Active Fault Tolerant Control of Piecewise Affine Systems with Reference Tracking and Input Constraints

M. Gholami*†, V. Cocquempot2, H. Schiøler1 and T. Bak1

1Department of Electronic Systems, Control and Automation Section, Aalborg University, DK-9220, Denmark
2LAGIS-CNRS, UMR 8219, Lille1 University, 59655 Villeneuve d’Ascq cedex, France

SUMMARY

An active fault tolerant control (AFTC) method is proposed for discrete-time piecewise affine (PWA) systems. Only actuator faults are considered. The AFTC framework contains a supervisory scheme which selects a suitable controller in a set of controllers such that the stability and an acceptable performance of the faulty system are held. The design of the supervisory scheme is not considered here. The set of controllers is composed of a normal controller for the fault free case, an active fault detection and isolation (AFDI) controller for isolation and identification of the faults, and a set of passive fault tolerant controllers (PFTCs) modules designed to be robust against a set of actuator faults. In this research, the piecewise nonlinear model is approximated by a piecewise affine (PWA) system. The passive fault tolerant controllers (PFTC) are state feedback laws. Each one is robust against a fixed set of actuator faults and is able to track the reference signal while the control inputs are bounded. The PFTC problem is transformed into a feasibility problem of a set of linear matrix inequalities (LMIs). The method is applied on a large-scale live-stock ventilation model. Copyright © 2012 John Wiley & Sons, Ltd.

Received …

KEY WORDS: Active fault tolerant controller; Passive fault tolerant controller; piecewise affine system; state feedback controller; LMIs; actuator faults

1. INTRODUCTION

Complex and industrial control systems consist of a large number of components which are strongly interconnected. Malbehaviour of a component may not only degrade the overall performance of the
system but also results in the loss of system reliability or safety. In applications such as climate control systems of livestock buildings, this is unacceptable as it may lead to the loss of animal life. Therefore, it is desirable to develop control systems which are capable of tolerating component malfunctions while still maintaining desirable performances, stability and survivability properties of the system. These control systems are known as fault tolerant control systems.

Fault tolerant control systems are classified into two broad categories; passive fault tolerant control systems (PFTCS) and active fault tolerant control systems (AFTCS), [1]: In PFTCS, the control law is fixed and does not change when a fault occurs. In fact, the control system is designed to be robust against a set of faults. The methods to design such control systems are derived from classical robust control, where the controller is designed to be insensitive to system uncertainties and disturbances, see [2], [3], and [4]. One drawback of such PFTC techniques is that the set of expected faults must be a priori known. Moreover, this set can not be too large to obtain a solution. To avoid this drawback, active fault tolerant control (AFTC) techniques may be used. In these techniques, a fault estimation scheme is needed to detect and identify the fault as soon as it occurs. The controller uses this fault information to adapt itself to the current situation. As in [5], AFTC systems are classically divided into three layers. The first layer is related to the control loop, the second layer corresponds to the FDI algorithm and accommodation scheme and the last layer corresponds to the supervisor system.

Many AFTC and PFTC results which were reported in the literature are devoted to linear continuous or discrete-time systems. However, complex industrial systems either show nonlinear behavior or contain both discrete and continuous components. One of the modeling frameworks which is relevant for such systems is piecewise affine (PWA) models. This framework has been applied in several areas, such as, switched production systems [6], aerospace systems [7], etc.

A PFTC approach is presented in [8], where, an $H_\infty$ state feedback controller is proposed for a class of continuous-time switched nonlinear systems subjected to actuator faults. The sufficient condition for asymptotic stability of the closed-loop system using the multiple Lyapunov function is given. In [9], a state feedback controller is designed for continuous-time piecewise affine (PWA) systems while an upper bound of a cost function is minimized. The design of the controller which is robust to actuator faults is cast as a set of linear matrix inequalities (LMIs). In [10], a new method for passive fault tolerant control of discrete-time PWA systems is presented. The approach is based on a reliable piecewise linear quadratic regulator (LQR) state feedback control that is tolerant against actuator faults. An upper bound of a performance cost is minimized, and the control design problem
is transformed into a convex optimization problem with LMI constraints. In [11], $H_\infty$ analysis is used to design a state feedback controller against actuator faults.

For AFTC systems, the reader is referred to [12], where the authors design a linear output feedback controller against multiple actuator failures for discrete-time switched linear systems. It is assumed that a FDI block detects and isolates the fault on-line. Authors modified the method for polytopic linear parameter varying (LPV) systems in [13]. The approach is based on a static output feedback controller, and the stability of the closed-loop system is preserved. The idea of [14] is the same as AFTC of switched linear systems where the system switches to a new system due to actuator failure such that the overall stability and performance of the system are held. The fault is detected by an adaptive filter, and it is assumed that the system is always controllable with the healthy actuators. [15] presents an AFTC method for a class of periodic switched nonlinear systems subjected to both continuous and discrete faults. The continuous fault is diagnosed by an adaptive filter and the discrete fault is diagnosed by a sliding mode observer.

In [16], an AFTC approach based on virtual sensors and actuators for continuous time PWA system subject to actuator and sensor faults is proposed. The basic idea is that the faults are hidden from the normal controller of the system. Sufficient conditions for stability and performance of the closed-loop system are given as a solution of a set of linear matrix inequalities (LMIs). The controller is designed to be insensitive to model uncertainties.

In AFTC approaches, the FDI block requires a delay to detect and identify a fault. In this delay period the fault has not been identified, and the faulty system is controlled by an irrelevant controller, the normal controller. This fact might lead to the instability of the overall closed-loop system. To reduce this time delay, we present an AFTC approach. In this approach, we substitute the FDI block with two fault detection (FD) and active fault detection and isolation (AFDI) blocks i.e., first a simple FD observer detects quickly any abnormal behaviour of the system, and after fault detection, a supervision scheme makes the system switch from normal controller to AFDI controller to isolate and identify the faults, while the overall closed-loop system is maintained stable. The AFDI controller generates a so-called excitation input, which shapes the input to the system, in order to decide whether the output represents a normal or a faulty behavior and if it is possible to decide which kind of fault has occurred. There are main advantages of our AFDI in comparison with FDI. First one is to identify the faults that may be hidden due to the regulatory actions of controllers during normal operation of the system. These kinds of faults are not identified by the FDI block. Second is to isolate and identify the faults in systems with slow response. Identifying faults in
such systems with the FDI block is impossible or takes a long period of time. As the results, our 
AFTC technique reduces the delay period, when the system is controlled by irrelevant controller, 
by the help of AFDI block. The readers are referred to [17], [18] and [19] for more details on AFDI 
controller design.

After detection and isolation of the faults, the supervisor makes the control system switch to the 
corresponding predefined PFTC which is designed off-line to be robust to a set of known faults. 
In this paper, the design methods of the AFDI controller and the supervision algorithm are not 
described in details; we mainly focus here on the design of the PFTC laws.

The illustration of the original AFTC scheme is given in Fig. 1 in section II where it includes a 
supervision layer that governs the switching mechanism between the AFDI controller and a family 
of pre-defined PFTC for discrete-time PWA systems.

In the previous PFTC researches for discrete-time PWA system, as in [10], the considered 
PWA systems switch only due to state variables. However, in general, PWA systems may switch 
based on both input and state trajectories. The other drawback of the previous PFTC researches 
is that however, most plants suffer from physical constraints, such as actuator saturation, and such 
constraints are not taken into account in the PFTC problem.

In this paper, the considered PWA systems switch due to both inputs and states trajectories. Also 
physical constraints on the inputs such as actuator saturation are taken into account. The inputs 
constraints are integrated when designing the passive fault tolerant controller. The PFTC is based 
on a state feedback controller such that the closed-loop system is asymptotically stable and is able to 
track the reference signal correctly in healthy and in actuator failure situations. A common Lyapunov 
function candidate is used to evaluate the stability of the system. The problem is cast as a set of linear 
matrix inequalities (LMI) and solved with YALMIP/ SeDumi solver, see [20].

The paper is organized as follows. Section II presents the preliminaries and the general schematic 
of our active fault tolerant control method. The fault model of piecewise affine systems is given in 
section III. Section IV discusses the control design for PWA systems. The extension of synthesis for 
fault tolerant control of piecewise affine systems is discussed in section V. Section VI is dedicated to 
the simulation results for the climate control system. The conclusion and future works are presented 
in section VII.
2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, first we describe the problem of nonlinear model approximation into piecewise affine model, then we give a general view of our active fault tolerant control scheme.

2.1. PWA Model for Nonlinear Model Approximation

In the following, the procedure to transform a nonlinear model into a PWA model is given. Consider a discrete-time piecewise nonlinear model of the form

\[ x(k+1) = f_i(x(k), u(k), k), \text{for } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{X}_i \] (1)

\[ y_m(k) = Cx(k) \] (2)

where \( u(k) \in \mathbb{R}^m \) is the control input, and \( x(k) \in \mathbb{R}^n \) is the state, \( C \in \mathbb{R}^{p \times n} \), and \( y_m(k) \in \mathbb{R}^p \) is the output. All variables are at time \( k \), the set

\[ \mathcal{X}_i = \left\{ \begin{bmatrix} x^T & u^T \end{bmatrix}^T : g_i(x, u) \leq K_i \right\} \] (3)

are manifolds (possibly un-bounded) in the state-input space, \( i = 1, \cdots, s \) is the set of indices of regions \( \mathcal{X}_i \), \( f_i \) is a vector field of the state space description, \( K_i \in \mathbb{R}^{l_i} \) and \( g_i \in \mathbb{R}^{l_i} \) is a known function.

The piecewise nonlinear model is approximated by an uncertain piecewise affine (PWA) model which is expressed in the following form:

\[ x(k+1) = (A_i + \Delta A_i)x(k) + B_i u(k) + a_i + \Delta a_i \text{ for } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{X}_i \] (4)

\[ y_m(k) = Cx(k) \] (5)

where \( A_i \in \mathbb{R}^{n \times n} \), \( B_i \in \mathbb{R}^m \) and \( a_i \in \mathbb{R}^n \) are constant matrix and represent the nominal PWA model, which are obtained from the nonlinear model, \( \Delta A_i \in \mathbb{R}^{n \times n} \) and \( \Delta a_i \in \mathbb{R}^n \) are the uncertainty terms. Let \( \mathbb{X} \subseteq \mathbb{R}^{n+m} \) be the set of every possible vectors \( \begin{bmatrix} x(k)^T & u(k)^T \end{bmatrix}^T \). \( \mathcal{X}_i \)
denotes polyhedral regions of \( \mathbb{X} \) which is obtained from \( g_i \) and \( a_i \in \mathbb{R}^n \). Each polyhedral region is represented by:

\[
\mathcal{X}_i = \left\{ x^T u^T \right\}^T \left| F_i^x x + F_i^u u \leq f_i^x u \right\} \tag{6}
\]

It is assumed that the regions are defined with known matrices \( F_i^x \in \mathbb{R}^{l_i \times n}, F_i^u \in \mathbb{R}^{l_i \times m}, \) and \( f_i^x u \in \mathbb{R}^{l_i} \). \( \mathcal{I} = \{1, \ldots, s\} \) is the set of indices of regions \( \mathcal{X}_i \). All possible switchings from region \( \mathcal{X}_i \) to \( \mathcal{X}_j \) are defined by the set \( \mathcal{S} \):

\[
\mathcal{S} = \{(i,j) : i,j \in \mathcal{I} \text{ and } \exists \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} \in \mathbb{X} \mid \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{X}_i \text{ and } \begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} \in \mathcal{X}_j \} \tag{7}
\]

\( \mathcal{I} \) is divided in two partitions. First partition is \( \mathcal{I}_0 \), which is the index set of the regions that contain the origin and \( a_i = 0 \). The second partition is \( \mathcal{I}_1 \) which is the index set of the regions that do not contain the origin.

We define an upper bound for the uncertainties as follows:

\[
\Delta \! A_i^T \Delta A_i < U_{A_i}^T U_{A_i} \quad \text{for } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{X}_i \tag{8}
\]

\[
\Delta a_i^T \Delta a_i < U_{a_i}^T U_{a_i} \tag{9}
\]

In order to obtain the uncertain PWA model (4) three steps should be carried out:

1. The polyhedral region \( \mathcal{X}_i \) (6) is obtained by approximation of the manifold \( \mathcal{X}_i \) (3).

2. A nominal PWA system is obtained by approximation of the nonlinear piecewise model (1).

3. The uncertainty bounds (8) are determined such that the original nonlinear system is contained in the uncertain PWA system (4).

The three steps are carried out by solving three problems.

The first step for approximation of the nonlinear system (1) by the uncertain PWA system (4) is presented in first problem.
Problem 1

The matrices \( F^x_i, F^u_i \), and scalar \( f^{xu}_i \) which specify the polyhedral region \( X_i \) are obtained as follows:

\[
g_i(x, u) - K_i \approx F^x_i x + F^u_i u - f^{xu}_i \quad (10)
\]

which will be reformulated as a convex optimization problem [21]:

\[
\min_{F^x_i, F^u_i, f^{xu}_i} \sum_{k=1}^{N_s} e^T(x_k)Q_i e(x_k) \quad (11)
\]

s.t. \( e(x(k)) = g_i(x(k), u(k)) - K_i - F^x_i x(k) - F^u_i u(k) + f^{xu}_i \)

\[
i = 1, \ldots, s, \quad k = 1, \ldots, N_s,
\]

where \( Q_i \) is given weighting matrices of appropriate dimensions, \( x(k) \) are the sampling points, \( s \) is the number of the polyhedral regions.

The second step for approximation of the nonlinear system (1) by the uncertain PWA system (4) is to obtain a nominal PWA system as:

\[
x(k+1) = A_i x(k) + B_i u(k) + a_i \quad \text{for} \quad \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in X_i, \quad (12)
\]

\[
y_m(k) = C x(k) \quad (13)
\]

In the following problem, it is illustrated how to obtain the nominal PWA system (12):

Problem 2

The matrices \( A_i, B_i, a_i \) which specify the state space description of (12) are obtained as follows:

\[
f_i(x(k), u(k), k) \approx A_i x + B_i u + a_i \quad (14)
\]

which will be reformulated as a convex optimization problem:

\[
\min_{A_i, B_i, a_i} \sum_{k=1}^{N_s} e^T(x(k)) R_i e(x(k)) \quad (15)
\]

s.t. \( e(x(k)) = f_i(x(k), u(k), k) - A_i x(k) - B_i u(k) - a_i \)

\[
(A_i - A_j)x^*(k) + (B_i - B_j)u^*(k) + (a_i - a_j) = 0,
\]

\[
i = 1, \ldots, s, \quad j \in N_i, \quad k = 1, \ldots, N_s,
\]
where $R_i$ is given weighting matrices of appropriate dimensions, $\mathcal{N}_i$ is the set of indices of the neighboring regions of region $i$. $x^*(k)$ and $u^*(k)$ are the sampling points corresponding to the boundary between two neighboring regions and obtained from $(F^x_i - F^x_j)x^*(k) + (F^u_i - F^u_j)u^*(k) + (f^x_i - f^x_j)x^*(k) + F^u_i u^*_\text{max}(k) - f^x_i x^*(k) + F^u_i u^*_\text{min}(k) - f^x_i x^*(k) = 0$, $F^x_i x^*_\text{max}(k) + F^u_i u^*_\text{max}(k) - f^x_i x^*(k) = 0$ and $F^x_i x^*_\text{min}(k) + F^u_i u^*_\text{min}(k) - f^x_i x^*(k) = 0$, $x^*_\text{max}(k), x^*_\text{min}(k), u^*_\text{max}(k)$ and $u^*_\text{min}(k)$ are maximum and minimum values of the state and input of the system which are obtained from physical knowledge of the system.

The last step is to obtain the uncertainty bounds (8), $W_{A_i} = U^T A_i U A_i$ and $W_{a_i} = U^T a_i U a_i$, which is described through the following problem:

**Problem 3**

Assuming the sampling points $x_k$, $k=1, \cdots, N_s$. The matrices $W_{A_i}$ and $W_{a_i}$ are obtained by solving following convex optimization problem:

$$\min_{W_{A_i}, W_{a_i}} \sum_{k=1}^{N_s} \text{trace}[W_{A_i} + W_{a_i}]$$

s.t. $$\Delta A_i x(k) + \Delta a_i = e(x(k)) = f_i(x(k), u(k), k) - A_i x(k) - B_i u(k) - a_i$$

$$\begin{bmatrix} W_{A_i} & \Delta A_i^T \\ \Delta A_i & I \end{bmatrix} > 0, \quad \begin{bmatrix} W_{a_i} & \Delta a_i^T \\ \Delta a_i & I \end{bmatrix} > 0,$$

$i = 1, \ldots, s, j \in \mathcal{N}_i$.

Note that, in this work the model uncertainties have been neglected considering that the nominal approximation (12) of the nonlinear system (1) is acceptable to design the AFTC for the animal stables, however in the future research the model uncertainties, noise or disturbance will be included.

### 2.2. Active Fault Tolerant Control Framework

Our AFTC scheme includes a family of control laws and a switching mechanism to switch between the control laws, which is done by the supervision algorithm as in Fig. 1. The control objectives are to stabilize and provide an acceptable performance of the system in normal situation as well as in faulty cases. The AFTC procedure is as follows:

- **Normal Control law:** when no fault in the system is detected, the supervisor selects this control law in the closed-loop system to satisfy the control objectives.
• **FD block:** the fault diagnosis (FD) block is an observer which estimates the output of the system at every sample instant in order to detect an abnormal behaviour of the system and inform the supervisor. For the details of the fault diagnoser, the readers are referred to observer design section of the following papers: [17], [18] and [19].

• **AFDI Controller:** once, after the supervisor receives a message from FD block denoting an abnormal behaviour of the system, the supervisor selects the active fault diagnosis control law in order to isolate and identify the current faults in the system. When the AFDI controller isolates and identifies the faults, it informs the supervisor.

• **Family of PFTC:** it consists of a family of passive fault tolerant controllers designed to be robust against specific set of faults. Once, the fault is isolated and identified by the AFDI block, the supervisory switches to the appropriate passive fault tolerant controller such that the stability and acceptable performance of the closed-loop system are satisfied.

Designing the normal control law is similar to designing passive fault tolerant control laws. The design procedure will be detailed in section V. Supervisor scheme is simple, and it includes basically a set of if-then-else rules. The precise design of this scheme is not detailed in the paper. The design of the AFDI is also not detailed in the following. The reader is referred to the corresponding literatures as for instances paper [17] and [18]. We focus in section V on the PFTC design method. Before that,
the fault model is described in the next section and the general framework of state feedback control for PWA systems is given in section IV.

3. FAULT MODEL OF PIECEWISE AFFINE SYSTEMS

Actuator faults are considered. $u_j$ is the actuator output. The partial loss of actuator efficiency can be formulated as

$$u_F^j = (1 - \alpha_j)u_j, \quad 0 \leq \alpha_j \leq \alpha_{Mj},$$  \hfill (17)

where $\alpha_j$ is the percentage of efficiency loss of the actuator $j$ and $\alpha_{Mj}$ is the maximum loss. $\alpha_j = 0$ corresponds to the nominal system, $\alpha_j = 1$ corresponds to 100% loss of the actuator and $0 \leq \alpha_j \leq 1$ corresponds to partial loss. Let us define $\alpha$ as

$$\alpha = \text{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}. \hfill (18)$$

Then

$$u_F = \Gamma u, \hfill (19)$$

where $\Gamma = (I_m \times m - \alpha)$, $I$ is the identity matrix. Thus $u_F$ represents the control signal that is applied in normal or faulty situations. The PWA model of the system with the fault $F_i$ is

$$x(k + 1) = A_i x(k) + B_i \Gamma u(k) + a_i \quad \text{for} \quad \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{X}_i \hfill (20)$$

4. STATE FEEDBACK CONTROL DESIGN

In order to cast the control problem as a set of LMIs, the polyhedral region $\mathcal{X}_i$ is over approximated by an ellipsoid when $i \in \mathcal{F}_1$.

Proposition 1

([22]) Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a parallelepiped with non-empty interior.

$$\mathcal{X} = \{x \in \mathbb{R}^n | |b_l^T x - \tilde{x}_l| \leq d_l, \ l = \{1, \cdots, n\} \}$$  \hfill (21)
and let
\[
\tilde{T} = \begin{bmatrix}
\frac{b_1^T}{d_1} & -\tilde{x}_1/d_1 \\
\vdots & \vdots \\
\frac{b_n^T}{d_n} & -\tilde{x}_1/d_n
\end{bmatrix}
\tag{22}
\]

where \(b_i\) is a constant vector, \(\tilde{x}_i\) and \(d_i\) are scalars which define the parallelepiped. Then, the ellipsoid of minimal volume that contains \(X\) is given by
\[
\tilde{x}^T \tilde{T}^T \tilde{T} \tilde{x} \leq n
\tag{23}
\]

where \(\tilde{x} = [x^T \; 1]^T\), where 1 is a scalar.

The ellipsoid of minimal volume can be reformulated as
\[
\xi = \{ x \mid \| Ex + f \| \leq n \}
\tag{24}
\]

where \(E \in \mathbb{R}^{l \times n}\) is a matrix and \(f \in \mathbb{R}^l\) is a vector which can be obtained as follow from (23):

First we write the ellipsoide (24) as
\[
(Ex + f)^T (Ex + f) \leq n.
\tag{25}
\]

Then substitute \(\tilde{T}\) in equation (23)
\[
\begin{bmatrix} x^T & 1 \end{bmatrix} \begin{bmatrix}
\frac{b_1^T}{d_1} & -\tilde{x}_1/d_1 \\
\vdots & \vdots \\
\frac{b_n^T}{d_n} & -\tilde{x}_1/d_n
\end{bmatrix}^T \begin{bmatrix}
\frac{b_1^T}{d_1} & -\tilde{x}_1/d_1 \\
\vdots & \vdots \\
\frac{b_n^T}{d_n} & -\tilde{x}_1/d_n
\end{bmatrix} \begin{bmatrix} x^T & 1 \end{bmatrix} \leq n.
\tag{26}
\]

The left-side of inequalities (25) and (26) are equivalent, which result in obtaining \(E\) and \(f\).

4.1. Reference Model

Here, the aim is to design a state feedback controller for a piecewise affine system such that the closed-loop system is able to track the reference \(r(k)\). The control structure is displayed on Fig. 2. \(K_i\) and \(K_r\) are the controller gains to be designed. \(\Sigma_r\) is the model of the reference \(r\) and its state.
The state space representation is given as:

\[
x_r(k + 1) = A_r x_r(k) + B_r (r(k) - C x(k))
\]  

(27)

where \(x_r(k) \in \mathbb{R}^{n_r}\) is the state vector, here \(p = n_r\). A well known asymptotic tracking of a reference \(r(k)\) is achieved by putting an integral action in the closed-loop, i.e. by fixing:

\[
A_r = I_{n_r \times n_r}, 
B_r = T_s \times I_{n_r \times p},
\]

where \(T_s\) is the sampling time of the system.

### 4.2. Controller Structure

Let a piecewise linear state feedback control be specified as:

\[
u(k) = K_i x(k) + K_r x_r(k) \quad for \quad \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] \in X_i,
\]  

(28)

where \(K_i\) and \(K_r\) are controller gains which are designed to stabilize exponentially the closed-loop PWA system. Since the index \(i\) is not a priori known, it is not possible to calculate \(u(k)\). Hence, the problem is changed into the following structure

\[
u(k) = K x(k) + K_r x_r(k) = \bar{K} \bar{x} \quad for \quad \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] \in X_i,
\]  

(29)

which means that we consider the same controller in all regions \(X_i\) with \(i \in \mathcal{I}\).

With considering reference model (27) and piecewise affine model (4) and applying the control law (29) the following closed-loop system is obtained:

\[
\bar{x}(k + 1) = \begin{bmatrix} x(k + 1) \\ x_r(k + 1) \end{bmatrix} = A_i \bar{x}(k) + \bar{B}_r r(k) + \bar{a}_i \quad for \quad \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] \in X_i,
\]  

(30)
where $A_i = \bar{A}_i + \bar{B}_i \bar{K}$, $\bar{A}_i = \begin{bmatrix} A_i & 0_{n \times n_r} \\ -B_r C & A_r \end{bmatrix}$, $\bar{B}_i = \begin{bmatrix} B_i \\ 0_{n_r \times m} \end{bmatrix}$, $\bar{B}_r = \begin{bmatrix} 0_{n \times n_r} \\ B_r \end{bmatrix}$, $\bar{K} = [K \ K_r]$

and $\bar{a}_i = \begin{bmatrix} \bar{a}_i \end{bmatrix}$, $0$ is a null matrix or vector.

Let describe now how to over-approximate a polyhedral region $X_i$ by an ellipsoid using Proposition 1. Let $X_i$ be reformulated as

$$X_i = \{ \bar{x}^T u^T \in \mathbb{R}^{n+n_r+m} | \| (b_i^T + c_i^T \bar{K}) \bar{x} - \bar{x}_i \| \leq d_i \}$$

which is equivalent to

$$X_i = \{ \bar{x}^T \in \mathbb{R}^{n+n_r} | \| (b_i^T + c_i^T \bar{K}) \bar{x} - \bar{x}_i \| \leq d_i \}$$

Then, the ellipsoid of minimal volume that contains $X_i$ is given by

$$\tilde{\xi}_i = \{ \bar{x} | \| (E_i + F_i \bar{K}) \bar{x} + f_i \| \leq 2 \}$$

The following theorem gives the sufficient conditions for stability of a piecewise affine system.

**Theorem 1**

(23) System (30) is exponentially stable if there exist matrices $P_i = P_i^T > 0$, $\forall i \in \mathcal{I}$, such that the positive definite function $V(x(k)) = x^T(k)P_i x(k)$, $\forall x \in X_i$, satisfies $V(x(k+1)) - V(x(k)) < 0$.

5. PASSIVE FAULT TOLERANT CONTROL OF PIECEWISE AFFINE SYSTEMS

The control objective is to track the reference $r(k)$ when the system (30) is subject to a fault $F_i$. The system (30) subject to the fault $F_i$ is given by:
\[
\bar{x}(k+1) = \begin{bmatrix}
  x(k+1) \\
  x_r(k+1)
\end{bmatrix} = A_{if} \bar{x}(k) + \bar{B}_r r(k) + \bar{a}_i \\
\text{for } \begin{bmatrix}
  x(k) \\
  u(k)
\end{bmatrix} \in \mathcal{X}_i, \tag{35}
\]

where \(A_{if} = \bar{A}_i + \bar{B}_i \Gamma \bar{K}, \xi_i = \{\bar{x} | (E_i + F_i \Gamma \bar{K}) \bar{x} + f_i \| \leq 2\}\).

### 5.1. Fault Tolerant Controller without Input Constraints

**Lemma 1**

Let \(M, N, H\) be real matrices. If \(H^T H \leq I\), \(I\) is an identity matrix with an appropriate dimension, then for every scalar \(\epsilon > 0\) the following inequality holds:

\[
MHN + N^T H^T M^T \leq \epsilon MM^T + \epsilon^{-1} N^T N \tag{36}
\]

**Definition 1**

The control law (29) is a passive fault-tolerant control if the closed-loop system (35), which is subject to fault \(F_i\), is exponentially stable i.e. the following inequality for system (35) is satisfied:

\[
V(x(k+1)) - V(x(k)) < 0 \ \forall \ [x(k) \in \mathcal{X}_i, \text{and} \ x(k+1) \in \mathcal{X}_j|(i,j) \in \mathcal{S}]\].

The following Theorem gives sufficient conditions for the passive fault tolerance.

**Theorem 2**

The fault tolerant linear controller (29) stabilizes the system (35), if there exist a symmetric matrix \(Q = Q^T > 0\), a matrix \(Y\) and positive constants \(\mu_i, \epsilon_i\) such that:

\[
\begin{bmatrix}
  Q & \ast & \ast & \ast \\
  \bar{A}_i Q + \bar{B}_i \bar{Y} & Q + \frac{1}{2} \mu_i \bar{a}_i \bar{a}_i^T - \epsilon_i \bar{B}_i \bar{B}_i^T & \ast & \ast \\
  E_i Q + F_i \bar{Y} & \frac{1}{2} \mu_i f_i f_i^T - \epsilon_i F_i F_i^T & \mu_i (\frac{1}{2} f_i f_i^T - I) - \epsilon_i F_i F_i^T & \ast \\
  \alpha Y & 0 & 0 & \epsilon_i I
\end{bmatrix}
\]

\[
> 0 \ \forall \ [x(k) \in \mathcal{X}_i, \text{and} \ x(k+1) \in \mathcal{X}_j|(i,j) \in \mathcal{S}], i \in \mathcal{I}_1, \tag{37}
\]

\[
\begin{bmatrix}
  Q & \ast \\
  \bar{A}_i Q + \bar{B}_i \bar{Y} & Q - \epsilon_i \bar{B}_i \bar{B}_i^T \\
  \alpha Y & 0 & \epsilon_i I
\end{bmatrix}
\]

> 0, \tag{38}
∀ \[x(k) \in X_i, \text{ and } x(k+1) \in X_j\] \((i, j) \in S\), \(i \in I_0\)

Then the linear feedback gains are given by:

\[
\bar{K} = YQ^{-1},
\]  

(39)

Proof

Let consider a common Lyapunov candidate function as \(V(x(k)) = x(k)^T P x(k)\), \(P > 0\) for \(x(k) \in X_i\). The sufficient stability condition is:

\[
V(x(k+1)) - V(x(k)) < 0, \ \forall \ [x(k) \in X_i, \text{ and } x(k+1) \in X_j].
\]  

(40)

First, we assume those switchings with \(i \in I_1\). To treat with the affine term \(\bar{a}_i\), the ellipsoidal approximation of regions as in (34) is considered. The equivalent form of (40) for the PWA system (35) is:

\[
[A_{id} x(k) + \bar{a}_i]^T P [A_{id} x(k) + \bar{a}_i] - x(k)^T P x(k) < 0, \ \forall \ x(k) \in X_i,
\]  

(41)

which is equal to:

\[
\Theta = \begin{bmatrix} x(k) \end{bmatrix}^T \begin{bmatrix} A_{id}^T P A_{id} - P & * \\ \bar{a}_i^T P A_{id} & \bar{a}_i^T P \bar{a}_i \end{bmatrix} \begin{bmatrix} x(k) \\ 1 \end{bmatrix} < 0, \ \forall \ x(k) \in X_i,
\]  

(42)

where \(A_{id} = \bar{A}_i + \bar{B}_i \Gamma \bar{K}\) and 1 is scalar. In the following, we integrate the matrix inequality \(\Theta\) with the regions condition \(X_i\) using the S-procedure, see [25], in order to simplify equation (42). To achieve this aim, the \(X_i\) is written in terms of matrix inequality using ellipsoidal approximation as:

\[
\begin{bmatrix} x(k) \end{bmatrix}^T \begin{bmatrix} \mathcal{E}_i^T \mathcal{E}_i & * \\ f_i^T \mathcal{E}_i & f_i^T f_i - 2 \end{bmatrix} \begin{bmatrix} x(k) \\ 1 \end{bmatrix} \leq 0,
\]  

(43)

where \(\mathcal{E}_i = E_i + F_i \Gamma \bar{K}\). Using the S-procedure, the equation (42) is satisfied if there exist multipliers \(\lambda_i > 0\) such that :

\[
\Theta - \lambda_i \begin{bmatrix} x(k) \end{bmatrix}^T \begin{bmatrix} \mathcal{E}_i^T \mathcal{E}_i & * \\ f_i^T \mathcal{E}_i & f_i^T f_i - 2 \end{bmatrix} \begin{bmatrix} x(k) \\ 1 \end{bmatrix} < 0
\]  

(44)
The above inequality is equal to:

$$\begin{bmatrix} A_i^T P A_i - P & \lambda_i \bar{E}_i^T \bar{E}_i \\ \bar{a}_i^T P A_i & \bar{a}_i^T P \bar{a}_i \end{bmatrix} - \lambda_i \begin{bmatrix} E_i^T E_i & * \\ f_i^T f_i & f_i^T f_i - 2 \end{bmatrix} < 0, \quad (45)$$

This inequality is nonlinear, in the following we transform it to linear terms. The inequality is rearranged as:

$$\begin{bmatrix} P + \lambda_i E_i^T E_i & * & * \\ \lambda_i f_i^T f_i & \lambda_i (f_i^T f_i - 2) & * \\ A_i f & \bar{a}_i & P^{-1} \end{bmatrix} > 0. \quad (46)$$

Applying Schur complement to the above equation yields to:

$$\begin{bmatrix} P + \lambda_i E_i^T E_i & * & * \\ \lambda_i f_i^T f_i & \lambda_i (f_i^T f_i - 2) & * \\ A_i f & \bar{a}_i & P^{-1} \end{bmatrix} > 0. \quad (47)$$

By pre- and Post-multiplying the above equation with

$${\text{diag}} \{ I, \begin{bmatrix} 0 & * \\ I & 0 \end{bmatrix} \}, I \text{ is a identity matrix with an appropriate dimension, it is obtained:}$$

$$\begin{bmatrix} P + \lambda_i E_i^T E_i & * & * \\ \lambda_i f_i^T f_i & \lambda_i (f_i^T f_i - 2) & * \\ A_i f & \bar{a}_i & P^{-1} \end{bmatrix} > 0. \quad (48)$$

By applying the Schur complement, it is obtained that:

$$\begin{bmatrix} P + \lambda_i E_i^T E_i & * \\ A_i f & P^{-1} \end{bmatrix} - \begin{bmatrix} \lambda_i E_i^T f_i \\ \bar{a}_i \end{bmatrix} \lambda_i^{-1} (f_i^T f_i - 2)^{-1} \begin{bmatrix} E_i^T E_i & \bar{a}_i^T \end{bmatrix} > 0. \quad (49)$$

Using the matrix inversion Lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \quad (50)$$
, $Q = P^{-1}$ and $\mu_i = \lambda_i^{-1}$, the inequality (49) can be reformulated as:

$$\Delta + \mu_i (\frac{1}{2} + \frac{1}{4} f_i^T (I - \frac{1}{2} f_i f_i^T)^{-1} f_i) \Lambda > 0$$  \hspace{1cm} (51)

where $\Delta$ is the first term of inequality (49) and $\Lambda = [\mu_i^{-1} f_i^T \varepsilon_i \bar{a}_i^T]$. The above inequality can be written as:

$$\Delta + \frac{1}{2} \mu_i \Lambda + \frac{1}{4} \left[ \varepsilon_i^T f_i f_i^T \right] (I - \frac{1}{2} f_i f_i^T)^{-1} [\mu_i^{-1} f_i f_i^T \varepsilon_i \bar{a}_i^T] > 0. \hspace{1cm} (52)$$

Which is equal to:

$$\Delta + \frac{1}{2} \mu_i \Lambda + \left[ \mu_i^{-1} (\varepsilon_i - (I - \frac{1}{2} f_i f_i^T) \varepsilon_i) \right] (I - \frac{1}{2} f_i f_i^T)^{-1} \left[ \mu_i^{-1} f_i f_i^T \varepsilon_i \bar{a}_i^T \right] > 0. \hspace{1cm} (53)$$

Let us define

$$\Lambda_{11} = \mu_i^{-1} \varepsilon_i^T (I - \frac{1}{2} f_i f_i^T)^{-1} \varepsilon_i - \mu_i^{-1} \varepsilon_i - \mu_i^{-1} \varepsilon_i + \mu_i^{-1} \varepsilon_i, \hspace{1cm} (54)$$

$$\Lambda_{12} = \frac{1}{2} \varepsilon_i^T (I - \frac{1}{2} f_i f_i^T)^{-1} f_i \bar{a}_i^T - \frac{1}{2} \varepsilon_i^T f_i \bar{a}_i^T,$$

$$\Lambda_{21} = \frac{1}{2} \bar{a}_i f_i^T (I - \frac{1}{2} f_i f_i^T)^{-1} \varepsilon_i - \frac{1}{2} \bar{a}_i f_i^T \varepsilon_i,$$

$$\Lambda_{22} = \frac{1}{4} \mu_i \bar{a}_i f_i^T (I - \frac{1}{2} f_i f_i^T)^{-1} f_i \bar{a}_i^T.$$

Then, the equation (53) is rearranged as:

$$\begin{bmatrix} Q^{-1} + \mu_i^{-1} \varepsilon_i^T \varepsilon_i & A_{1f}^T \\ A_{1f} & Q \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \mu_i^{-1} \varepsilon_i^T f_i f_i^T \varepsilon_i & \frac{1}{2} \varepsilon_i^T f_i \bar{a}_i^T \\ \frac{1}{2} \bar{a}_i f_i^T \varepsilon_i & \frac{1}{2} \mu_i \bar{a}_i \bar{a}_i^T \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} > 0 \hspace{1cm} (54)$$
which can be rewritten as

\[
\begin{bmatrix}
Q^{-1} & * \\
A_i & Q + \frac{1}{2} \mu_i \bar{a}_i \bar{a}_i^T
\end{bmatrix} - \\
\left[
\begin{array}{c}
\mathcal{E}_i^T \\
\frac{1}{2} \mu_i \bar{a}_i \bar{a}_i^T
\end{array}
\right] \mu_i^{-1} (I - \frac{1}{2} f_if_i^T)^{-1} \left[
\begin{array}{c}
\mathcal{E}_i \\
\frac{1}{2} \mu_i f_i f_i^T
\end{array}
\right] > 0.
\] (55)

By applying the Schur complement, it is obtained that:

\[
\begin{bmatrix}
Q^{-1} & * & * \\
\bar{A}_i + \bar{B}_i \Gamma \bar{K} & Q + \frac{1}{2} \mu_i \bar{a}_i \bar{a}_i^T & * \\
E_i + F_i \Gamma \bar{K} & \frac{1}{2} \mu_i f_i f_i^T & \mu_i (\frac{1}{2} f_i f_i^T - I)
\end{bmatrix} > 0,
\] (56)

By substituting \(\Gamma = I - \alpha\), the left side of (56) is expressed as:

\[
\begin{bmatrix}
Q^{-1} & * & * \\
\bar{A}_i + \bar{B}_i \Gamma \bar{K} & Q + \frac{1}{2} \mu_i \bar{a}_i \bar{a}_i^T & * \\
E_i + F_i \Gamma \bar{K} & \frac{1}{2} \mu_i f_i f_i^T & \mu_i (\frac{1}{2} f_i f_i^T - I)
\end{bmatrix} - \\
\begin{bmatrix}
\bar{K}^T \alpha^T \\
0
\end{bmatrix}
\begin{bmatrix}
0 & B_i^T & F_i^T
\end{bmatrix} - \\
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\left[
\begin{array}{c}
\alpha \bar{K} \\
F_i
\end{array}
\right].
\] (57)

Using Lemma 2 in [10] with \(H = -I\), it is obtained that:

\[
(57) \geq \Delta - \epsilon_i^{-1} \begin{bmatrix}
\bar{K}^T \alpha^T \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\alpha \bar{K} \\
0 \\
F_i
\end{bmatrix} - \epsilon_i \begin{bmatrix}
0 \\
\bar{B}_i \\
F_i
\end{bmatrix} \begin{bmatrix}
0 & B_i^T & F_i^T
\end{bmatrix},
\] (58)

where \(\Delta\) is the first term of inequality (57). We have \(\alpha \leq \alpha_M\), therefore it holds that:

\[
(58) \geq \\
\begin{bmatrix}
Q^{-1} - \epsilon_i^{-1} \bar{K}^T \alpha^T M \bar{K} & * & * \\
\bar{A}_i + \bar{B}_i \Gamma \bar{K} & Q + \frac{1}{2} \mu_i \bar{a}_i \bar{a}_i^T - \epsilon_i \bar{B}_i \bar{B}_i^T & * \\
E_i + F_i \Gamma \bar{K} & \frac{1}{2} \mu_i f_i f_i^T - \epsilon_i F_i F_i^T & \Lambda
\end{bmatrix},
\] (59)

where \(\Lambda = \mu_i (\frac{1}{2} f_i f_i^T - I) - \epsilon_i F_i F_i^T\).
Applying Schur complement, pre- and post-multiplying (59) by $\text{diag}\{Q, I, I\}$, and defining $Y = \bar{K}Q$, we obtain the LMI (37).

For the regions that contain the origin i.e. $i \in I_0$, we have $\frac{1}{2}f_i f_i^T - I < 0$ which means that the LMI (37) is not feasible. For these regions, the LMI (38) is considered and there is no need to include the region information. Therefore, the following matrix inequality must be satisfied:

$$[(\bar{A}_i + \bar{B}_i \Gamma \bar{K}_i)^T P[(\bar{A}_i + \bar{B}_i \Gamma \bar{K}_i)] - P < 0, \forall [x(k) \in X_i, \text{ and } x(k+1) \in X_j],$$

(60)

which can be shown to be equivalent to (38). The proof is similar to the previous part. □

5.2. Fault Tolerant Controller with input constraints

Here we develop the previous results to deal with the input constraints. The satisfaction of the constraints are expressed in terms LMIs.

**Theorem 3**

Assume that there exist a symmetric matrix $Q = Q^T > 0$, a matrix $Y$ and positive constants $\mu_i > 0, \epsilon_i > 0$, such that the LMIs (37) and (38) are satisfied and

$$\begin{bmatrix} -1 & * \\ x(0) & -Q \end{bmatrix} < 0$$

(61)

$$\begin{bmatrix} -u^2_{v,\text{max}}Q & * & * & * \\ Y & -I & * & * \\ E_i + F_iY & 0 & \mu_i(\frac{1}{2}f_i f_i^T - I) & * \\ \alpha Y & 0 & 0 & -\epsilon_i I \end{bmatrix} \leq 0 \forall i \in \mathcal{I}_1,$$

(62)

$$\begin{bmatrix} -u^2_{v,\text{max}}Q & * \\ Y & -I \end{bmatrix} < 0 \forall i \in \mathcal{I}_0, \quad v = 1, 2, \ldots, m$$

(63)

where $u_{v,\text{max}}$ is a known control input bound and $m$ is the number of inputs, then the fault tolerant controller gain is obtained as $\bar{K} = YQ^{-1}$, and the closed-loop system (35) subject to fault $F_i$ has these properties:

(i) It is exponentially stable
(ii) The input constraints are satisfied, i.e.

\[ |u_v| \leq u_{v,\text{max}} \quad v = 1, 2, \ldots, m \]  

(64)

\textbf{Proof}

Property (i) can be proved following the same way as the proof of Theorem 2. We just focus on the proof of property (ii). In order to treat the norm constraints on the input \( u = \bar{K}x \), the stabilizability is specified in terms of holdable ellipsoids i.e. the ellipsoid \( \Omega = \{ x \in \mathbb{R}^n | x^T Q^{-1} x \leq 1 \} \) is holdable for the system (35) if there exist a state feedback gain \( \bar{K} \) and LMIs (37) and (38) such that \( \Omega \) is invariant for the system (35) [25]. In the following we integrate the condition \( \Omega \), the approximated ellipsoids of \( X_i \) and equation (64) as a matrix inequality.

Using (64), the S-procedure, with multipliers \( \lambda_i = \mu_i^{-1} > 0 \) for approximated ellipsoids for each region \( X_i \) and \( \lambda_{ie} = u_{v,\text{max}}^2 > 0 \) for holdable ellipsoids, we can obtain that:

\[
(u^T v - u_{v,\text{max}}^2 - u_{v,\text{max}}^2 x^T Q^{-1} x - 1) - \lambda_i \begin{bmatrix} x(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} x^T E_i \\ f_i^T E_i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leq 0,
\]

(65)

which is equivalent to

\[
\begin{bmatrix} \bar{K}^T \bar{K} - u_{v,\text{max}}^2 Q^{-1} - \lambda_i E_i^T E_i & * \\ -\lambda_i f_i^T E_i & -\lambda_i (f_i^T f_i - 2) \end{bmatrix} \leq 0.
\]

(66)

As in proof of Theorem 2, by using the Schur complement, the following LMI is obtained:

\[
\begin{bmatrix} -u_{v,\text{max}}^2 Q^{-1} & * & * \\ -\bar{K} & -I & * \\ E_i & 0 & -\lambda_i^{-1} (\frac{1}{2} f_i f_i^T - I) \end{bmatrix} \leq 0
\]

(67)

which is the same as:

\[
\begin{bmatrix} -u_{v,\text{max}}^2 Q & * & * \\ Y & -I & * \\ E_i Q + F_i \Gamma Y & 0 & -\lambda_i^{-1} (\frac{1}{2} f_i f_i^T - I) \end{bmatrix} \leq 0.
\]

(68)

As for the previous proof and using Lemma 2 and \( H = -I \), we can obtain the LMI (62). In the same way we can obtain the LMI (63) for \( x \in X_i \) with \( i \in \mathcal{J}_0 \). \( \square \)
6. APPLICATION EXAMPLE: ACTIVE FAULT TOLERANT CLIMATE CONTROL OF A LIVE-STOCK BUILDING

The method is applied to a climate control systems of a live-stock building, which was studied in previous research, [26]. The general schematic of the large scale live-stock building equipped with hybrid climate control system is illustrated in Fig. 3. In a large scale stable, the indoor airspace is incompletely mixed; therefore it is divided into conceptually homogeneous parts called zones. In our model, there are three zones which are not similar in size. Zone 1, the one on the left, is the biggest and Zone 2, the middle one, is the smallest. Due to the indoor and outdoor conditions, the airflow direction varies between adjacent zones. In more details, each flow direction depends on its relevant condition (invariant condition), and as long as the condition is met by the states, the system behavior is expressed according to the appropriate dynamic equations. Once the state violates the invariant condition and satisfies a new one, the system behavior is defined with a new equation. Therefore, the system behavior is represented by a finite number of different dynamic equations. The model is divided into subsystems as follows: Inlet model for both windward and leeward, outlet model, and stable heating system, and finally the dynamic model of temperature based on the heat balance equation.

![Airflow diagram](image_url)

Figure 3. The top view of the test stable

6.1. Inlet Model

An inlet is built into an opening in the wall, and it consists of a hanged flap for adjusting amount and direction of incoming air. In [27] and [26] the following approximated model is suggested for
airflow $q_{z}^{in}$ into the zone $z$

$$q_{z}^{in} = k_z(\alpha_z + \text{leak})\Delta P_{\text{inlet}}^{z}$$  \hspace{1cm} (69)$$

$$\Delta P_{\text{inlet}}^{z} = 0.5C_pV_{\text{ref}}^{2} - P_z + \rho g(1 - \frac{T_z}{T_o})(H_{NLP} - H_{\text{inlet}})$$  \hspace{1cm} (70)$$

where $P_z$ is the pressure inside zone $z$, $k_z$ and $\text{leak}$ are constants, $\alpha_z$ is the opening angle of the inlets, $\Delta P_{\text{inlet}}^{z}$ is the pressure difference across the opening area and wind effect, $\rho$ is the outside air density, $V_{\text{ref}}$ is the wind speed, $C_p$ stands for the wind pressure coefficient. $H$ stands for height and $H_{NLP}$ is the neutral pressure level which is calculated from mass balance equation. $T_z$ and $T_o$ are temperature inside and outside the stable, and $g$ is the gravity constant.

6.2. Outlet Model

The outlet is a chimney with an electrically controlled fan and plate inside. A simple linear model for the airflow out of zone $z$ is given by:

$$q_{z}^{\text{fan}} = V_{\text{fan}}^{z}c_z - d_z\Delta P_{\text{outlet}}^{z}$$  \hspace{1cm} (71)$$

$$\Delta P_{\text{outlet}}^{z} = 0.5C_pV_{\text{ref}}^{2} - P_z + \rho g(1 - \frac{T_z}{T_o})(H_{NLP} - H_{\text{outlet}})$$  \hspace{1cm} (72)$$

$$\sum_{z=1}^{3} q_{z}^{in} \rho \frac{\Delta P_{\text{inlet}}^{z}}{|\Delta P_{\text{inlet}}^{z}|} + \sum_{z=1}^{3} q_{z}^{\text{fan}} \rho = 0$$  \hspace{1cm} (73)$$

where $c_z$ and $d_z$ are constants, $V_{\text{fan}}^{z}$ is fan voltage and the number of zones is 3. It must be noted that the entire space of stable is divided into three conceptual zones where $P_z$ related to each zone can be calculated from applying equation (69-73) for each zone.

More details about the relevant conditions for the airflow direction are illustrated in Fig. 4.
The stationary flows, $q_{z-1,z}^{st}$ and $q_{z,z+1}^{st}$, which moves through the zonal border of two adjacent zones is given by:

$$q_{z-1,z}^{st} = m_1(P_{z-1} - P_z)$$
$$q_{z,z+1}^{st} = m_2(P_z - P_{z+1})$$
$$q_{z-1,z}^{st} = \{ q_{z-1,z}^{st} \}^+ - \{ q_{z-1,z}^{st} \}^-$$

where $m_1$ and $m_2$ are constants coefficients. The use of curly brackets is defined as:

$$\{ q_{z-1,z} \}^+ = \max(0, q_{z-1,z}^{st}), \quad \{ q_{z-1,z} \}^- = \min(0, q_{z-1,z}^{st})$$

### 6.3. Stable Heating Model

The following model is used to represent heating:

$$Q_{heater}^z = C_1(T_z - T_{win}^z)C_2$$

$$C_1 = \dot{m}_{heater}c_{pwater}$$

$$C_2 = \exp\left[ \frac{-U_{heater}A_{pipe}}{\dot{m}_{heater}c_{pwater}} \right] - 1$$

where $\dot{m}_{heater}$ is the mass flow rate of heating system, heat capacity is presented by $C_{pwater}$, $T_{win}$ is temperature of incoming flow of heating system, $U_{heater}$ is the overall average heat transfer coefficient, and $A_{pipe}$ is the cross area of the pipe in the heating system. In order to derive more precise stable heating model, $C_2$ is estimated from the laboratory experiments.

### 6.4. Modeling Climate Dynamics

The following formulation for the dynamical model of the temperature for each zone inside the stable is driven by thermodynamic laws. The dynamical model includes four piecewise nonlinear models which describe the heat exchange between adjacent zones:

$$M_zc_z\frac{\partial T_z}{\partial t} = Q_{z-1,z} + Q_{z,z-1} + Q_{z,z+1} + Q_{z+1,z} + Q_{in,z} + Q_{out,z} + Q_{conv,z} + Q_{source,z}$$

$$Q = \dot{m}c_pT_z, \quad Q_{z-1,z} = \{ q_{z-1,z}^{st} \}^+ \rho c_p T_{z-1}, \quad Q_{z,z-1} = \{ q_{z-1,z}^{st} \}^- \rho c_p T_z$$
where $Q_{in,z}$ and $Q_{out,z}$ represent the heat transfer by mass flow through inlet and outlet, $Q_{z-1,z}$ denotes heat exchange from zone $z-1$ to zone $z$ which are caused by stationary flow between zones. $Q_{conv} = UA_{wall}(T_z - T_o)$ is the convective heat loss through the building envelope, $Q_{source,z}$ is the heat source, $\dot{m}$ is the mass flow rate, $c_z$ and $c_p$ are the heat capacity and $M_z$ is the mass of the air inside zone $z$.

The state space model is given by

$$\dot{T} = f_j(T, U, q) \quad \text{for} \quad \begin{bmatrix} T \\ U \end{bmatrix} \in \mathcal{X}_j, \quad j = 1, \ldots, 4$$

(83)

$$q = h_3(T, P, U) = \begin{bmatrix} q_{in}^{z}, q_{1}^{z}, q_{2}^{z}, q_{3}^{z}, q_{4}^{zout} \end{bmatrix}^T, \quad z = 1, \ldots, 3$$

(84)

$$h_2(P, T, U) = 0, \quad U = \begin{bmatrix} \alpha_z, V_{fan}^{ij}, Q_{source,z} \end{bmatrix}^T$$

(85)

$$y = CT, \quad T = [T_1, T_2, T_3] = x.$$ 

(86)

where $f_j$ is dedicated to each piecewise state space model, $T$ represents the state vector, $h_2$ denotes the mass balance equation (73) for obtaining the indoor pressure in each zone and $U$ is the system inputs. In the Nomenclature, the parameters of the system for better understanding of the model are provided.

6.5. PWA Approximation and Model Validation

In the following, the nonlinear model (83) is approximated by a discrete-time PWA system with 4 regions based on the airflow direction, i.e. each nonlinear model of the region $j$ is approximated by a linear model. The models are derived for the following polyhedral regions:

<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 1.2$</td>
<td>the outside air density</td>
</tr>
<tr>
<td>$V_{ref} = 14$</td>
<td>the ambient wind speed</td>
</tr>
<tr>
<td>$C_P = 0.21$</td>
<td>the wind pressure coefficient</td>
</tr>
<tr>
<td>$H = 1.42$</td>
<td>Height</td>
</tr>
<tr>
<td>$H_{NLP}$</td>
<td>NLP stands for the neutral pressure level</td>
</tr>
<tr>
<td>$g = 9.8$</td>
<td>gravity constant</td>
</tr>
<tr>
<td>$C_{P_{uation}} = 0.21$</td>
<td>the wind pressure coefficient</td>
</tr>
<tr>
<td>$c_p = 1007 \frac{Kg.K}{J}$</td>
<td>heat capacity</td>
</tr>
<tr>
<td>$T_o = 2^\circ C$</td>
<td>temperature outside of the stable</td>
</tr>
<tr>
<td>$V_z = 1000 m^3$</td>
<td>volume of the zone $z$</td>
</tr>
</tbody>
</table>
\( X_1 = \{ [x^T u^T]^T | F_1^x x + F_1^u \geq f_1, F_2^x x + F_2^u \geq f_2 \} \), \hfill (87) \\
\( X_2 = \{ [x^T u^T]^T | F_1^x x + F_1^u < f_1, F_2^x x + F_2^u < f_2 \} \), \hfill (88) \\
\( X_3 = \{ [x^T u^T]^T | F_1^x x + F_1^u < f_1, F_2^x x + F_2^u \geq f_2 \} \), \hfill (89) \\
\( X_4 = \{ [x^T u^T]^T | F_1^x x + F_1^u \geq f_1, F_2^x x + F_2^u < f_2 \} \), \hfill (90) \\

where

\[
F_1^x = \begin{bmatrix} 1.0817 & -0.0457 & -0.9938 \end{bmatrix} \\
F_2^x = \begin{bmatrix} -1.1144 & 0.0490 & 1.0187 \end{bmatrix} \\
F_1^u = \begin{bmatrix} 0.2323 & -0.0072 & 0.2323 & 0.2323 & -0.0072 \\
0.2323 & -0.072 & 0.1349 & -0.0719 & -0.0064 \end{bmatrix}, \\
F_2^u = \begin{bmatrix} -0.2558 & 0.0074 & -0.2558 & -0.2558 & 0.0074 \\
-0.2558 & 0.0742 & -0.12 & 0.0742 & 0.0074 \end{bmatrix}, \\
f_1 = 0.4058, f_2 = -0.4575 \hfill (91) \\

The discrete-time PWA model is described by:

\[
A_1 = \begin{bmatrix} 1.6361 & 0.0480 & -0.7716 \\
1.5782 & 0.5522 & -0.9983 \\
0.7747 & 0.0462 & 0.0990 \end{bmatrix}, A_2 = \begin{bmatrix} 1.1145 & -0.0300 & -1.0590 \\
1.6452 & 0.1010 & -1.4342 \\
0.3008 & 0.0191 & -0.2324 \end{bmatrix}, \hfill (92) \\
A_3 = \begin{bmatrix} 1.6340 & 0.0259 & -0.7150 \\
1.5474 & 0.8335 & -1.4790 \\
0.7674 & 0.0314 & 0.1456 \end{bmatrix}, A_4 = \begin{bmatrix} 1.6274 & 0.0049 & -0.6987 \\
1.6242 & 0.8163 & -1.4751 \\
0.7623 & 0.0051 & 0.1640 \end{bmatrix}. \hfill (93)
\[
B_1 = \begin{bmatrix}
-0.1163 & 0.0459 & -0.1163 & -0.1163 & 0.0459 \\
0.5718 & -0.3768 & 0.5718 & 0.5718 & -0.3768 \\
-0.1147 & 0.0353 & -0.1147 & -0.1147 & 0.0353 \\
-0.1163 & 0.0018 & -0.0567 & 0.0018 & 0.0070 \\
0.5718 & -0.1518 & 0.2724 & -0.1518 & -0.0056 \\
-0.1147 & 0.0022 & -0.0553 & 0.0022 & 0.0071
\end{bmatrix}, \quad (94)
\]

\[
B_2 = \begin{bmatrix}
0.1137 & -0.0044 & 0.1137 & 0.1137 & -0.0044 \\
-0.0104 & 0.1057 & -0.0104 & -0.0104 & 0.1057 \\
0.0581 & 0.0258 & 0.0581 & 0.0581 & 0.0258 \\
0.1137 & -0.0697 & 0.2883 & -0.0697 & 0.0023 \\
-0.0104 & 0.0183 & 0.8276 & 0.0183 & 0.1275 \\
0.0581 & 0.0097 & 0.0939 & 0.0097 & 0.0273
\end{bmatrix}, \quad (95)
\]

\[
B_3 = \begin{bmatrix}
-0.0677 & -0.0127 & -0.0677 & -0.0677 & -0.0127 \\
0.2031 & 0.0778 & 0.2031 & 0.2031 & 0.0778 \\
-0.0697 & -0.0188 & -0.0697 & -0.0697 & -0.0188 \\
-0.0677 & -0.0103 & -0.0080 & -0.0103 & 0.0078 \\
0.2031 & -0.0594 & -0.0506 & -0.0594 & -0.0012 \\
-0.0697 & -0.0098 & -0.0087 & -0.0098 & 0.0075
\end{bmatrix}, \quad (96)
\]

\[
B_4 = \begin{bmatrix}
-0.0393 & -0.0380 & -0.0393 & -0.0393 & -0.0380 \\
0.0851 & 0.1683 & 0.0851 & 0.0851 & 0.1683 \\
-0.0414 & -0.0434 & -0.0414 & -0.0414 & -0.0434 \\
-0.0393 & -0.0133 & -0.0234 & -0.0133 & 0.0086 \\
0.0851 & -0.0568 & 0.0160 & -0.0568 & 0.0029 \\
-0.0414 & -0.0130 & -0.0241 & -0.0130 & 0.0085
\end{bmatrix}, \quad (97)
\]

\[
a_1 = \begin{bmatrix}
0.4749 \\
-0.9236 \\
0.4214
\end{bmatrix}, \quad a_2 = \begin{bmatrix}
-0.0676 \\
2.2442 \\
0.3784
\end{bmatrix}, \quad a_3 = \begin{bmatrix}
0.2356 \\
0.3694 \\
0.2500
\end{bmatrix}, \quad a_4 = \begin{bmatrix}
0.3510 \\
-0.5021 \\
0.3682
\end{bmatrix}. \quad (98)
\]
The large scale stable has a slow dynamic behavior with time constants around 30 minutes. The actuator settings (control signals) for ventilation systems are a Pseudo-Random Digital Signal (PRDS) with time granularity of 30 minutes and an amplitude variation. The validation of the PWA model is carried out for open loop and with the inputs signals which were not used in the estimation process. Then the output of the PWA model is compared with the original piecewise nonlinear model. Fig. 5 presents the associated data of the indoor temperature of the stable for each zone. It illustrates that there is a neglectable discrepancy attributed to linearization error. This error is acceptable for indoor temperature of the animal stables.

![Graph of indoor temperature for every zone](image)

**Figure 5. Simulation results for indoor temperature for every zone**

### 6.6. Simulation Results of Fault Tolerant Controllers

The simulation results are performed for the initial nonlinear system. Here, first the simulation results for a passive fault tolerant controller (PFTC) are illustrated, thereafter the general active fault tolerant controller (AFTC) is implemented and the results are shown.
We assume that the PFTC objective is to tolerate actuator faults. The climate control system contains 10 actuators, 6 inlets, 3 fans and a heating system, note here it is assumed that the heating system for each zone is not independent from others and the three heating systems are controlled by a three-way valve. Each of inlets consists of 6 or 12 connected inlets. In order to show the performance of the PFTC, 3 of the 6 inlets and 1 of the 3 fans are assumed to become faulty at a given time with 95% efficiency loss. \( x(0) = [8^\circ C, 8^\circ C, 8^\circ C] \) and the aim is to regulate the temperature of each zone around 3\(^\circ\) C. The PFTC based on Theorem 3 is designed for temperature regulation while the control inputs due to the physical restrictions are bounded. Here \( \mathcal{J}_0 = 1 \) and \( \mathcal{J}_1 = 2, 3, 4 \). The LMIs problem is solved with YALMIP/SeDuMi solver. The linear controller and common Lyapunov function are obtained as:

\[
\bar{K} = \begin{bmatrix}
-0.0520 & -0.0058 & 0.0558 & 0.0000 & 0.0001 & 0.0000 \\
-0.0082 & -0.0029 & 0.0106 & 0.0000 & 0.0001 & 0.0000 \\
-0.0520 & -0.0058 & 0.0559 & 0.0000 & 0.0001 & 0.0000 \\
-0.2620 & -0.0256 & 0.3088 & 0.0030 & 0.0030 & -0.0051 \\
-0.0493 & 0.0040 & 0.0038 & -0.0011 & 0.0038 & -0.0032 \\
-0.1817 & -0.0248 & 0.2241 & -0.0066 & 0.0033 & 0.0042 \\
4.4924 & 0.1510 & -4.2053 & -0.0287 & -0.0443 & -0.0264 \\
1.8256 & 0.2009 & -2.0118 & -0.0048 & -0.0027 & -0.0044 \\
0.0202 & 0.0023 & -0.0207 & -0.0000 & -0.0000 & -0.0000 \\
-3.8247 & -0.3018 & 0.3386 & 0.0454 & 0.1155 & 0.0499
\end{bmatrix}
\]

(99)

\[
\bar{P} = \begin{bmatrix}
0.15341 & -0.00063 & -0.14801 & -0.00014 & -0.00023 & 0.00014 \\
-0.00061 & 0.00070 & -0.00031 & -0.00004 & -0.00002 & -0.00001 \\
-0.14800 & -0.00033 & 0.14612 & 0.00013 & 0.00023 & -0.00014 \\
-0.00015 & -0.00005 & 0.00014 & 0.00001 & 0.00002 & -0.00004 \\
-0.00021 & -0.00003 & 0.00022 & 0.00004 & 0.00001 & 0.00003 \\
0.00015 & -0.00002 & -0.00014 & -0.00003 & 0.00003 & 0.00001
\end{bmatrix}
\]

(100)
Fig. 6 shows the temperature of each zone, the fault tolerant controller is able to track the reference signal after 30 hours when there is no fault in the system, i.e., at first the controller cools down the stable using the entire ventilation system then it regulates the temperature around $3^\circ C$. As it is clear in graph, the temperature of each zone drops down to around $0.5^\circ C$ for a short period of time. The reason is that the controller uses the entire ventilation system, which includes 5 large fans and 64 inlets, to cooling down. As the result, the strong circulation of outdoor air with temperature of $0^\circ C$ inside the live-stock building decreases the temperature very fast. The heating system is not as fast as the cooling system, and indoor temperature will remain around $0.5^\circ C$ for a short time. By using a larger heating system, it is possible to reduce this period of time or avoid the temperature to drop down to $0.5^\circ C$.

Figure 6. Simulation results with a PFT controller designed to tolerate $95\%$ actuator failure for the fault-free system
In the next step, it is assumed that 3 of 6 inlets and 1 of 3 fans lose 95% of their efficiency after 30 minutes. Fig. 7 shows that the controller with a small oscillation is still able to track precisely the reference signal after 40 hours.

Figure 7. Simulation results with a PFT controller designed to tolerate 95% actuator failure for the faulty system

Due to physical limitations the control input can not exceed a given boundary. In our application case, the boundaries are as follows; angle of the inlets is $-1 \leq a_{inlet} \leq 1$, voltage of the fan is $-10 \leq V_{fan} \leq 10$, and temperature of heating system is $-50 \leq T_{heating} \leq 50$.

When these boundaries are not taken into account in the control design procedure, the obtained control signals are plotted in Fig. 8. It is obvious that the control limits are exceeded. When integrating these control boundaries in the design method, we obtain the results reported in Fig. 9, 10 and 11. The simulation results of the reference tracking error for both PFT controllers with and
Figure 8. Control inputs when no input constraints are fulfilled in the control design problem. Angle of the inlets, voltages of the fans and temperature of the heating system. The inlet 1 to 3 and fan 3 lose 95% of their efficiency.

Figure 9. Angle of the inlets when input constraints are fulfilled in the PFT control design problem. The inlet 1 to 3 and fan 3 lose 95% of their efficiency.

without input boundaries are illustrated in Fig. 12. Obvious, these boundaries result in a large error at first, however the error tends to zero after a while.

In this step the global AFTC scheme and a classical AFTC scheme are applied on the climate control system. In our AFTC, as in Fig. 13, first, the system is controlled by a normal controller, which is shown in blue dash line in Fig. 13, as in general schematic of AFTC, Fig. 1. The controller regulates the temperature around 3°C. At time 166.6 hours, a fault happens, and it is obvious from Fig. 13, the normal controller is not able to track the references. The fault detection (FD) block
detects an abnormal behaviour of the system after one hour, thereafter the supervisor makes the system switch to AFDI controller to detect and isolate the faults. The AFDI controller excites the system to identify the faults as is shown red colour in the figure. The AFDI controller identifies the faults after three hours, then the supervisor switches to an appropriate PFTC. The simulation results show that the PFTC, which is shown in green colour, is able to track the references after one oscillation. The different control signals with respect to input constraints are given in Fig. 14, 15, and 16.

In the classical AFTC, as in Fig. 17, first the system is controlled by a normal controller. After occurrence the fault at time 166.6, a fault detection and isolation (FDI) block will detect and isolate the faults while the closed loop system is controlled by the same normal controller. After 24 hours the FDI block identifies that inlet 1 and fan 3 are faulty while three inlets were faulty. As the results, the supervisor switches to a wrong PFTC. The simulation results confirm that the controller is not able to regulate the temperature around $3^\circ C$. 
7. CONCLUSIONS AND FUTURE WORKS

In this paper, we design an active fault tolerant control (AFTC) scheme for discrete-time PWA systems based on a supervision algorithm and a set of passive fault tolerant controllers (PFTC), which are designed off-line, each PFTC is robust against a set of known faults. When a fault occurs in the system, the fault detection (FD) block sends an information to the supervisor about the occurrence of a fault. The supervision scheme, which consists of a set of logic rules, e.g. if-then-else rules, switches from normal controller to active fault detection and isolation (AFDI) controller to detect and isolate precisely the faults. After fault isolation, the supervisor makes the system switch to the appropriate passive fault tolerant controller such that the system remains stable and a degraded performance is held. The design of the supervision scheme is simple and it has not been detailed here. By using a common Lyapunov function for stability analysis, a state feedback controller is designed such that the closed-loop system is able to track the reference signal in healthy situation.
Figure 13. Simulation results of the AFT controller which includes a normal controller, AFDI and family of PFT controllers. The AFTC is able to regulate the temperature around $3^\circ C$ when three inlets and one of the fans loose 95% their efficiency.

as well as in the faulty actuator case. In many industrial systems, the control inputs can not take any value and they should be less than a given bound. Here, the input limitations are also integrated in the control design. The results show that the closed-loop system tracks the reference signal precisely while the actuators are subject to 95% efficiency loss. However the input limitation slows down the controller and causes some tracking error, this error tends to zero after some hours.

Note that, the results show that in the systems with slow response, the FDI block might not be able to identify the fault correctly and which could lead the supervisor scheme to switch to a wrong PFTC where the closed-loop system is not able to track the references.
Figure 14. Angle of the inlets for different normal, AFDI and the PFT controllers. The inlet 1 to 3 and fan 3 lose 95% of their efficiency.

Figure 15. Voltages of the fans for different normal, AFDI and the PFT controllers. The inlet 1 to 3 and fan 3 lose 95% of their efficiency.

Figure 16. Temperature of the heating system for different normal, AFDI and the PFT controllers. The inlet 1 to 3 and fan 3 lose 95% of their efficiency.

For future works, it is addressed that a system is not always full-state observable, therefore it is recommended to use an output-feedback controller instead of a state-feedback one. The model uncertainties should also be investigated.
Figure 17. Simulation results of a classical AFT controller which includes a normal controller, and family of PFT controllers. This AFTC is unable to regulate the temperature around $3^\circ C$ when three inlets and one of the fans loose 95% their efficiency.

REFERENCES


