Wiener number of hexagonal jagged-rectangles

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Abstract

The Wiener number of a connected graph is equal to the sum of distances between all pairs of its vertices. A graph formed by a row of \( n \) hexagonal cells is called an \( n \)-hexagonal chain. Wiener number of an \( n \times m \) hexagonal rectangle was found by the authors. An \( n \times m \) hexagonal jagged-rectangle whose shape forms a rectangle and the number of hexagonal cells in each chain alternate between \( n \) and \( n-1 \). In the paper, we obtain the Wiener numbers of three types of \( n \times m \) hexagonal jagged-rectangles.

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1. Introduction

An important invariant of connected graphs is called the Wiener number (or Wiener index) \( W \). This number is equal to the sum of distances between all pairs of vertices of the respective graph. American physico-chemist Harold Wiener first examined the quantity \( W \) in 1947 and 1948 [18–22]. He conceived this index in an attempt to formulate a mathematical model capable of describing molecular shapes. Wiener, and after him numerous researchers, including Wiener, reported the existence of correlation between \( W \) and a variety of physico-chemical properties of alkanes. For recent reviews on this matter and references to previous work in this area, see [8, 12]. The Wiener number was extensively studied also in the mathematical literature (see, for instance, [4, 11, 13, 23]). For generalization of the Wiener number, refer to [3, 9].

Despite large number of works on the theory of the Wiener number, some basic problems still remain open. For example, no recursive method is known for the calculation of \( W \) of a general graph, especially of polycyclic graphs. This is particularly frustrating in chemical applications, where the majority of molecular graphs is polycyclic. There is a significant breakthrough with regard to this problem by designing a
method for finding the expression for \( W(H_n) \) (see [14]), where \( H_n \) is a hexagonal system consisting of one central hexagon, surrounded by \( n - 1 \) layers of hexagonal cells, \( n \geq 2 \). Note that \( H_n \) is a molecular graph, corresponding to benzene \((n = 1)\), coronene \((n = 2)\), circumcoronene \((n = 3)\), etc. \( H_n \) was much examined in the theory of benzene hydrocarbons (see, for instance, [1, 5, 7]). Other types of hexagonal systems, hexagonal parallelogram \( Q_{n,m} \) (it was denoted by \( Q_{m,n} \) in [16]), hexagonal rectangles \( R_{n,m} \), hexagonal triangles \( T_n \), hexagonal bitrapeziums \( S_{n,m} \) and hexagonal trapeziums \( T_{n,m} \) were considered in [15-17].

In [17], one type of \( n \times m \) hexagonal rectangle was studied. In this paper, we shall study other types of \( n \times m \) hexagonal rectangles. We shall call these \textit{hexagonal jagged-rectangles}, or simply \textit{HJR}. In the hexagonal rectangle of [17], every hexagonal chain is of the same length \( n \). In the HJR the number of hexagonal cells in each chain alternative between \( n \) and \( n - 1 \). These result in 3-types of HJR. If the top and bottom rows are longer we shall call it HJR of type \( I \) and denoted by \( I^{n,m} \). If the top and bottom rows are shorter we shall call it HJR of type \( K \) and denoted by \( K^{n,m} \). The last one is called HJR of type \( J \) and denoted by \( J^{n,m} \). Note that \( J^{n,1} = T_{n,2} \) and \( K^{n,1} = S_{n,2} \) in [15]; \( I_{n,m}^{1,m} \), \( J_{n,m}^{1,m} \) and \( K_{n,m}^{1,m} \) are same as \( I_m \), \( J_m \) and \( K_m \) in [10], respectively.

Notation and terminology of graph theory not defined in this paper is the same as the one described in the book of Bondy and Murty [2].

2. Preliminary results

\textbf{Definition.} Let \( G = (V, E) \) be a graph. For \( v, w \in V \), let \( \rho(v, w) \) be the distance between \( v \) and \( w \). The \textit{Wiener number} of \( G \) is defined by \( W(G) = \frac{1}{2} \sum_{v, w \in V} \rho(v, w) \).

\textbf{Definition.} Let \( G = (V, E) \) be an infinite graph where \( V = \mathbb{Z} \times \mathbb{Z} \) and \( \{(x_1, y_1), (x_2, y_2)\} \in E \) if (1) \( y_1 = y_2 \) and \( |x_1 - x_2| = 1 \), or, (2) \( x_1 = x_2 \), \( |y_1 - y_2| = 1 \) and \( x_1 + y_1 + x_2 + y_2 \equiv 1 \) (mod 4) (see below). This graph is called the \textit{wall} which was defined in [14].

Let \( G = (V, E) \) be the wall. For \( n \geq 2 \) we identify the \( n \times m \) hexagonal rectangles \( I^{n,m} \), \( J^{n,m} \) and \( K^{n,m} \) as subgraphs of \( G \), where

\[
V(I^{n,m}) = V_I = \{(x, y) | 0 \leq x \leq 2n, 0 \leq y \leq 2m - 1\},
\]

\[
V(J^{n,m}) = V_J = V_I \cup \{(x, -1) | 1 \leq x \leq 2n - 1\}
\]
and

\[ V(K'^{n,m}) = V_K \cong V_j \cup \{(x, 2m) \mid 1 \leq x \leq 2n - 1\}. \]

We shall call these three kinds of hexagonal rectangles to be *hexagonal jagged-rectangles of type I, J and K*, respectively. In Sections 3–5 we shall find recurrence relations between their Wiener numbers.

The following lemma is a useful tool for computing the distance between two vertices in some "convex" subgraphs of the wall. It was proved by Shiu and Lam [14].

**Lemma A.** Suppose \( d \geq b \). The distance between two vertices \((a, b)\) and \((c, d)\) in the wall is

\[
\rho((a, b), (c, d)) = \begin{cases} 
2(d - b) & \text{if } |c - a| \leq (d - b) \text{ and } \\
2(d - b) + 1 & \text{if } |c - a| \leq (d - b), \\
2(d - b) - 1 & \text{if } |c - a| \leq (d - b), \\
(d - b) + |c - a| & \text{if } |c - a| > (d - b).
\end{cases}
\]

Moreover, a shortest path between \((a, b)\) and \((c, d)\) lies in the rectangle spanned by \((a, b)\) and \((c, d)\).

For convenience, we write \( \rho(V_1, V_2) = \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} \rho(v_1, v_2) \) for any subsets \( V_1, V_2 \) of the vertex set of the considered graph. If \( V_1 = \{v\} \) we write \( \rho(V_1, V_2) \) by \( \rho(v, V_2) \).

**3. Relationship of Wiener numbers of HJRs of Type I and Type J (part 1)**

In this section we shall formulate the Wiener number of HJR of type I in terms of the Wiener number of HJR of type J.
Let
\[ A = \{(a,0) \mid 0 \leq a \leq 2n\} \quad \text{and} \quad B = V_I \setminus (A \cup \{(0,1),(2n,1)\}). \]

Then
\[
W(I_n^{m,1}) = \frac{1}{2} \rho(V_I, V_I)
\]
\[
\quad = \frac{1}{2} [\rho(B,B) + \rho(A,A) + \rho(\{(0,1),(2n,1)\}, \{(0,1),(2n,1)\})
\quad + 2\rho(A,B) + 2\rho(A,\{(0,1),(2n,1)\}) + 2\rho(B,\{(0,1),(2n,1)\})]
\quad = \frac{1}{2} [\rho(B,B) + \rho(A,A) + \rho(\{(0,1),(2n,1)\}, \{(0,1),(2n,1)\})
\quad + 2\rho(A,V_I \setminus A) + 2\rho(\{(0,1),(2n,1)\}, V_I \setminus A)]
\quad = W(J_{n,m}^{1,1}) + W(P_{2n+1}) + \rho(A, V_I \setminus A)
\quad \quad + \rho(\{(0,1),(2n,1)\}, V_I \setminus A) - \rho((0,1),(2n,1)),
\]

where \( P_r \) is the path with \( r \) vertices and \( W(P_r) = \frac{1}{6} r(r-1)(r+1) \), [4, 6].

Let \( T(v) = \sum_{w \in V_I \setminus A} \rho(v,w), \ v \in A. \)

Since \( T((2n,1)) = T((0,1)) = T((0,0)) = (2n+1)(2m-1) \) and \( \rho((0,1),(2n,1)) = 2n \), we have
\[
W(I_n^{m,1}) = W(J_{n,m}^{1,1}) + W(P_{2n+1}) + \rho(A, V_I \setminus A)
\quad \quad + 2T((0,0)) - 2(2n+1)(2m-1) - 2n.
\]

To compute \( T(v) \) we consider the following three cases.

Case 1: \( 1 \leq m \leq \frac{1}{2} (n+1). \)

Since \( T((a,0)) = T((2n-a,0)) \), \( \rho(A, V_I \setminus A) = 2 \sum_{a=0}^{2m-1} T((a,0)) + \sum_{a=2m}^{2n-2m} T((a,0)). \)

(a) If \( 0 \leq a \leq 2m-1 \), \( T((a,0)) \) is equal to the sum of the following four summands:

By using Lemma A and a direct computation we have

(i) \( \sum_{y=1}^{a} \sum_{x=0}^{a-y} (a - x + y) = \frac{1}{2} a(1 + a)^2; \)

(ii) \( \sum_{y=1}^{a} \sum_{x=a+y}^{2m-1} (x - a + y) = \frac{1}{2} (-1 + 2m)(-a + a^2 + 4m - 2am - 4m^2)
\quad \quad + 2n - 4an + 4mn + 4n^2; \)

(iii) \( \sum_{y=1}^{a} \sum_{x=a-y+1}^{y+a-1} \rho((a,0),(x,y)) = \sum_{y=1}^{a} \{2y(2y-1) - (-1)^a y\}
\quad \quad = \frac{1}{2} a(1 + a)\{2 - 3(-1)^a + 8a\}; \)

(iv) \( \sum_{y=a+1}^{2m-1} \sum_{x=0}^{y+a-1} \rho((a,0),(x,y)) = \sum_{y=a+1}^{2m-1} \{2y(a + y) - (-1)^a \left[ \frac{a + y}{2} \right]\} \)
\[
\begin{align*}
\frac{1}{12} (8a - 15a^2 - 20a^3 + 8m - 36am - 60a^2 + 48am^2 + 64m^3) & \quad \text{if } a \text{ is even} \\
\frac{1}{12} (-3 - 16a - 33a^2 - 20a^3 + 8m - 12am - 36m^2 + 48am^2 + 64m^3) & \quad \text{if } a \text{ is odd}
\end{align*}
\]
\[
= \frac{1}{12} \{ (-1)^a \left( \frac{3}{2} + 12a + 9a^2 - 12am - 12m^2 \right) - \frac{3}{2} - 4a - 24a^2 \\
- 20a^3 + 8m - 24am - 48m^2 + 48am^2 + 64m^3 \}.
\]

where \([x]\) is the least integer which is greater than \(x\).

Summing up the above results we have
\[
T((a, 0)) = \frac{1}{12} \{ (-1)^a \left( \frac{3}{2} + 6a + 3a^2 - 12am - 12m^2 \right) - \frac{3}{2} + 4a - 6a^2 \\
+ 2a^3 - 16m - 24am + 12a^2m + 24m^2 + 24am^2 + 16m^3 \\
- 12n + 24an - 48amn + 48m^2n - 24n^2 + 48mn^2 \}.
\]

Hence we have
\[
\sum_{a=0}^{2m-1} T((a, 0)) = m(-1 + 2m - 6m^2 + 10m^3 - 4n + 8mn - 4n^2 + 8mn^2).
\]

(b) If \(2m \leq a \leq 2n - 2m\), \(T((a, 0))\) equals to the sum of the following three summands:

(i) \[\sum_{y=1}^{2m-1} \sum_{x=0}^{a-y} (a - x + y) = \frac{1}{2}(-1 + 2m)(a + a^2 + 4m + 2am - 4m^2);\]

(ii) Same as Case 1(a)(ii);

(iii) \[\sum_{y=1}^{2m-1} \sum_{x=a-y+1}^{a-1} \rho((a, 0), (x, y)) = \sum_{y=1}^{2m-1} \{2y(2y - 1) - (-1)^a y\} = \frac{1}{3} m(1 - 2m)(10 + 3(-1)^a - 16m).\]

Thus we have
\[
T((a, 0)) = \frac{1}{3}(-1 + 2m)(3a^2 + 2m - 3(-1)^a m + 4m^2 + 3n - 6an + 6mn + 6n^2),
\]
\[
\sum_{a=2m}^{2n-2m} T((a, 0)) = \frac{1}{3}(3m - 14m^2 + 48m^3 - 64m^4 - 4n + 22mn - 36m^2n + 16m^3n \\
- 12n^2 + 36mn^2 - 24m^2n^2 - 8n^3 + 16mn^3)
\]

and
\[
\rho(A, V_l \setminus A) = \frac{1}{3}(-3m - 2m^2 + 12m^3 - 4m^4 - 4n - 2mn \\
+ 12m^2n + 16m^3n - 12n^2 + 12mn^2 + 24m^2n^2 - 8n^3 + 16mn^3).
\]
Combining the results of part (a) and (b), we finally obtain the desired relationship for case 1:

$$W(I^{n,m}) = W(J^{n,m-1}) + \frac{1}{3}(6 - 23m + 4m^2 + 20m^3 - 4m^4 - 2n - 26mn + 36m^2n$$

$$+ 16m^3n - 18n^2 + 36mn^2 + 24m^2n^2 - 4n^3 + 16mn^3). \quad (3.1)$$

**Case 2:** $\frac{1}{2}(n + 1) \leq m \leq n$. Since $T((a, 0)) = T((2n - a, 0))$,

$$\rho(A, Y\setminus A) = 2 \sum_{a=0}^{2n-2m+1} T((a, 0)) + 2 \sum_{a=2n-2m+2}^{n} T((a, 0)) - T((n, 0)).$$

(a) $0 \leq a \leq 2n - 2m + 1$. All the summands of $T((a, 0))$ are the same as in Case 1(a).

Hence we have

$$\sum_{a=0}^{2n-2m+1} T((a, 0)) = \frac{1}{3}(-3 - 23m + 26m^2 - 18m^3 - 2m^4 - 5n - 30mn + 96m^2n$$

$$- 48m^3n - 2n^2 - 24mn^2 + 48m^2n^2 + 2n^3 + 2n^4).$$

(b) $2n - 2m + 2 \leq a \leq n$. $T((a, 0))$ is equal to the sum of the following 5 summands:

\begin{itemize}
  \item[(i)] Same as Case 1(a)(i);
  \item[(ii)] $\sum_{y=1}^{2n-a} \sum_{x=a+y}^{2n} (x - a + y) = \frac{1}{2}(-a + 2n)(1 - a + 2n)^2$;
  \item[(iii)] Same as Case 1(a)(iii);
  \item[(iv)] $\sum_{y=a+1}^{2n-a} \sum_{x=0}^{y+a-1} \rho((a, 0), (x, y)) = \sum_{y=a+1}^{2n-a} \left\{ 2y(a + y) - (-1)^a \left\lfloor \frac{a + y}{2} \right\rfloor \right\}$

  $$= \frac{1}{3}(-a + n)\{2 - 3(-1)^a + 6a - 3(-1)^a a$$

  $$+ 4a^2 + 12n - 3(-1)^a n + 4an + 16n^2\};$$
\end{itemize}
(v) \[
\sum_{y=2n-a+1}^{2m-1} \sum_{x=0}^{2n} \rho((a,0),(x,y)) = \sum_{y=2n-a+1}^{2m-1} \left\{ 2y(2n+1) - (-1)^y(n + \frac{1}{2}) + \frac{(-1)^y}{2} \right\}
\]
\[
= \frac{1}{4} \left\{ (-1)^y(1 - 2a - 4m + 8n - 4an - 8mn + 8n^2) - 1 + 4a - 4a^2 - 8m + 16m^2 - 8n + 24an - 8a^2n - 16mn + 32m^2n - 32n^2 + 32an^2 - 32n^3 \right\}.
\]

Also
\[
T((a,0)) = \frac{1}{12} \left\{ (-1)^y(3 + 6a^2 - 12m + 12n - 12an - 24mn + 12n^2) - 3 - 24m + 48m^2 - 4n + 12a^2n - 48mn + 96m^2n - 24an^2 + 16n^3 \right\},
\]
\[
\sum_{a=2n-2m+2}^{m} T((a,0)) = \frac{1}{34} \left\{ (-1)^y(3 - 12m + 12n - 24mn + 6n^2) + 21 - 12m - 168m^2 + 192m^3 + 20n + 184mn - 624m^2n + 448m^3n - 32n^2 + 240mn^2 - 288m^2n^2 - 44n^3 + 64mn^3 - 16n^4 \right\}
\]
and
\[
\rho(A, V_i \setminus A) = \frac{1}{3}(-3m - 2m^2 + 12m^3 - 4m^4 - 4n + 2mn + 12m^2n + 16m^3n - 12n^2 + 12mn^2 + 24m^2n^2 - 8n^3 + 16mn^3).\]

Summarizing above results, we get the following relationship for case 2:
\[
W(I^n,m) = W(J^n,m-1) + \frac{1}{3}(6 - 23m + 4m^2 + 20m^3 - 4m^4 - 2n - 26mn + 36m^2n + 16m^3n - 18n^2 + 36mn^2 + 24m^2n^2 - 4n^3 + 16mn^3). \tag{3.2}
\]
which is identical to (3.1).

Case 3: \(n + 1 \leq m\). When \(0 \leq a \leq n\), all the summands of \(T((a,0))\) are the same as in Case 2(b). Since \(\rho(A, V_i \setminus A) = 2 \sum_{a=0}^{n} T((a,0)) - T((n,0))\), we have
\[
W(I^n,m) = W(J^n,m-1) + \frac{1}{3}(6 - 39m + 36m^2 + 14n - 90mn + 96m^2n + 14n^2 - 24mn^2 + 48m^2n^2 + 16n^4 + 4n^4). \tag{3.3}
\]

4. Relationship of Wiener numbers of HJRs of Type I and Type J (part 2)

In this section we shall express the Wiener number of HJR of type \(J\) in terms of the Wiener number of HJR of type \(I\). Let \(A = \{(a, -1) \mid 1 \leq a \leq 2n - 1\}\). Similar to Section 3, we have
\[
W(J^n,m) = W(I^n,m) + \rho(A, V_j) - W(P_{2n-1}).
\]
Case 1: \(1 \leq m \leq \frac{1}{2} n\). Similar to Section 3 we have 
\[
\rho(A, V_j) = 2 \sum_{a=1}^{2m} T((a, -1)) + \sum_{a=2m+1}^{2n} T((a, -1)),
\]
where \(T(v) = \sum_{w \in V_j} \rho(v, w) \) for \(v \in A\).

(a) If \(1 \leq a \leq 2m\), \(T((a, -1))\) is equal to the sum of the following four summands:

By using Lemma A and a direct computation we have

(i) \[
\sum_{y=0}^{a-1} \sum_{x=0}^{a-1-y} (a - x + y + 1) - \rho((a, -1), (0, -1)) = \frac{1}{2} a^2 (3 + a);
\]

(ii) \[
\sum_{y=-1}^{2m-1} \sum_{x=a+1+y}^{2n} (x - a + y + 1) - \rho((a, -1), (2n, -1))
\]
\[
= \frac{1}{2} (1 + 2m) (-a + a^2 + 2m - 2am - 4m^2 + 2n - 4an + 4mn + 4n^2) + a - 2n;
\]

(iii) \[
\sum_{y=0}^{a-1} \sum_{x=a-y}^{y+a} \rho((a, -1), (x, y)) = \sum_{y=0}^{a-1} \{2(y + 1)(2y + 1) + (-1)^y(y + 1)\}
\]
\[
= \frac{1}{6} a(1 + a) \{-2 + 3(-1)^a + 8a\};
\]

(iv) \[
\sum_{y=a}^{2m-1} \sum_{x=0}^{y+a} \rho((a, -1), (x, y))
\]
\[
= \sum_{y=a}^{2m-1} \left\{2(y + 1)(a + y + 1) + (-1)^y \left\lfloor \frac{a + y + 1}{2} \right\rfloor \right\}
\]
\[
= \begin{cases} 
\frac{1}{12} (-10a - 33a^2 - 20a^3 + 20m + 36am) \\
+ 60m^2 + 48am^2 + 64m^3) & \text{if } a \text{ is even} \\
\frac{1}{12} (-3 + 2a - 15a^2 - 20a^3 - 4m) \\
+ 12am + 36m^2 + 48am^2 + 64m^3) & \text{if } a \text{ is odd}.
\end{cases}
\]

Summing up the above results we can get \(T((a, -1))\).

Hence we have \(\sum_{a=1}^{2m} T((a, -1)) = \frac{1}{3} m(-1 + 18m + 46m^2 + 30m^3 - 12n + 12n^2 + 24mn^2)\).

(b) If \(2m + 1 \leq a \leq 2n - 2m - 1\), \(T((a, -1))\) is equal to the sum of the following three summands:

(i) \[
\sum_{y=1}^{2m-1} \sum_{x=0}^{a-1-y} (a - x + y + 1) - a
\]
\[
= \frac{1}{2} (-a + a^2 + 2m + 4am + 2a^2m + 4am^2 - 8m^2);
\]

(ii) Same as Case 1(a)(ii);
(iii) \[
\sum_{y=0}^{2m-1} \sum_{x=a-y}^{y+a} \rho((a, -1)(x, y)) = \sum_{y=0}^{2m-1} \{2(y + 1)(2y + 1) + (-1)^a(y + 1)\} = \frac{1}{3}m(1 + 2m)\{-2 + 3(-1)^a + 16m\}.
\]

Thus we have
\[
\sum_{a=2n+1}^{2n-2m-1} T((a, -1)) = \frac{1}{3}(-9m - 50m^2 - 96m^3 - 64m^4 + 4n + 22mn + 12m^2n + 16m^3n - 12n^2 - 12mn^2 - 24m^2n^2 + 8n^3 + 16mn^3)
\]

and
\[
\rho(A, V_f) = \frac{1}{3}(-11m - 14m^2 - 4m^3 - 4m^4 + 4n - 2mn + 12m^2n + 16m^3n - 12n^2 + 24m^2n^2 + 8n^3 + 16mn^3). 
\]

Therefore we have the following relationship for Case 1:
\[
W(I^{n,m}) = W(I^{n,m}) + \frac{1}{3}(-11m - 14m^2 - 4m^3 - 4m^4 + 2n - 2mn - 2n^2 - 6n^2 + 12mn^2 + 24m^2n^2 + 4n^3 + 16mn^3).
\]

Case 2: \(\frac{1}{2}n < m < n\). Similarly, we have
\[
\rho(A, V_f) = 2 \sum_{a=1}^{2n-2m} T((a, -1)) + 2 \sum_{a=2n-2m+1}^{n} T((a, -1)) - T((n, -1)).
\]

(a) \(1 \leq a \leq 2n - 2m\). All the summands of \(T((a, -1))\) are the same as Case 1(a).

After direct computation we have
\[
\sum_{a=1}^{2n-2m} T((a, -1)) = \frac{1}{3}(m - 10m^2 - 34m^3 - 2m^4 - n + 18mn + 2m^2n - 48m^3n + 48m^2n^2 + 10n^3 + 2n^4) .
\]

Note that when \(m = n\) the above formula is zero.

(b) \(2n - 2m + 1 \leq a \leq n\). \(T((a, -1))\) is equal to the sum of the following five summands:

(i) Same as Case 1(a)(i);

(ii) \[
\sum_{y=-1}^{2n-a-1} \sum_{x=a+y+1}^{2n} (x - a + y + 1) - \rho((a, -1), (2n, -1)) = \frac{1}{2}(-a + 2n)^2(3 - a + 2n);
\]

(iii) Same as Case 1(a)(iii);
(iv) \[ \sum_{y=a}^{2n-a-1} \sum_{x=0}^{y+a} \rho((a,-1),(x,y)) \]

\[ = \sum_{y=a}^{2n-a-1} \left\{ 2(y+1)(a+y+1) + (-1)^a \left( \frac{a+y+1}{2} \right) \right\} \]

\[ = \frac{1}{3} \left\{ (-a+n)(2+3(-1)^a(1+a+n)+6a+4a^2+12n+4an+16n^2) \right\}; \]

(v) \[ \sum_{x=0}^{2n-2n-a} \sum_{y=2n-a}^{2n-1} \rho((a,-1),(x,y)) \]

\[ = \sum_{y=2n-1}^{2m-1} \left\{ 2(y+1)(2n+1) + (-1)^a(n+\frac{1}{2}) + \left( \frac{(-1)^y}{2} \right) \right\} \]

\[ = \frac{1}{4} \left\{ (1+4a+4m-4n+8mn-8n^2) -1 + 4a - 4a^2 + 8m + 16m^2 - 8n + 24an - 8a^2n + 16mn + 32m^2n + 32n^2 + 32an^2 + 32n^3 \right\}. \]

Thus we have

\[ \sum_{a=2n-2m+1}^{n} T((a,-1)) = \frac{1}{24} \left\{ \frac{3(-1)^a(1+4m+8mn-2n^2)-3-28m+72m^2+256m^3+8n-104mn-48m^2n+448m^3n+28n^2+48mn^2-288m^2n^2-44n^3+64mn^3-16n^4)}{24} \right\}. \]

Therefore

\[ W(J^{n,m}) = W(I^{n,m}) + \frac{1}{3} \left\{ -11m-14m^2-4m^3-4m^4+2n-2mn+12m^2n+16m^3n-6n^2+12mn^2+24mn^2+4n^3+16mn^3 \right\}. \] (4.2)

Clearly, it is the same as Case 1.

Case 3: \( n \leq m \). When \( 1 \leq a \leq n \), all summands of \( T((a,-1)) \) are the same as in Case 2(b). Then \( \rho(A, I^m) = 2 \sum_{a=1}^{n} T((a,-1)) - T((n,-1)) \). Hence the relationship for Case 3 is

\[ W(J^{n,m}) = W(I^{n,m}) + \frac{1}{3} \left\{ -9m-12m^2-6mn-4n^2+24mn^2+48m^2n^2+4n^4 \right\}. \] (4.3)

5. Relationship of Wiener numbers of HJR of Type J and Type K

In this section we shall formulate the Wiener number of HJR of type K in terms of the Wiener number of HJR type J. Let \( A = \{(a,-1) \mid 1 \leq a \leq 2n - 1 \} \) and \( B = \{(a,2m) \mid 1 \leq a \leq 2n - 1 \} \).
Clearly, $K^{n,m} - A \cong J^{n,m}$. Similar to Section 3 we have
\begin{align*}
W(K^{n,m}) &= W(J^{n,m}) + \rho(A, V_J) - W(P_{2n-1}) \\
&= W(J^{n,m}) + \rho(A, V_J) + \rho(A, B) - W(P_{2n-1}).
\end{align*}

We compute $\rho(A, B)$ first.

Case 1: $1 \leq m \leq \frac{1}{2} n - 1$. $\rho(A, B) = 2 \sum_{a=1}^{2m+1} \rho(\{(a, -1)\}, B) + \sum_{a=2m+2}^{2n-2m-2} \rho(\{(a, -1)\}, B)$.  
(a) When $1 \leq a \leq 2m + 1$,
\begin{align*}
\rho(\{(a, -1)\}, B) &= \sum_{x=1}^{2m+1} \rho((a, -1), (x, 2m)) + \sum_{x=2m+a+1}^{2n+1} (x - a + 2m + 1) \\
&= 2(2m + 1)(2m + a) + (-1)^{m} \left[ \frac{2m + a}{2} \right] \\
&\quad + \sum_{x=2m+a+1}^{2n-1} (x - a + 2m + 1) \\
&= \frac{1}{4} \{( -1)^{a}(1 + 2a + 4m) - 5 + 6a + 2a^2 - 4m \\
&\quad + 8an + 8m^2 + 4n - 8an + 16mn + 8n^2 \}.
\end{align*}

(b) When $2m + 2 \leq a \leq 2n - 2m - 2$,
\begin{align*}
\rho(\{(a, -1)\}, B) &= \sum_{x=1}^{2m-a} (a - x + 2m + 1) + \sum_{x=a-2m}^{2m+a} \rho((a, -1), (x, 2m)) \\
&\quad + \sum_{x=2m+a+1}^{2n-1} (x - a + 2m + 1) \\
&= \sum_{x=1}^{2m-a} (a - x + 2m + 1) + 2(2m + 1)(4m + 1) + (-1)^{a}(2m + 1) \\
&\quad + \sum_{x=2m+a+1}^{2n-1} (x - a + 2m + 1) \\
&= (-1)^{a}(1 + 2m) + a^2 + 2m + 4m^2 + n - 2an + 4mn + 2n^2.
\end{align*}

Thus $\rho(A, B) = \frac{1}{4} \{( -3 - 16m - 24m^2 - 8m^3 - 2n + 24m^2n + 24mn^2 + 8n^3 \}$. And
\begin{align*}
W(K^{n,m}) &= W(J^{n,m}) + \frac{1}{3} \{( -3 - 27m - 38m^2 - 12m^3 - 4m^4 - 2mn \\
&\quad + 36m^2n - 6n^2 + 36mn^2 + 24m^2n^2 + 12n^3 + 16mn^3 \}.
\end{align*}
Case 2: $\frac{1}{2}n \leq m \leq n - 2$.

$$
\rho(A, B) = 2 \sum_{a=1}^{2n-2m-2} \rho((a, -1), B) + \sum_{a=2n-2m-1}^{2m+1} \rho((a, -1), B).
$$

(a) When $1 \leq a \leq 2n - 2m - 2$, $\rho((a, -1), B)$ is same as Case 1(a).

(b) When $2n - 2m - 1 < a < 2m + 1$,

$$
\rho((a, -1), B) - \sum_{x=1}^{2n-1} \rho((a, -1), (x, 2m))
$$

$$
= 2(2n - 1)(2m + 1) + (-1)^a(n - \frac{1}{2}) - \frac{1}{2}.
$$

Thus we have

$$
W(K^{n,m}) = W(J^{n,m}) + \frac{1}{3}(-3 - 27m - 38m^2 - 12m^3 - 4m^4 - 2mn
$$

$$
+ 36m^2n - 6n^2 + 36mn^2 + 24m^2n^2 + 12n^3 + 16mn^3).
$$

Clearly, it is the same as Case 1.

Case 3: $m \geq n - 1$.

$$
\rho(A, B) = \sum_{a=1}^{2n-1} \{2(2n - 1)(2m + 1) + (-1)^a(n - \frac{1}{2}) - \frac{1}{2}\}
$$

$$
= (2n - 1)(8mn + 4n - 4m - 3).
$$

and

$$
W(K^{n,m}) = W(J^{n,m}) + \frac{1}{3}(9 + 3m - 12m^2 - 30n - 54mn + 20n^2
$$

$$
+ 72mn^2 + 48m^2n^2 + 4n^4).
$$

In fact, (5.2) and (5.3) are the same when $m = n - 1$.

6. Main results

In this section, the explicit formulae for the Wiener numbers of the three types of HJR are obtained by first considering $J^{n,m}$. There are two cases.

Case 1: If $1 \leq m \leq n$ then by (3.1), (3.2), (4.1) and (4.2) we have

$$
W(J^{n,m}) = W(J^{n,m-1}) + \frac{2}{3}(3 - 17m - 5m^2 + 8m^3 - 4m^4 - 14mn
$$

$$
+ 24m^2n + 16m^3n - 12n^2 + 24mn^2 + 24m^2n^2 + 16mn^3). \quad (6.1)
$$

By (4.1) and $W(I^{n,1}) = \frac{1}{3}(16n^3 + 36n^2 + 26n + 3)$, see [6, 15–17], we have

$$
W(J^{n,1}) = 12n^3 + 22n^2 + 18n - 10.
$$
Note that it can be seen that $W(J^{n,1}) = W(T_{n,2})$ where $T_{n,m}$ is defined in [17].

Solving (6.1) we get

\[
W(J^{n,m}) = \frac{2}{15} \left( 40m^2n^3 + 40mn^3 + 10n^3 + 40m^3n^2 + 120m^2n^2 \\
+ 20mn^2 - 15n^2 + 20m^4n + 80m^3n + 45m^2n - 15mn \\
+ 5n - 4m^5 + 5m^3 - 45m^2 - 31m \right).
\]  

(6.2)

Case 2: If $n \leq m$ then by (3.3) and (4.3) we have

\[
W(J^{n,m}) = W(J^{n,m-1}) + \frac{2}{3} (1 + 2n)(3 - 24m + 12m^2 + n + 24m^2n + 3n^2 + 2n^3).
\]  

(6.3)

Solving (6.3) with the initial value obtained from (6.2) with $m = n$:

\[
W(J^{n,n}) = \frac{2}{15} (96n^5 + 240n^4 + 80n^3 - 75n^2 - 26n),
\]

we get

\[
W(J^{n,m}) = \frac{2}{15} \left( -4n^5 + 20mn^4 + 40mn^3 - 5n^3 + 80m^3n^2 + 120m^2n^2 \\
+ 65mn^2 + 80m^3n - 45mn + 9n + 20m^3 - 30m^2 - 35m \right).
\]  

(6.4)

In fact the above formula holds when $m = n - 1$. Thus we have the following theorem:

**Theorem 6.1.** The Wiener number of $J^{n,m}$ is

\[
\begin{cases} 
\frac{2}{15} \left( 40m^2n^3 + 40mn^3 + 10n^3 + 40m^3n^2 + 120m^2n^2 + 20mn^2 \\
- 15n^2 + 20m^4n + 80m^3n + 45m^2n - 15mn + 5n - 4m^5 \\
+ 5m^3 - 45m^2 - 31m \right) & \text{if } 1 \leq m \leq n, \\
\frac{2}{15} \left( -4n^5 + 20mn^4 + 40mn^3 - 5n^3 + 80m^3n^2 + 120m^2n^2 \\
+ 65mn^2 + 80m^3n - 45mn + 9n + 20m^3 - 30m^2 - 35m \right) & \text{if } n - 1 \leq m.
\end{cases}
\]

By substituting $W(J^{n,m})$ into (3.1) and (3.2) we get

**Theorem 6.2.** The Wiener number of $I^{n,m}$ is

\[
\begin{cases} 
\frac{1}{15} m(80mn^3 + 80m^2n^2 + 120mn^2 - 20n^2 + 40m^3n + 80m^2n \\
+ 30mn - 20n - 8m^4 + 20m^3 + 30m^2 - 20m - 7) & \text{if } 1 \leq m \leq n, \\
\frac{1}{15} \left( -8n^5 + 40mn^4 - 20n^4 + 80mn^3 - 10n^3 + 160m^3n^2 \\
+ 10mn^2 + 20n^2 + 160m^3n - 60mn + 18n + 40m^3 - 25m \right) & \text{if } n \leq m.
\end{cases}
\]

Finally by substituting $W(J^{n,m})$ into (5.1) to (5.3) we get
Theorem 6.3. The Wiener number of $K^m,n$ is

\[
\begin{aligned}
\frac{1}{15} (80m^2n^3 + 160mn^3 + 80n^3 + 80m^3n^2 + 360m^2n^2 \\
+ 220mn^2 - 60n^2 + 40m^4n + 240m^3n + 270m^2n - 40mn \\
+ 10n - 8m^5 - 20m^4 - 50m^3 - 280m^2 - 197m - 15) & \quad \text{if } 1 \leq m \leq n - 1, \\
\frac{1}{15} (-8n^5 + 40mn^4 + 20n^4 + 80mn^3 - 10n^3 + 160m^3n^2 \\
+ 480m^2n^2 + 490mn^2 + 100n^2 + 160m^3n - 360mn \\
- 132n + 40m^3 - 120m^2 - 55m + 45) & \quad \text{if } n - 1 \leq m.
\end{aligned}
\]

We have verified the formulae above by direct computer evaluations for $1 \leq m, n \leq 10$.

References


