Optimal Dynamic Advertising Strategy
Under Age-Specific Market Segmentation

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Abstract. We consider the model proposed by Faggian and Grosset for determining the advertising efforts and goodwill in the long run of a company under age segmentation of consumers. Reducing this model to optimal control sub problems we find the optimal advertising strategy and goodwill.

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THE FAGGIAN-GROSSEY ADVERTISING MODEL

The advertising is the most used tool to communicate with the consumers. It affects both the present and future sales of the firm, and hence also the present and future revenues. For that reason Nerlove and Arrow, suggesting in [4] their model for finding optimal advertising strategy, treat the advertising expenditures as investment and the gained as a result of advertising goodwill as capital. The segmentation is another basic marketing strategy for creating competitive position of the firm. It consists of dividing the market into groups of users with similar needs and behavior. A basic characteristic of age-specific segmentation is the change of the composition of the segments in the course of time. As noted in [2], the investment through advertisements in 16 year old customers after two years is shown in the capital of 18 year old customers. Therefore, the segmentation of the goodwill is interpreted as a vintage capital in the model suggested of Faggian and Grosset in [2], similar to that introduced in [1] model of investment in capital goods.

We’ll consider the following control problem of finding the optimal advertising strategy of a company:

\[
J(u) = \int_0^\infty \int_0^\infty e^{-\rho t} \left[ \pi(a)G(t,a) - c(a,u(t,a)) \right] \, da \, dt \rightarrow \max
\]

subject to

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) G(t,a) = -\delta(a) G(t,a) + u(t,a)
\]

\[G(t,0) = 0 \quad \text{for all} \ t \in [0, \infty)\]
Here $t$ is the time and $a$ is the consumer’s age. We denote by $Q = [0, \infty) \times [0, \omega]$ the field in which the time and the age is running. It turns out that it is convenient to divide this field to the subfields $Q_1 = Q \cap \{(t, a) \mid t - a < 0\}$ and $Q_2 = Q \cap \{(t, a) \mid t - a \geq 0\}$ as shown in the figure 1 (a). The control variable is $u(t, a)$ and it represents the advertising efforts for attracting customers of the age segment $a$ at the moment $t$. The phase variable $G(t, a)$ represents the goodwill of the firm at the moment $t$ for the customers of age $a$. We assume that the firm wants to maximize the present value of the stream of profits, with $\rho > 0$ the discount rate. This present value is represented by the functional in Eq. 1. The integrand of this functional is the discounted instantaneous profit from the age segment $a$ at the moment $t$. The profit from this segment is the difference between the net revenue $\pi(a)G(t, a)$ and the advertising expenditures $c(a, u)$ for attracting customers from this age segment. We assume that $\pi(a)$ and $c(a, u)$ are continuous functions, and $c(a, u)$ is strictly convex with respect to $u$.

The dynamics of the goodwill is described by the Eq. 2, in which by $\frac{\partial}{\partial t} + \frac{\partial}{\partial a}G(t, a)$ is denoted the directional derivative of the phase variable along the characteristic direction $(1, 1)$. In other words

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)G(t, a) = \lim_{\varepsilon \to 0} \frac{G(t + \varepsilon, a + \varepsilon) - G(t, a)}{\varepsilon}$$
The left-hand side of the Eq. 2 represents the fact that the consumers are getting older in the course of time. The right-hand side states that at any point of $Q$ the change of the size of the segment $a$ is the difference between the directed to this age segment advertising efforts and the depreciation of the age segment.

The boundary condition Eq. 3 and the initial condition Eq. 4 mean, respectively, that the goodwill value of the segment 0 is always 0 and in the initial moment the initial goodwill value for all segments is given by the non-negative bounded piecewise continuous function $g(a)$.

The model represented by Eq. 1 – Eq. 5 is a slightly generalization of the proposed by Faggian and Grosset model. Here we have allowed the rate of the capital depreciation to depend on the age of the consumers. A more detailed description of the model can be found in [2].

### THE OPTIMAL SOLUTION OF THE MODEL

Faggian and Grosset solved their model by rephrasing as a control problem for an ordinary differential equation in the infinite-dimensional Hilbert space $L^2(0, \omega)$ as shown in the paper [2]. Here we will solve this model by reducing to finite horizon control problems for ordinary differential equations in finite dimensional space and by using sufficient conditions for optimality.

First let us convert the model to characteristic coordinates $(t, x)$ by introducing the variable $x = t - a$, where $x$ indicates the time of birth. In these coordinates the model is

$$J(u) = \int_0^\infty \int_{t-\omega}^t e^{-\rho t} [\pi(t-x)G(t,t-x) - c(t-x, u(t,t-x))] dx \, dt \rightarrow \max$$  \hspace{1cm} (6)

subject to

$$\frac{d}{dt} G(t,t-x) = -\delta(t-x)G(t,t-x) + u(t,t-x) \quad \text{for each } x \in [\omega, \infty)$$  \hspace{1cm} (7)

$$G(x,0) = 0 \quad \text{for } x \geq 0$$  \hspace{1cm} (8)

$$G(0,-x) = g(-x) \quad \text{for } x \in [-\omega, 0)$$  \hspace{1cm} (9)

$$u(t,t-x) \geq 0$$  \hspace{1cm} (10)

Here the dynamics is described by ordinary differential equations. The field $Q$ and its subfields $Q_1$ and $Q_2$ are displayed in the figure 1 (b).

First we will consider the above model, provided that the control variable is bounded, that is instead of Eq. 10 it is fulfilled the condition
for some \( N > 0 \). Under this condition, the arguments of the functions \( \pi(a) \) and \( c(a,u) \) are limited. So from the continuity of these functions it follows that their values are also limited.

For each fixed \( x \) Eq. 7 – Eq. 9 constitute a Cauchy problem for a linear differential equation. We know that the solution of each of these problems can be explicitly written ([5], p. 15) and that solution depends continuously on parameters and the initial condition. It follows from these properties and from the boundedness and measurability of the control \( u(t,a) \) that the solution \( (t,a) \to G(t,a) \) is a nonnegative measurable function. Furthermore we obtain from the mentioned above explicitly written solution

\[
G(t,t-x) = G_0 \exp \left( -\int_{t_0}^{t} \delta(\tau-x) d\tau \right) + \int_{t_0}^{t} \exp \left( -\int_{t_0}^{\tau} \delta(\sigma-x) d\sigma \right) u(\tau,\tau-x) d\tau \leq \]

\[
\leq G_0 + N(t-t_0)
\]

where \( G_0 = 0 \), \( t_0 = x \) if \( x \geq 0 \) and \( G_0 = g(-x) \), \( t_0 = 0 \) if \( x \in [-\omega, 0) \). In the first case \( t - t_0 = t - x = a \in [0, \omega] \). We see that \( G(t,a) \) does not exceed a linear function of \( t \). It follows from this fact and from the presence of the multiplier \( e^{-\rho t} \) that the integrand in Eq. 6 is summable on the field \( Q \). Using the Fubini’s theorem ([3], p. 298) we change the order of integration in the objective functional in Eq. 6. Thus the last equation become

\[
J(u) = \int_{-\omega}^{\omega} \int_{0}^{x+\omega} e^{-\rho t} \left[ \pi(t-x)G(t,t-x)-c(t-x, u(t, t-x)) \right] dtdx +
\]

\[
+ \int_{0}^{\infty} \int_{x}^{x+\omega} e^{-\rho t} \left[ \pi(t-x)G(t,t-x)-c(t-x, u(t, t-x)) \right] dtdx \to \max
\]

Here by the first term of the right-hand side we have expressed the double integral of Eq. 6 on the subfield \( Q_1 \) and by the second term – on the subfield \( Q_2 \).

Let us introduce for a fixed \( x \) the denotations \( \overline{u}(t) = u(t, t-x) \), \( \overline{\pi}(t) = \pi(t-x) \), \( \overline{G}(t) = G(t, t-x) \), \( \overline{c}(t,u) = c(t-x, u) \) and \( \overline{\delta}(t) = \delta(t-x) \). It is sufficient for the maximization of the upper functional for each fixed \( x \) to maximize the corresponding inner integral. Consequently the problem, described by Eq. 12, Eq. 7 – Eq. 9 and Eq. 11, can be decomposed into the following two problems:
Problem 1. To find the optimal pair \((\bar{u}, \bar{G})\) for each fixed \(x \in [-\omega, 0]\) for the next optimal control problem:

\[
J_1(\bar{u}) = \int_0^{x+\omega} e^{-\rho t} \left( \bar{\pi}(t) \bar{G}(t) - \bar{c}(t, \bar{u}(t)) \right) dt \to \max
\]

\[
\frac{d}{dt} \bar{G}(t) = -\bar{\delta}(t) \bar{G}(t) + \bar{u}(t)
\]

\[
\bar{G}(0) = g(-x) \geq 0
\]

\[0 \leq \bar{u}(t) \leq N\]

Problem 2. To find the optimal pair \((\bar{u}, \bar{G})\) for each fixed \(x \in [0, \infty)\) for the problem

\[
J_2(\bar{u}) = \int_x^{x+\omega} e^{-\rho t} \left( \bar{\pi}(t) \bar{G}(t) - \bar{c}(t, \bar{u}(t)) \right) dt \to \max
\]

\[
\frac{d}{dt} \bar{G}(t) = -\bar{\delta}(t) \bar{G}(t) + \bar{u}(t)
\]

\[
\bar{G}(x) = 0
\]

\[0 \leq \bar{u}(t) \leq N\]

These two problems are with free-end points and they differ from each other by the time interval and the initial condition. We will find their optimal solutions by using the Arrow’s sufficient conditions for optimality (see in [6] p. 107). For this purpose let us introduce the following Hamiltonian which corresponds to those two problems

\[
H(\bar{G}, \bar{u}, \psi, t) = e^{-\rho t} \left( \bar{\pi}(t) \bar{G}(t) - \bar{c}(t, \bar{u}(t)) \right) - \bar{\delta}(t) \psi \bar{G} + \psi \bar{u}
\]  \hspace{1cm} (13)

Where the adjoint variable \(\psi\) is the solution of the following Cauchy problem for the adjoint equation

\[
\dot{\psi}(t) = -e^{-\rho t} \bar{\pi}(t) + \bar{\delta}(t) \psi(t), \quad \psi(x+\omega) = 0
\]  \hspace{1cm} (14)

Since the cost function \(c(a, u)\) is convex with respect to \(u\), the Hamiltonian is concave with respect to \(\bar{u}\). Therefore a necessary and sufficient condition for its maximization in the point \(\hat{u}\) is \(\partial H(\bar{G}, \hat{u}, \psi, t)/\partial u (\bar{u} - \hat{u}) \leq 0\) for each

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admissible \( \hat{u} \). We obtain from here that if on a fixed \( t \) the control \( \hat{u}(t) \) which maximizes the Hamiltonian is in \((0, N)\), the partial derivative of \( H \) must be equal to zero, that is in this case \( \hat{u}(t) \) must be determined by the equation

\[
\hat{c}_\alpha' \left( t, \hat{u}(t) \right) = e^{\rho t} \psi(t) \quad \text{if} \quad 0 < \hat{u}(t) < N \tag{15}
\]

where \( \psi(t) \) is the solution of the Cauchy problem of Eq. 14. The maximized Hamiltonian \( \hat{H}(\hat{G}, \psi, t) = H(\hat{G}, \hat{\alpha}, \psi, t) \) is linear, therefore concave, with respect to the phase variable \( \hat{G} \). Therefore the Arrow’s sufficient condition for optimality ([16], p. 107) is satisfied and \( \hat{u}(t) \) is the optimal control for both problems.

In the foregoing considerations \( N \) was an arbitrary positive number. Taking the limit as \( N \) approaches infinity, we obtain that if the optimal control \( \hat{u}(t) \) for the model Eq. 6 – Eq. 10 is positive, it is determined by Eq. 15.

To describe completely the optimal control, we must find the adjoint variable \( \psi(t) \).

Due to the linearity of the differential equation in Eq. 14, we obtain the following explicit solution to this Cauchy problem (see in [5] p. 15):

\[
\psi(t) = \int_t^{x+\omega} \exp \left( \int_\tau^t \bar{\delta}(s) ds \right) e^{-\rho \tau} \bar{\pi}(\tau) d\tau \tag{16}
\]

From Eq. 15 and Eq. 16, returning to the original coordinates and changing the variables by which we integrate, we obtain that for the values of the arguments \( t \) and \( a \) for which the optimal control \( \hat{u}(t,a) \) of the problem Eq. 1 – Eq. 5 has positive values, it is determined by the following equation:

\[
c_u(a, \hat{u}(t,a)) = \int_a^\omega \exp \left( \int_\alpha^a \delta(\sigma) d\sigma \right) e^{\rho(a-\alpha)} \pi(\alpha) d\alpha \quad \text{if} \quad \hat{u}(t,a) > 0 \tag{17}
\]

Now it remains to find the solution for the optimal goodwill. We find the next representations for the goodwill from the Cauchy problems for the dynamics of both problems with the corresponding initial conditions:

\[
\bar{G}(t) = g(-x) \exp \left( \int_0^t \bar{\delta}(\sigma) d\sigma \right) + \int_0^t \exp \left( \int_\tau^t \bar{\delta}(\sigma) d\sigma \right) \hat{u}(\tau) d\tau \quad \text{for} \quad x < 0
\]
\[
\tilde{G}(t) = \int_x^t \exp \left( \int_0^\tau \tilde{\delta} (\sigma) d\sigma \right) \tilde{u} (\tau) d\tau 
\]

for \( x \geq 0 \)

Returning to the original coordinates and changing the variables by which we’ve integrated, from the above two equations we obtain the following representation for the optimal phase variable (optimal goodwill) for the problem Eq. 1 – Eq. 5:

\[
G(t,a) = g(a-t) \exp \left( - \int_{a-t}^a \delta (\sigma) d\sigma \right) + 
\]

\[ + \int_{a-t}^a \exp \left( - \int_\alpha^a \delta (\sigma) d\sigma \right) \dot{u} (\alpha - a + t, \alpha) d\alpha \]

and

\[
G(t,a) = \int_0^a \exp \left( - \int_\alpha^a \delta (\sigma) d\sigma \right) \dot{u} (\alpha - a + t, \alpha) d\alpha 
\]

for \((t,a) \in Q_1\) \( (18) \) and

\[ G(t,a) = \int_0^a \exp \left( - \int_\alpha^a \delta (\sigma) d\sigma \right) \dot{u} (\alpha - a + t, \alpha) d\alpha \]

for \((t,a) \in Q_2\) \( (19) \)

In this paragraph we found the optimal pair \((\dot{u}, \tilde{G})\) of control and phase trajectory for the model Eq. 1 – Eq. 5. The pair found is represented by Eq. 17, Eq. 18 and Eq. 19. In the case of linear quadratic costs, that is if

\[
c(a, u(t,a)) = \eta(a) u(t,a) + \frac{k(a)}{2} u^2(t,a) \]

we can easily get the solution, which was studied in detail in [2].

REFERENCES