Performance Analysis of $M$-PAM and $M$-QAM with Selection Combining in Independent but Non-Identically Distributed Rayleigh Fading Paths

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Abstract—A compact expression for pdf of SC combiner on independent but not necessarily identical distributed Rayleigh fading channels is derived. Using this expression, new exact closed form expressions for average SER and average BER of $M$-PAM and $M$-QAM are studied. The proposed mathematical analysis is verified by various simulation results which demonstrate the accuracy of the theoretical approach.

Index Terms—$M$-QAM, $M$-PAM, Selection Combining (SC), bit error rate (BER), symbol error rate (SER), Rayleigh-fading channels.

I. INTRODUCTION

Diversity is an effective technique used in wireless communication systems to combat the performance degradation caused by fading. It can alleviate the deleterious effect of fading by means of multiple reception of the same information bearing signals. Receiving several replicas of the same signal requires some kind of combining techniques in order to obtain a single representation of the desired symbol. The most important diversity reception methods employed in digital communication receivers are selection combining (SC) which selects the signal from that diversity path with the largest instantaneous SNR; equal-gain combining (EGC) which coherently combines all available paths weighting each with an equal gain; and maximal-ratio combining (MRC) which also coherently combines all available paths but weighs each with the respective gain of the path. Among them, SC gives the most inferior performance, MRC gives the best and the optimum performance, and EGC has a performance quality in between the others. SC and MRC are the two extremes of complexity quality tradeoff. Although optimum performance is highly desirable, practical systems often sacrifice some performance in order to reduce their complexity. Instead of using MRC which requires exact knowledge of the channel state information, a system may use SC which simply requires SNR measurements. This leads to a simple receiver structure that is hardware feasible and cheaper to implement. Another benefit of using SC as opposed to MRC is reduced power consumption at the receiver. In the early analyses of SC, the assumption was made that the fading on the $N$ received paths is Rayleigh distributed and both independent and identically distributed (i.i.d.) from path to path. Under this assumption, the average symbol error rate (SER) for $M$-QAM has been investigated [1] as well as the average bit error rate (BER) for $M$-PSK modulation [2]. However, in certain environments, it may be more appropriate to consider independent but non-identically distributed (i.n.d.) channels. There are two reasons for interest in the case of i.n.d. diversity channels. In the first place, the diversity branches in a practical system are frequently unbalanced because of differing noise figures, different feeder-line lengths, etc. In the second place, unbalanced systems perform quite effectively, and some systems, which use unequal antennas in space diversity, for example, have been designed to be unbalanced [3], [4]. To the best of our knowledge, there is no closed form expression of average SER and average BER of $M$-PAM and $M$-QAM over i.n.d. Rayleigh fading channels with SC. Hence in this paper, we derive compact expressions for the pdf of the selective combiner output which are then used to derive SER and BER of $M$-PAM as well as $M$-QAM over i.n.d. Rayleigh fading channels by exploiting the methods discussed in [5]. The rest of this paper is organized as follows. In section II, we introduce the model under study. Section III shows the formulas allowing for evaluation of the average SER and BER. Section IV, we contrast the simulations and the results yielded by theory. Finally, the paper is closed in section V.

II. SYSTEM MODEL

We consider a diversity reception system for $M$-PAM and $M$-QAM over $N$ independent but non-identically distributed Rayleigh fading channels. The complex baseband equivalent signals received over the $j$-th channel can be written as

$$r_j = h_j s + n_j, \quad j = 1, 2, \ldots, N$$

(1)

where $h_j = \alpha_j e^{i\phi_j}$ is a zero-mean complex Gaussian random variable with a Rayleigh-distributed amplitude $\alpha_j$ and a uniformly distributed phase angle $\phi_j$. Due to Rayleigh fading, the channel power, denoted by $\alpha_j^2$ is independent and exponential random variables whose means is $\lambda_j$. $s$ is...
the complex baseband transmitted signal. \(n_j\) is a zero-mean complex Gaussian random variable representing the AWGN with variance \(N_0\) which is the one-sided power spectral density in \(V^2/Hz\).

We assume matched filter detection and perfect channel estimation for the systems. The error rates of modulation schemes in slow and flat Rayleigh fading channels can be derived by averaging the error rate for the AWGN channel over the pdf of the SNR in Rayleigh fading.

\[
P(\varepsilon) = \int_0^\infty P(\varepsilon | \gamma) f_\gamma(\gamma) d\gamma
\]

where \(P(\varepsilon | \gamma)\) is the error rate conditioned on \(\gamma\), \(f_\gamma(\gamma)\) is the pdf of the instantaneous SNR per bit \(\gamma\). For SC, \(\gamma\) is defined as \(\gamma = (E_b/N_0) \max(\alpha_1^2, \alpha_2^2, \ldots, \alpha_M^2, \ldots, \alpha_N^2)\), where \(E_b\) is the average energy per bit defined as \(E_b = E_s / \log_2(M)\) with \(E_s\) is the average symbol energy.

**III. DERIVATION AND ANALYSIS**

In this section, important performance criteria for the SC system operating over i.n.d. Rayleigh fading conditions will be studied.

**A. pdf**

With selection combining, the branch with the largest bit energy-to-noise ratio is always selected so the instantaneous bit energy-to-noise ratio at the output of selector is

\[
\gamma = \max\{\gamma_1, \gamma_2, \ldots, \gamma_j, \ldots, \gamma_N\}
\]

where \(N\) is the number of branches. If the branches are independently faded then order statistics gives the cumulative distribution function (CDF).

\[
F_\gamma(\gamma) = P[\gamma_j \leq \gamma_1, \ldots, \gamma_j \leq \gamma_2, \ldots, \gamma_j \leq \ldots, \gamma_j \leq \ldots, \gamma_j \leq \gamma_N] = \prod_{j=1}^N F_{\gamma_j}(\gamma)
\]

where \(F_{\gamma_j}(\gamma) = P(\gamma_j \leq \gamma)\) is the corresponding CDF of \(\gamma_j\).

We know that when the strongest diversity branch is selected from a total \(N\) available i.n.d. diversity branches, the joint pdf of \(\gamma\) for \(N\)-branch SC is given by differentiating (4):

\[
f_\gamma(\gamma) = \frac{\partial}{\partial \gamma} \prod_{j=1}^N F_{\gamma_j}(\gamma)
\]

From (5), we have:

\[
f_\gamma(\gamma) = \sum_{j=1}^N f_{\gamma_j}(\gamma) \prod_{k=1}^N P(\gamma_k \leq \gamma) \prod_{k \neq j} P(\gamma_k \leq \gamma)
\]

where, for the Rayleigh fading channel case:

\[
f_{\gamma_j}(\gamma_j) = \frac{1}{\gamma_j} e^{-\gamma_j/\gamma_1}, \quad \gamma_j \geq 0, j = 1, 2, \ldots, N
\]

and

\[
P(\gamma_k \leq \gamma) = 1 - e^{-\gamma/\gamma_k}
\]

Substituting (8) and (7) in (6), we obtain:

\[
f_\gamma(\gamma) = \sum_{j=1}^N \frac{1}{\gamma_j} e^{-\gamma_j/\gamma_1} \prod_{k=1}^N (1 - e^{-\gamma/\gamma_k})
\]

\[
= \prod_{i_1=1}^N \gamma_{i_1}^{-1} e^{-\gamma_{i_1}} + (-1)^{j-1} \sum_{i_1 < i_2} \left( \sum_{k=1}^N \gamma_{i_1}^{-1} e^{-\gamma_{i_1} \sum_{k=1}^N \gamma_{i_1}^{-1}} \right) + \ldots
\]

\[
= \prod_{i_1=1}^N \gamma_{i_1}^{-1} e^{-\gamma_{i_1}} + (-1)^{N-2} \sum_{i_1, i_2, \ldots, i_N=1} \left( \sum_{k=1}^N \gamma_{i_1}^{-1} e^{-\gamma_{i_1} \sum_{k=1}^N \gamma_{i_1}^{-1}} \right) + \ldots
\]

\[
= \prod_{i_1=1}^N \gamma_{i_1}^{-1} e^{-\gamma_{i_1}} + (-1)^{N-1} \sum_{i_1, i_2, \ldots, i_N=1} \left( \sum_{k=1}^N \gamma_{i_1}^{-1} e^{-\gamma_{i_1} \sum_{k=1}^N \gamma_{i_1}^{-1}} \right)
\]

where

\[
C_j = \sum_{k=1}^N \gamma_{i_1}^{-1}
\]

For example with \(N = 2\):

\[
f_\gamma(\gamma) = \frac{1}{\gamma_1} e^{-\gamma/\gamma_1} (1 - e^{-\gamma/\gamma_2}) + \frac{1}{\gamma_2} e^{-\gamma/\gamma_2} (1 - e^{-\gamma/\gamma_1})
\]

\[
= \frac{1}{\gamma_1} e^{-\gamma/\gamma_1} + \frac{1}{\gamma_2} e^{-\gamma/\gamma_2} - \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) e^{-\gamma (1/\gamma_1 + 1/\gamma_2)}
\]

\[
= (-1)^0 \sum_{i_1=1}^2 \frac{1}{\gamma_{i_1}} e^{-\gamma/\gamma_{i_1}} + (-1)^1 \sum_{i_1, i_2=1}^2 \sum_{k=1}^N \gamma_{i_1}^{-1} e^{-\gamma_{i_1} \sum_{k=1}^N \gamma_{i_1}^{-1}}
\]

\[
= \sum_{j=1}^N (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_N=1} C_j e^{-C_j \gamma}
\]

where

\[
C_1 | i_1=1 = \gamma_{i_1}^{-1}, C_1 | i_1=2 = \gamma_{i_2}^{-1}, C_2 | i_2=1 = \gamma_{i_1}^{-1}
\]

\[
B. Symbol Error Rate
\]

1) Mary-PAM: For M-PAM in which \(M = 2^m\) with \(m = 1, 2, \ldots\), the SER in the AWGN channel is given in [6, p. 266, eq. (5.246)] as

\[
P_s^{PAM}(\varepsilon | \gamma) = \frac{2(M - 1)}{M} Q\left(\sqrt{\alpha \gamma}\right)
\]

where \(\alpha = 6 \log_2 M/(M^2 - 1)\).
For SC with i.n.d Rayleigh fading channels, using (2), (9) & (11), we obtain:

\[ P_{s}^{PAM}(\varepsilon) = \int_{0}^{\infty} \frac{2(M-1)}{M} Q\left(\sqrt{\alpha\gamma}\right) \times \]
\[ \sum_{j=1}^{N} \left( (-1)^{j-1} \times \int_{0}^{\infty} \frac{\pi/2}{\sin^2(\theta)} d\theta \right) C_{j} e^{-C_{j} \gamma} d\gamma \]
\[ = \frac{2(M-1)}{M} \sum_{j=1}^{N} \left( (-1)^{j-1} \times \int_{0}^{\infty} Q\left(\sqrt{\alpha\gamma}\right)C_{j} e^{-C_{j} \gamma} d\gamma \right) \]
\[ = \frac{2(M-1)}{M} \sum_{j=1}^{N} \left( (-1)^{j-1} \times \int_{0}^{\infty} \frac{\pi/2}{\sin^2(\theta)} d\theta \right) C_{j} e^{-C_{j} \gamma} d\gamma \]

where \( Q(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\exp\left(-\frac{x^2}{2\sin^2(\theta)}\right)}{\sin^2(\theta)} d\theta \) is defined in [5, p.85, eq. (4.2)]. Interchange the order of integration and some tedious manipulations, we obtain

\[ P_{s}^{PAM}(\varepsilon) = \frac{2(M-1)}{M} \sum_{j=1}^{N} \left( (-1)^{j-1} \times \int_{0}^{\infty} \frac{\pi/2}{\sin^2(\theta)} d\theta \right) C_{j} e^{-C_{j} \gamma} d\gamma \]

Using (2), (9) and (14), we have:

\[ P_{s}^{QAM}(\varepsilon) = \int_{0}^{\infty} \left[ 2pQ\left(\sqrt{\gamma}\right) - p^2Q^2\left(\sqrt{\gamma}\right) \right] \times \]
\[ \sum_{j=1}^{N} \left( (-1)^{j-1} \sum_{i_{1}, i_{2}, \ldots, i_{j}=1}^{N} C_{j} e^{-C_{j} \gamma} \right) d\gamma \]
\[ = \sum_{j=1}^{N} \left( (-1)^{j-1} \sum_{i_{1}, i_{2}, \ldots, i_{j}=1}^{N} \right. \left. \int_{0}^{\infty} pQ\left(\sqrt{\gamma}\right)C_{j} e^{-C_{j} \gamma} d\gamma \right) \]

2) M-ary Square QAM: The SER for square M-QAM in the AWGN channel is given in [6, p. 278 eq. (5.279)] as

\[ P_{s}^{QAM}(\varepsilon | \gamma) = 2pQ\left(\sqrt{\gamma}\right) - p^2Q^2\left(\sqrt{\gamma}\right) \]

where \( p = 2(1 - 1/\sqrt{M}) \), \( q = 3\log_{2} M/(M - 1) \).
Integration, we obtain using (2), (9) & (22), we get

\[ \frac{1}{M \log_{2} M} \sum_{k=1}^{\log_{2} M} \sum_{i=0}^{(1-2^{-k})M-1} A_{i}^{k} \times \]

\[ \text{erfc} \left( \sqrt{B_{i} \gamma} \right) \left\{ \left( -1 \right)^{j-1} \sum_{i_{1},i_{2},...,i_{j}=1}^{N} \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin^{2} \theta d\theta}{\sin^{2} \theta + B_{i} C_{j}} \right\} d\gamma \]

\[ = \frac{1}{M \log_{2} M} \sum_{k=1}^{\log_{2} M} \sum_{i=0}^{(1-2^{-k})M-1} \left\{ \left( -1 \right)^{j-1} \times \right\} \]

\[ \sum_{i_{1},i_{2},...,i_{j}=1}^{N} \text{erfc} \left( \sqrt{B_{i} \gamma} \right) C_{j} e^{-C_{j} \gamma} d\gamma \]

(20)

Substituting \( \text{erfc}(x) = \frac{2}{\pi} \int_{0}^{\pi/2} \exp \left( -\frac{x^{2}}{\sin^{2} \theta} \right) d\theta \) in (20) defined in [5, p. 121, eq. (4A.6)] and interchanging the order of integration, we obtain

\[ P_{b}^{\text{PAM}}(\varepsilon) = \frac{1}{M \log_{2} M} \sum_{k=1}^{\log_{2} M} \sum_{i=0}^{(1-2^{-k})M-1} \left\{ A_{i}^{k} \times \right\} \]

\[ \sum_{i_{1},i_{2},...,i_{j}=1}^{N} \left( 1 - \sqrt{B_{i} C_{j}} \right) \]

\[ = \frac{1}{\sqrt{M \log_{2} M}} \sum_{k=1}^{\log_{2} M} \sum_{i=0}^{(1-2^{-k})M-1} D_{i}^{k} \times \]

\[ \text{erfc} \left( \sqrt{F_{i} \gamma} \right) \]

\[ = \frac{1}{\sqrt{M \log_{2} M}} \sum_{k=1}^{\log_{2} M} \sum_{i=0}^{(1-2^{-k})M-1} D_{i}^{k} \times \]

\[ \text{erfc} \left( \sqrt{F_{i} \gamma} \right) \]

(21)

2) M-ary Square QAM: For M-QAM in which \( M = 4^{m} \) with \( m = 1, 2, ... \), the BER in the AWGN is given in [7] as

\[ P_{b}^{\text{QAM}}(\varepsilon | \gamma) = \frac{1}{\sqrt{M \log_{2} M}} \times \]

\[ \sum_{k=1}^{\log_{2} M} \sum_{i=0}^{(1-2^{-k})M-1} D_{i}^{k} \text{erfc} \left( \sqrt{F_{i} \gamma} \right) \]

(22)

where

\[ D_{i}^{k} = \left( -1 \right)^{i+1} \frac{i^{2k-1}}{\sqrt{M}} \left( 2^{k-1} - \frac{i^{2k-1}}{\sqrt{M}} + 1 \right) \]

(23)

\[ F_{i} = \frac{(2i+1)^{2}3 \log_{2} M}{2(M-1)} \]

(24)

Similarly, for M-PAM with i.n.d. Rayleigh fading channels, using (2), (9) & (22), we get

\[ P_{b}^{\text{PAM}}(\varepsilon) = \frac{1}{\sqrt{M \log_{2} M}} \sum_{k=1}^{\log_{2} M} \sum_{i=0}^{(1-2^{-k})M-1} D_{i}^{k} \times \]

\[ \text{erfc} \left( \sqrt{F_{i} \gamma} \right) f_{\gamma}(\gamma) d\gamma \]

(25)

\[ \sum_{j=1}^{N} (-1)^{j-1} \sum_{i_{1},i_{2},...,i_{j}=1}^{N} \left( 1 - \sqrt{G_{i} C_{j}} \right) \]

where \( G = F_{i} |_{i=0} = 3 \log_{2} (M) / [2(M-1)] \)

IV. NUMERICAL RESULTS

In this section, some examples of the average SER, BER of M-PAM and M-QAM for the case i.n.d. diversity branches in Rayleigh fading channel are given. Results computed using our theoretical analysis and Monte Carlo simulation are compared. For ease of analysis, it is assumed that \( \lambda_{j} \) with \( j = 1, \ldots, N \) are uniformly distributed between 0 and 1.

From Fig. 1 to Fig. 4, we study the average SER and BER performance for different levels of M-PAM & M-QAM modulation and different number of diversity branches. It is seen that our analytical results and the simulation results are in excellent agreement. In Fig. 5 we compare two expressions for average SER of M-QAM derived by two approaches, i.e. eq. (25) and eq. (26). It can be seen that the result obtained from approximation expression are tight with that of exact one in high region of SNR. However with 1024-QAM, there is a

Fig. 1. Exact SER for M-PAM

For high SNR, the first term \( (i = 0) \) is dominant in (25). Thus, for high SNR the BER of M-ary square QAM over i.n.d. Rayleigh channel can be approximated to a certain degree of accuracy by neglecting some of the higher order term in (25), that is

\[ P_{b}^{\text{QAM}}(\varepsilon) = \frac{1}{\sqrt{M \log_{2} M}} \sum_{j=1}^{N} \left[ (-1)^{j-1} \times \right. \]

\[ \sum_{i_{1},i_{2},...,i_{j}=1}^{N} \left( 1 - \sqrt{G_{i} C_{j}} \right) \]

\[ = \frac{1}{\sqrt{M \log_{2} M}} \sum_{j=1}^{N} \left[ (-1)^{j-1} \times \right. \]

\[ \sum_{i_{1},i_{2},...,i_{j}=1}^{N} \left( 1 - \sqrt{G_{i} C_{j}} \right) \]

(26)
small gap between two curves with SNRs are lower than 12 dB because the condition of the first term of eq. (26) has not been satisfied.

V. CONCLUSION

The performance of SC over slow and frequency-nonselective fading channels of the case of i.n.d. diversity branches has been analyzed. Novel expressions for SER and BER for M-PAM and M-QAM have been derived. Simulations results are in excellent agreement with the derived expressions. The agreement with some known results validates the analysis. The expressions are general and offer a convenient way to evaluate any system which exploits SC technique with M-PAM and M-QAM.

REFERENCES


