm} (8)

Finally we seek the minimum of $\rho$. We call $\omega_{\text{min}}$ and $h_{\text{min}}$ to the coordinates of the minimum. Next we discuss if it corresponds to a periodic orbit. We studied the following
• The curve $\rho$ as a function of $\omega$ for the fixed value $h_{\text{min}}$. This curve should have a sharp peak for a single value of $\omega$. That is, that there should be a drop of several orders of magnitude in the error against small variations in $\omega$. The value at the minimum must approach 0.

• The surface $\tilde{y}(\omega, h)$, which we call $h-\omega$-surface together with the horizontal plane with value $\tilde{y}(\omega_{\text{min}}, h_{\text{min}})$. The intersection of the two surfaces must contain a horizontal segment approximately parallel to the $h$ axis.

• The $h$–curves $\tilde{y}(\omega_{\text{min}}, h)$. This is an explicit two-dimensional representation of the previous point. The $h$–curve must show typical aspect always seen in the HAM.

All these calculations can be easily made with Mathematica.

3. Results and discussion

3.1. Anharmonic Oscillator

We consider the equation of an anharmonic oscillator with a cubic term

$$x'' + x + x^3 = 0.$$  \hspace{1cm} (9)

This is the equation of motion of a Hamiltonian system with a nonlinear center. The period of the oscillations can be explicitly obtained by the following expression

$$T(E) = \sqrt{2} \int_{x_-}^{x_+} \frac{dx}{\sqrt{E - x^2 - x^4}},$$  \hspace{1cm} (10)

where $E$ is the energy, given by $E = a^2/2 + a^4/4$. Here $a$ is the amplitude of the cycle. The limits of the integral are

$$x_\pm = \pm \sqrt{-1 + \sqrt{1 + 4E}}.$$  \hspace{1cm} (11)

This integral can also be expressed in terms of elliptic functions.

In order to apply the previously described method to this equation we have written it as a system in the plane and introduced the frequency $\omega$. In this particular case the equation (2) is

$$\omega y_1' = y_2$$

$$\omega y_2' = -y_1 - 3y_1^2.$$  \hspace{1cm} (12)

We have taken different initial amplitudes. For each amplitude a different cycle is obtained. This is because it is a center. In particular we have calculated the periods of these cycles. Figure 1 shows $\rho$ as a function of the frequency, the $h-\omega$–surface, and the $h$–curves. The three graphics correspond to the initial amplitude $0.5$ and HAM of order 15.

We should note two facts. In a center the method does not allow us to accurately find a cycle of predetermined amplitude. In contrast, with an initial cycle of given amplitude, a periodic solution is obtained with an amplitude close to it. For example in Figure 2 it is shown the obtained cycle for initial amplitude 2. The resulting cycle has an amplitude of 1.66. In the same figure it is shown the frequency $(2\pi/T)$ as a function of the energy compared with the exact solution.

Another interesting observation is that the energy is not exactly conserved for the obtained solution. Nevertheless, the variation is small, of the order of 1% for order 15, as seen in Figure 2.

3.2. Van der Pol oscillator

We have considered the van der Pol equation

$$x'' + \epsilon(x^2 - 1)x' + x = 0.$$  \hspace{1cm} (13)

By changing coordinates to the plane and introducing the frequency $\omega$ the equation becomes

$$\omega y_1' = y_2$$

$$\omega y_2' = -y_1 - \epsilon(y_1^2 - 1)y_2.$$  \hspace{1cm} (14)
The proposed method allows us to find the stable limit cycle with great accuracy, even for non-small values of the parameter. Figure 3 shows the obtained cycles for \( \epsilon = 0.1 \) and \( \epsilon = 1.0 \). Order 15 was used in both cases. Numerically obtained cycles are also shown. Figure 4 shows the quadratic error as a function of the frequency, the \( h-\omega \)–surface and the \( h \)–curves correspond to the parameter value \( \epsilon = 1.0 \).

### 3.3. Hopf bifurcation

We have considered the following equation [Kuznetsov, 1995]

\[
x'' - (\beta - x'^2)x' + x = 0.
\]

In this case we obtain the following system

\[
\begin{align*}
\omega y'_1 &= y_2 \\
\omega y'_2 &= -y_1 + (\beta - y_2^2)y_2.
\end{align*}
\]

This system undergoes a supercritical Hopf bifurcation for the parameter value \( \beta = 0 \). As in the previous cases the method allows us to find cycles and frequencies very accurately. Cycles for \( \beta = 0.2 \) and \( \beta = 1.5 \) with HAM of order 12 are shown in Figure 5. Figure 6 shows the quadratic error as a function of frequency, the \( h-\omega \)–surface and the \( h \)–curves for \( \beta = 1.5 \).

It is interesting to analyze the amplitude of the emerging orbit as a function of the bifurcation parameter. This amplitude is shown in Figure 7. It is compared with that obtained with the program MATCONT. Comparison shows a very good agreement for a wide range of the parameter. The method has a very good performance even for parameter values away from the bifurcation. The global nature of the method is typical of HAM.
3.4. Multiple cycles and equilibria

In this last example we have taken the following system of equations

\[
\begin{align*}
x_1' &= \beta_1 x_1 x_2^3 - x_2 + \beta_2 x_1 (x_1^2 + x_2^2) - x_1 (x_1^2 + x_2^2)^2 \\
x_2' &= x_1 + \beta_1 x_2 + \beta_2 x_2 (x_1^2 + x_2^2) - x_2 (x_1^2 + x_2^2)^2.
\end{align*}
\]

(17)
Fig. 4. Van der Pol equation. a) Quadratic error as a function of $\omega$ for $\epsilon = 1$, b) $h-\omega$-surface, c) $h$-curves for $\omega = 0.938251$.

Fig. 5. Hopf bifurcation. —: HAM of order 12, · · · : numerical, a) $\beta = 0.2$, b) $\beta = 1.5$.

This system is similar to the normal form of a Bautin bifurcation [Kuznetsov, 1995]. However the additional factor $x_2^2$ in the first term of the first equation cause the appearance of new equilibria besides the trivial. In this way a much more complex dynamic is generated.

To apply the method here developed we transform the equation (17) in the following system

$$\begin{align*}
\omega y_1' &= \beta_1 y_1 y_2 - y_2 + \beta_2 y_1(y_1^2 + y_2^2) - y_1(y_1^2 + y_2^2)^2 \\
\omega y_2' &= y_1 + \beta_1 y_2 + \beta_2 y_2(y_1^2 + y_2^2) - y_2(y_1^2 + y_2^2)^2.
\end{align*}$$

(18)

The system has a Bautin bifurcation for $\beta_1 = \beta_2 = 0$. The bifurcation diagram in a neighbourhood of this point is shown in Figure 8, this has been calculated with the program MATCONT. The line $\beta_1 = 0$ corresponds to a Hopf bifurcation. Curve (d) is a saddle-node bifurcation of cycles. At this curve the system
Fig. 6. Hopf bifurcation. a) Quadratic error as a function of \( \omega \) for \( \beta = 1.5 \), b) \( h-\omega \) surface, c) \( h \)-curves for \( \omega = 0.887891 \).

Fig. 7. Amplitude of Hopf cycle, equation (15), \( \cdots \): HAM, —: MATCONT.

has a unique periodic orbit, while in the region (c) there are two cycles, one stable and one unstable. We choose a point in each of the regions (a), (c) and (d) of Figure 8 in order to find periodic cycles with the proposed method.

Figure 9 shows the limit cycle for \( \beta_1 = 0.5 \) and \( \beta_2 = 1 \). Cycle obtained with the HAM of order 12 is displayed together with the numerically obtained one.

Two cycles corresponding to the values \( \beta_1 = -0.5 \) and \( \beta_2 = 2 \) are shown in Figure 10. The inner cycle is obtained with the HAM of order 18. The outer cycle corresponds to order 9. Again numerical results are
Fig. 8. Bifurcation diagram for system (17).

Fig. 9. Multiple cycles and equilibria. Cycle for $\beta_1 = 0.5$ and $\beta_2 = 1$, —: HAM, ···: numeric.

shown. To obtain the different cycles the initial condition was varied. $a_0 = 0.355$ for the internal cycle and $a_0 = 1.4$ for the outer one.

Finally Figure 11 shows the limit cycle in the region (d), i.e. on the saddle–node bifurcation line of periodic orbits. The parameter values are $\beta_1 = -0.486$ and $\beta_2 = 1$. Numerically obtained cycle and HAM of order 9 are shown.

The criteria explained in the previous sections have been applied in all these examples. Figure 12 shows the quadratic error as function of the frequency, the $h-\omega$–surface and the $h$–curves corresponding to the cycle shown in Figure 9, corresponding to case (a) of Figure 8. Order 12 was used. The other cases are similar.
3.5. **No periodic orbits**

Finally we tested the method in some regions where there is no periodic cycles. The results are somewhat surprising and throw light on some aspect of heuristic search. We have considered the most interesting case, the previous system, i.e. equation (17).

First we take a point in the region (b) of Figure 8, we choose the values $\beta_1 = -1$ and $\beta_2 = 0$. For values of the parameters in this region the system has only the stable trivial equilibrium. By using a large value for the initial amplitude (in this case we take the value 2) the method shows no signs of convergence. In Figure 13 it can be seen that the minimum of $\rho$ is given for a negative frequency, and the value of of the
Fig. 12. Multiple cycles and equilibria. a) Quadratic error as a function of $\omega$ for $\beta_1 = 0.5$ and $\beta_2 = 1$ equation (17), b) $h$-$\omega$-surface, c) $h$-curves for $\omega = 0.95$.

error at the minimum is extremely large. The shape of the $h$-$\omega$-surface is not as expected, it appears that the best possible $h$-curves represent only random variations of the functions in a very small interval near zero.

However, if the initial amplitude becomes smaller the method displays a weak indication of convergence. Figure 14 shows that for an initial amplitude of 0.1 the error has a peak-shaped minimum. The $h$-curves also suggest a small region of convergence. However, we must emphasize that the obtained cycle has lower amplitude than the original, in this case more than 10 times lower, indicating the necessity of iterating the initial condition as described above. The process seems to converge to a cycle of zero amplitude. In other words, the equilibrium acts as a basin of attraction of cycles in a very small neighborhood.

The above phenomenon is found again when we seek a cycle in a region where there is an homoclinic curve. Of course this is a singular case and care should be taken when interpreting the results of any method of dynamics study. Figure 15 shows the homoclinic orbit for the values of $\beta_1 = 0.86575$ and $\beta_2 = 2$. In the same figure it can be seen a cycle obtained by the proposed method and the non-trivial equilibrium. The trajectory of such cycle seems to mimic the homoclinic orbit. In Figure 16 are frequency, $h$-$\omega$-surface and $h$-curves. The figures seem to indicate convergence but without being entirely conclusive, and the frequency obtained do not tend to zero as happens to cycles of the perturbed system near the homoclinic orbit.

In this case a small perturbation of the system can lead to the occurrence of a periodic orbit. In part, the presence of this cycle could justify the result, but the observation of the previous case and the fact that there is not a clear trend in the frequency leads us to think that the homoclinic orbit generates a basin of attraction of cycles producing the result.

3.6. Discussion

We have developed a method that involves the application of HAM to obtain a formal expression which is then studied to find the real solution of the problem. The method has been applied to several benchmark
cases. The method has a strong heuristic component.

To find the actual frequency of the periodic orbit it should be studied the landscape of the $h$-$\omega$–surface.
This is a very complex task. We reduced it to a series of steps in which we have to make certain observations and decide based on them. Also some parameters must be adjusted, such as the length of the interval $\Delta$ or the initial amplitude. These choices depend on the particular problem.

If the value of $\Delta$ is very large, then the resolution is lost in the search for the minimum quadratic error as a function of frequency. The peaks are much wider. Moreover, $h$–curves with very little support can not be detected. By contrast, if $\Delta$ is very small, then the sensitivity is lost in all calculations because
the values of the differences are very small.

We must emphasize that the HAM, in its traditional form, is not exempt from certain heuristic component. In particular with regard to the observation of the $h$-curves and its interpretation.

Being based on the HAM, the proposed method shares some of its characteristics, in particular, unlike traditional perturbative methods, it is able to tackle problems with strong nonlinearity. Also, as we have seen, it is able to deal with a wide variation in the parameters.

We have shown that the method developed here can find both isolated cycles and cycles around a nonlinear center. This is possible due to the particular choice of the linear operator and allowing the initial to vary freely. We suppose that it may be generalized to the search for periodic orbits in other systems such as differential equations with delays.

As we conjectured each solution may have a basin of attraction and eventually some non-periodic structures can confuse the results. A detailed study of this problem is beyond the proposal for this work.

In this work we have solved two-dimensional systems. Although in principle there is no reason to believe that the method does not work in more dimensions, however we believe that its implementation will require some adjustments. The problem is that in more dimensions the initial condition is harder to fix and, as we have seen, the convergence towards a solution strongly depends on this condition.

4. Conclusions

In this work we have developed a new way to apply the HAM to find periodic orbits in differential equations. We apply the method in the search for isolated cycles, isolated cycles emerging from a Hopf bifurcation and cycles in a nonlinear center. In all cases trigonometric polynomial approximations were obtained for such cycles. These are very good approximations as shown by the comparison with numerical results.

The method allows us to find solutions in a wide range of parameters. The frequencies obtained are in excellent agreement with those calculated by analytical methods based on the integration of the system, when they are available.

We believe that the method is a promising tool for the calculation of periodic orbits and for the study of bifurcations associated to them. Algebraically explicit character of the obtained solutions allows, among other things, to decide very easily about stability.

In the previous section we emphasized the advantages of the method as well as its potential disadvantages, the latter are mainly related to the choice of the initial condition, especially in more than two dimensions and interference on the results from other equilibria. Future work may focus in these issues.

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