Note

On the bandwidth of convex triangulation meshes

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Abstract

Hochberg et al. (1995) obtained the bandwidth of triangulated triangles. In this paper, we consider a more general class of graphs, called convex triangulation meshes and denoted by \( T_{l,m,n} \). We show that the bandwidth of \( T_{l,m,n} \) is \( \min\{l,m,n\} \).

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1. Introduction

Given a simple graph \( G = (V, E) \) a vertex numbering, or simply a numbering, of \( G \) is a bijection \( f: V \rightarrow \{1, 2, \ldots, |V|\} \), and the bandwidth of \( G \) for \( f \) is

\[
B(G,f) = \max \{|f(u) - f(v)|: uv \in E(G)\}.
\]

The bandwidth of \( G \) is defined as

\[
B(G) = \min \{B(G,f): f \text{ is a numbering of } G\}.
\]

It is a well-known NP-complete problem to determine \( B(G) \) in general. Much work has been done for specific types of graphs (see [2–6]). In structure analysis involving the finite element method, which is an origin of the bandwidth problem, the typical graphs that people most often encountered are triangulation meshes in which each element is a triangle (see Fig. 1).

The bandwidth of triangulated triangles has been settled in [6]. In this paper we study a more general class of graphs which includes the triangulated triangles as

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a sub-class. We called them convex triangulated meshes in the plane. The result obtained is similar to that of rectangular meshes, i.e. $P_m \times P_n$, in [4].

Notation and terminology of graph theory not defined in this paper will be the same as described in the book of Bondy and Murty [1].

2. Convex triangulation meshes

Consider three sets of parallel lines $\{H_1, H_2, \ldots, H_h\}$, $\{R_1, R_2, \ldots, R_r\}$, and $\{L_1, L_2, \ldots, L_l\}$ in the plane satisfying:

(a) All $H$-lines run horizontally (East–West). All $R$-lines run towards north-east (or south-west), cutting an $H$-line at an angle of $2\pi/3$. All $L$-lines run towards north-west (or south-east), cutting an $H$-line at an angle of $2\pi/3$.

(b) Two successive lines in the same set, i.e. $L_i$ and $L_{i+1}$, have unit distance between them when measured along another sets of lines;

(c) Each line has at least one point which is the intersection of three lines (called ‘triple intersection point’).

From the set of lines $\{H_1, H_2, \ldots, H_h\}$, $\{R_1, R_2, \ldots, R_r\}$, and $\{L_1, L_2, \ldots, L_l\}$, a graph is obtained by taking all triple intersection points as vertices and all unit line-segments joining two triple intersection points as edges. This is called a convex triangulation mesh, and denoted by $T_{h,r,l}$. Note that each convex triangulation mesh has an exterior cycle, which is formed by the set of all vertices adjacent to the unbounded face on the plane. For three different examples of $T_{5,5,5}$, see Figs. 1(a)–(c) above.

3. The main result

We shall obtain as our main result the following theorem. Note that the result is similar to that for rectangular meshes: $B(P_m \times P_n) = \min\{m, n\}$ (see [2, 4]).
Theorem 1.

\[ B(T_{k,r,l}) = \min\{h,r,l\} \]

To prove Theorem 1, we need some lemmas.

For a numbering \( f \), denote \( S_k = f^{-1}([1,2,\ldots,k]) \). For any subset \( S \) of \( V(G) \), the \textit{boundary (neighbour set)} of \( S \) is

\[ N(S) = \{ v \in V(G) \setminus S : \exists u \in S \text{ such that } uv \in E(G) \} \]

The following lemma due to Harper [5] is an efficient tool for getting sharp lower bounds (see Lemma 3 of [6]).

**Lemma 2.** For any numbering \( f \) of \( G \), \( B(G,f) \geq \max_{1 \leq k \leq n} |N(S_k)| \).

A graph \( G \) is called a \textit{plane near-triangulation} if it can be embedded in the plane so that all bounded faces are triangles. Let \( G \) be such a graph, and let \( v_1, v_2 \) and \( v_3 \) be three distinct vertices that lie on the exterior cycle \( C \). Then \( C \) is the union of three paths \( P_1, P_2 \) and \( P_3 \) of the form \( P_1 = (v_2, \ldots, v_3), P_2 = (v_3, \ldots, v_1) \) and \( P_3 = (v_1, \ldots, v_2) \) so that \( v_i \) is not on \( P_i \). A subset \( S \) of \( V(G) \) is called a \textit{connecting set} (with respect to \( u_1, u_2 \) and \( u_3 \)) if the subgraph \( G[S] \) (induced by \( G \) on \( S \)) includes a tree which contains vertices from each of \( P_1, P_2 \) and \( P_3 \). The following lemma of Hochberg et al. (see [6]), is crucial in dealing with bandwidth of triangulated triangles.

**Lemma 3.** If \( G \) is a plane near-triangulation and \( v_1, v_2 \) and \( v_3 \) are three distinct vertices on the exterior cycle, then \( B(G) \) is at least as large as the smallest connecting set.

It is clear that all convex triangulation meshes \( T_{l,m,n} \) defined above are plane near-triangulation. In particular, a mesh \( T_{l,m,n} \) is said to be regular if \( l = m = n \), as shown in Figs. 1(a)–(c). The following lemma is a generalization of Corollary 2 of [6].

**Lemma 4.**

\[ B(T_{m,m,m}) = m. \]

**Proof.** For a regular mesh \( T_{m,m,m} \), we choose \( v_1, v_2 \) and \( v_3 \) on the exterior cycle so that \( P_1 \) intersects all \( H \)-lines, \( P_2 \) intersects all \( L \)-lines, and \( P_3 \) intersects all \( R \)-lines as shown in Fig. 2.

Since the problem has been solved when \( T_{m,m,m} \) is a triangulated triangle, we may assume that \( T_{m,m,m} \) is a triangulated hexagon. So the path \( P_1 \) consists of two parts: one along \( R_m \) and the another along \( L_1 \). Similarly, \( P_2 \) consists of vertices on \( L_m \) and \( H_m \), and \( P_3 \) consists of vertices on \( R_1 \) and \( H_1 \).

Let \( S \) be a connecting set of \( G = T_{m,m,m} \) (with respect to the chosen \( v_1, v_2 \) and \( v_3 \)). If \( H_1 \cap S \neq \emptyset \) and \( H_m \cap S \neq \emptyset \), then by the connectedness of \( G[S] \), each \( H \)-line has at least
one vertex in $S$ for $1 \leq i \leq m$. Therefore, $|V(S)| \geq m$, which implies $B(G) \geq m$. The same argument is also valid for the case where $R_1 \cap S \neq \emptyset$ and $R_m \cap S \neq \emptyset$; and for the case where $L_1 \cap S \neq \emptyset$ and $L_m \cap S \neq \emptyset$.

By definition of a connecting set, we have $S \cap P_i \neq \emptyset$ for $i = 1, 2$ or $3$. Let $x_i \in S \cap P_i$, for $i = 1, 2$ or $3$. Suppose $x_1 \in R_m$, then we have $x_3 \notin R_1$, i.e. $x_3 \in H_1$, otherwise $B(G) \geq m$ by argument in the previous paragraph. This in term means that we have $x_2 \notin H_m$, i.e. $x_2 \in L_m$, by the same reasoning. Therefore, $x_1$, $x_2$, and $x_3$ lie on alternate segments of the hexagonal exterior cycle. Moreover, $S$ is a connecting set of a triangulated triangle $T_{k,k,k}$ defined in [6], where $k > m$, as shown in Fig. 3.
Hence \(|V(S)| \geq k > m\), which implies \(B(G) > m\). We will arrive at a similar conclusion if \(x_i \in L_1\). Combining this with conclusion in the previous paragraph, we may conclude that \(B(G) \geq m\).

Since we can obtain a numbering which attains this lower bound from the 'row-by-row' numbering, the assertion of the lemma holds.

**Proof of Theorem 1.** Suppose that \(\min\{l,m,n\} = m\). Then \(T_{l,m,n}\) includes a regular mesh \(T_{m,m,m}\). By Lemma 4, \(B(T_{l,m,n}) \geq B(T_{m,m,m}) = m\). In the meanwhile, a numbering which attains this lower bound can be obtained from the 'row-by-row' numbering. Hence the theorem follows.

**References**