On the Semantics of Perception-Based Fuzzy Logic Deduction

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In this article, we return to the problem of the derivation of a conclusion on the basis of fuzzy IF–THEN rules. The, so called Mamdani method is well elaborated and widely applied. In this article, we present an alternative to it. The fuzzy IF–THEN rules are here interpreted as genuine linguistic sentences consisting of the so called evaluating linguistic expressions. Sets of fuzzy IF–THEN rules are called linguistic descriptions. Linguistic expressions derived on the basis of an observation in a concrete context are called perceptions. Together with the linguistic description, they can be used in logical deduction, which we will call a perception-based logical deduction. We focus on semantics only and confine ourselves to one specific model. If the perception-based deduction is repeated and the result interpreted in an appropriate model, we obtain a piecewise continuous and monotonous function. Though the method has already proved to work well in many applications, the nonsmoothness of the output may sometimes lead to problems. We propose in this article a method for how the resulting function can be made smooth so that the output preserves its good properties. The idea consists of postprocessing the output using a special fuzzy approximation method called F-transform. © 2004 Wiley Periodicals, Inc.

1. INTRODUCTION

Although there are already hundreds of papers focusing on elaboration of fuzzy IF–THEN rules and on the theory of approximate reasoning (among many of them, let us only mention Refs. 1–3), we return in this article again to this problem. Usually, fuzzy IF–THEN rules are interpreted as special fuzzy relations (special formulas) that imprecisely characterize some dependence. The fuzzy relations are derived from fuzzy sets, which for better understandability, are labeled by some expressions of natural language but without pretension that these fuzzy sets are indeed an apt explication of the meaning of the latter. The reason is that in

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this case, the goal of approximate reasoning is to approximate some function being characterized by these fuzzy relations. Thence, the proper meaning of the labeling linguistic expressions is unimportant. The extensively used Mamdani approximate reasoning method (and its variants) is thus mainly manipulation with fuzzy relations and no formal deduction proceeds. One of the important arguments in favor of using it in fuzzy control is continuity and smoothness of its output.

However, we are convinced that the potential of fuzzy set theory is large enough to model the genuine meaning of at least part of the natural language expressions and that fuzzy IF–THEN rules can be taken and interpreted as special conditional sentences. We confine ourselves to the case when they consist of the so-called, evaluating linguistic expressions (expressions like “big,” “roughly medium,” etc.). The relation between expressions forming a fuzzy IF–THEN rule is understood as a linguistically characterized logical implication. When finding an appropriate formal logical system, we may translate fuzzy IF–THEN rules into formulas of it and, hence, we are open to formal manipulation common in logic.

We will deal with sets of fuzzy IF–THEN rules, which we call linguistic descriptions. Each linguistic description can be translated into a set of formulas of some chosen logical system. The translation, however, must fit the genuine linguistic meaning of the occurring linguistic expressions as well as possible.

There are two formal systems in which such a translation has been described, namely, fuzzy logic with evaluated syntax and a fuzzy intensional logic. In both of them, a linguistic description is used for establishing a formal logical theory within which logical deduction is possible.

To speak about logical deduction, we suppose to be given an observation that for us is some element (value) encountered in a concrete context (in the article, we speak about a possible world). Then, we can characterize the observation by an evaluating linguistic expression, which becomes our perception of the given value. Consequently, the perception can be translated into a formula of the above-mentioned formal system and, together with the linguistic description, a logical deduction can proceed. The derived conclusion is a formula that represents certain evaluating expression characterizing values in various contexts. Finally, choosing a specific context, we can find a value that is a result of the deduction. We will call this procedure a perception-based deduction (the inspiration comes from Ref. 9).

The perception-based deduction often leads at each step to firing of one fuzzy IF–THEN rule only. This is in accordance with our intuition and we argue that when a linguistic description of some situation is given, then our method gives results that are close to conclusions derived by people when facing the same situation. This fact, together with clear understandability of linguistic descriptions to people (because they use simple expressions of natural language) make perception-based deduction attractive for applications, especially when expert-related solutions are necessary (among them we can rank also fuzzy control).

Let us consider a specific model. When repeating the perception-based deduction for all elements of the model and interpret the results of formal derivation back in it, we obtain a function, which, unlike the Mamdani method, is in general only piecewise continuous and monotonous. Applications of perception-based deduction in fuzzy control demonstrated that this is not a principal obstacle (see,
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e.g., Ref. 10). However, because the output is not smooth enough, the behavior of the active element need not be desirable. Therefore, we proposed an improved perception-based deduction whose output is continuous and smooth but at the same time preserves its main advantages mentioned above. The main idea is to postprocess the output using the so-called F-transform. The latter is a fuzzy approximation technique, which can be applied in various tasks, for example, smooth filtering of data, solving differential equations, and so forth (see Refs. 11 and 12).

In this article, we prefer fuzzy intensional logic as the basic formal logical system because its means enable us to formulate the theory of meaning of natural language in a most elegant way. However, this article is not logical, and so the reader will not find here precise definitions of formal language, formula, interpretation, or description of all formal steps of the perception-based deduction. The interested reader is referred to Refs. 8 and 13. Instead, we confine ourselves to one specific model and focus only on the semantics. The reason is that we are interested more in the results that perception-based deduction can bring us and our goal is to improve them.

In Section 2, we first briefly review the concept of evaluating linguistic expressions, outline a mathematical model of their semantics, and show how to model the meaning of the above considered fuzzy IF–THEN rules. In Section 3 we show how perceptions can be modeled and introduce the perception-based logical deduction. In Section 4, we briefly describe the method of F-transform and then show how the perception-based deduction and F-transform can be joined together. In Section 5, we compare the behavior of the original and smooth perception-based deduction on some examples and demonstrate their power and advantages.

A fuzzy set $A$ in the universe $V$, in symbols $A \subseteq V$, is identified with a function $A : V \rightarrow [0,1]$ (the function $A$ is also called the membership function of the fuzzy set $A$). By $\mathcal{F}(V)$ we denote the set of all fuzzy sets on $V$.

Elements from $[0,1]$ are interpreted as truth values. We suppose that they form Łukasiewicz algebra (understood as a residuated lattice)

$$\mathcal{L}_L = ([0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1)$$

where $\vee$ is the operation of maximum, $\wedge$ that of minimum, $a \otimes b = 0 \vee (a + b - 1)$ is Łukasiewicz conjunction and $a \rightarrow b = 1 \wedge (1 - a + b)$ is Łukasiewicz implication ($a, b \in [0,1]$). As a special case, we introduce the operation of negation by $\neg a = a \rightarrow 0 = 1 - a$.

2. THE THEORY OF EVALUATING LINGUISTIC EXPRESSIONS

The perception-based logical deduction assumes that the linguistic description is formulated using the so-called evaluating linguistic expressions, for example “very large,” “extremely deep,” “roughly one thousand,” “more or less hot,” etc. In this article, we confine ourselves only to the simplest case of these expressions. More about them and their theory can be found in Refs. 6, 7, and 14.
2.1. The Class of Simple Evaluating Linguistic Expressions

The simple evaluating expressions have the form

\[(\text{linguistic hedge}) \cdot (\text{atomic evaluating expression})\]

where \textit{atomic evaluating expressions} are words of type “small,” “medium,” or “big.” Let us stress that these words should be taken as \textit{canonical} and can be replaced by any other cases such as “thin,” “thick,” “old,” “new,” and so forth. Specific are also fuzzy quantities, namely, “approximately $x_0$.” Atomic evaluating expressions usually form \textit{pairs of antonyms} and when completed by a middle term, such as “medium,” “average,” and so on, they form the so-called \textit{fundamental linguistic trichotomy}.

\textit{Linguistic hedges} (introduced by L.A. Zadeh) are special adverbs that modify the meaning of adjectives before which they stand (cf. also Refs. 2 and 16). We distinguish hedges with \textit{narrowing effect} (very, significantly, etc.) and \textit{widening effect} (more or less, roughly, etc.).

It is important that a missing linguistic hedge is understood as the presence of an \textit{empty linguistic hedge}. Hence, all simple evaluating expressions can be treated equally. In the following, we will use script letters $A, B, \ldots$ to denote evaluating expressions.

\textit{Evaluating linguistic predications} are expressions of the form “(noun) is $A$” where $A$ is an evaluating linguistic expression. Because in our considerations, we are usually not interested in specific nouns, we replace “(noun)” by some variable $X$ and assume that its values are real numbers.

The fuzzy IF–THEN rule is a conditional linguistic clause of natural language characterizing a relation between two evaluating predications, which has the form

\[ R := \text{IF } X \text{ is } A \text{ THEN } Y \text{ is } B \quad (1) \]

The part before THEN is called the \textit{antecedent} and the part after it is called the \textit{succeedent} (for simplicity, we confine ourselves to one antecedent variable only in this article; generalization to more variables is straightforward).

2.2. Informal Characterization of the Semantics of Evaluating Linguistic Expressions

We outline here some ideas about the construction of a mathematical model of the meaning of evaluating expressions.

In any model of the semantics of linguistic expressions, we must make a distinction between their intension and extensions in various possible worlds. A \textit{possible world} is a state of our world at a given moment and place. A possible world can also be understood as a \textit{particular context} in which the linguistic expression is used. Let us stress that the term “possible world” intension and extension (see further) have been introduced by Carnap. In this article, we prefer to use the term \textit{linguistic context} (or simply \textit{context}) instead of possible world.
Linguistic expressions can be generally taken as names of properties. Then an \textit{intension} of a linguistic expression is an abstract construction that conveys a property denoted by the expression. Consequently, we can take linguistic expressions to be names of intensions. Intension is invariant with respect to various contexts (possible worlds), and so, each linguistic expression is a name of just one intension.

An \textit{extension} of a linguistic expression is a class of objects determined by its intension in a given context (possible world). Thus, it depends on a particular context of use and it changes whenever the context (time, place) is changed.

For example, the expression “deep” is the name of an intension being a certain property of depth, which in a concrete context may mean 1 cm when a beetle needs to cross a puddle, 3 m in a small lake, but 3 km or more in the ocean.

Let us now formalize the previous reasoning. We will begin with more general definitions but soon make them specific only for the case of evaluating expressions.

We consider a sufficiently large set \(V\), the elements of which will be taken to form extensions of all thinkable linguistic expressions. In the case of evaluating expressions, we may put \(V = \mathbb{R}\). Furthermore, let \(W\) be a set of elements that will represent contexts. The original idea of Carnap\(^{17}\) was to define \textit{intension} as a \textit{function} from the set of contexts (possible worlds) to the set of extensions. Following it, we will define intension of an evaluating expression as a function

\[
A : W \rightarrow \mathcal{F}(V)
\]

that is, the \textit{extension} in the given context \(w \in W\) is a \textit{fuzzy set}

\[
A(w) \subseteq V
\]

In other words, extension is a functional value of the intension \(A\) in the given context \(w\). Let us remark that Carnap was led to his simple definition of intension by the generally accepted requirement that extension of an expression should be fully definable from its intension.

We will often write \(A_w\) instead of \(A(w)\). Summing up the previous considerations, we conclude that each linguistic expression is assigned some intension of the form 2.

For example, “small” is a name of an intension \(S_m : W \rightarrow \mathcal{F}(\mathbb{R})\). In each context \(w \in W\), the extension of “small” is a certain fuzzy set of real numbers.

It is difficult to specify what a concrete context means. Hence, in a mathematical model of linguistic meaning the set \(W\) is taken as a set of some abstract parameters representing various contexts without attempt to specify them more closely. Fortunately, the evaluating linguistic expressions seem to be simple enough to make possible an explicit definition of the set of contexts. Namely, we will put

\[
W = \{(v_L, v_S, v_R) | v_L, v_S, v_R \in [0, \infty) \text{ and } v_L < v_S < v_R\}
\]

\(^{a}\)Of course, homonyms behave as if they have more intensions, but these should be taken as various expressions having equal surface form.
Elements $u \in [0,\infty]$ may belong to various contexts. If $w \in W$ is a context, $w = (v_L, v_S, v_R)$, then $u \in w$ means that $u \in [v_L, v_R]$ and we say that $u$ belongs to the context $w$.

The justification for taking contexts as elements of the set $3$ is based on ideas of Vopěnka. Each context $w \in W$ is delineated by two distinguished points $v_L, v_R$, which represent a left limit and a right limit, respectively. All values that fall in the context $w$ lay between them. Because (in the given context!) nothing can be smaller than $v_L$ or bigger than $v_R$, these points are the “most typical” small value and the “most typical” big value, respectively. The properties of being “small” and “big” are the so-called primary recordable properties, which are vague. We can point a small value (e.g., $v_L$) but there does not exist the last small value. The only thing we know is that small values run somewhere toward a certain point that is the horizon of our seeing of small values. Everything that is beyond this point is surely not small. Note that this reasoning contains the sorites paradox.

Quite similarly, starting from $v_R$ and going in the opposite direction, we find a horizon of big values such that everything beyond it is surely not big. In our model, we will identify both horizons with one point $v_S$, which lies somewhere between $v_L$ and $v_R$. This point represents a central limit. However, we cannot be sure about the precise position of $v_S$. Therefore, we specify our uncertainty about this using degrees of truth expressing that “we find ourselves still before the horizon.”

Consequently, extensions of the evaluating expressions characterizing small values lie between $v_L, v_S$ and those characterizing big values lie between $v_S, v_R$ (with the direction from $v_R$ to $v_S$).

The expressions characterizing medium values are determined by the point $v_S$, which is the “most typical” medium and their extensions lay around it.

### 2.3. Mathematization of the Semantics of Evaluating Linguistic Expressions

To model mathematically the above reasoning, we start with a fuzzy logic model of the sorites paradox (for the detailed analysis of it see Refs. 7 and 19). The simplest semantic interpretation of the sorites paradox leads to a couple of linear functions $L, R : W \times \mathbb{R} \to [0,1]$ defined in each context $w \in W$ by

$$L_w(x) = \left( \frac{v_S - x}{v_S - v_L} \right)^*$$

$$R_w(x) = \left( \frac{x - v_S}{v_R - v_S} \right)^*$$

\[ \text{In this article, we will consider only positive numbers to fall into the meaning of extensions of evaluating expressions. Generalization to the whole } \mathbb{R} \text{ is possible and requires a fine model that reverses ordering when speaking about “negative small,” “negative very big,” and so forth.}\]

\[ \text{One grain does not form a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.}\]
where the asterisk means cut of all values to interval \([0,1]\). The functions \(L_w, R_w\) describe the idea of running toward the horizon because \(L_w(v_L) = 1\) and \(L_w(u) = 0\) for \(u \geq v_S\). If the truth value \(L_w(u) > 0\) for \(u \in [v_L, v_S]\), then \(u\) lies still before the horizon \(v_S\), and so, it may fall into extension of some evaluating expression characterizing small values. If \(L_w(u) = 0\), then \(u\) lies beyond the horizon. An analogous interpretation for big values has \(R_w\).

There are also values which are neither small nor big. These are characterized by the function

\[
M_w(x) = \neg L_w(x) \land \neg R_w(x) = \left(\frac{x - v_L}{v_S - v_L}\right)^* \land \left(\frac{v_R - x}{v_R - v_S}\right)^*
\]

The functions \(L, R, M\) are fundamental in further construction of our model of the meaning of evaluating expressions. For each possible world, extensions of the expressions characterizing the respective small, big, or medium values lie before the corresponding horizon. To mathematize this, we proceed as follows.

Let us consider a class \(\mathcal{H}_f\) of functions that we will call abstract hedges (horizon deformations). Elements of \(\mathcal{H}_f\) are continuous functions \(\nu_{a,b,c} : [0,1] \rightarrow [0,1]\) determined by some parameters \(a < b < c\) so that \(\nu_{a,b,c}(y) = 0\) for \(y \leq a\), \(\nu_{a,b,c}(y) = 1\) for \(c \leq y\), and it is increasing otherwise. We explicitly put

\[
\nu_{a,b,c}(y) = \begin{cases} 1, & c \leq y \\ 1 - \frac{(e - y)^2}{(e - b)(e - a)}, & b \leq y < c \\ \frac{(y - a)^2}{(b - a)(c - a)}, & a \leq y < b \\ 0, & y < a \end{cases}
\]

Note that there are infinitely many possible functions \(\nu_{a,b,c}\). Our goal was to consider the simplest nonlinear one because according to known psychological investigations, the shapes of membership functions should be nonlinear. However, we cannot reject the function \(\nu_{a,b,c}\) to be possibly even linear.

Now we define the following classes of functions that will serve as possible intensions of evaluating expressions. We will distinguish three type of functions according to their future role as intensions:

(i) S-intensions:

\[
\text{Sm} = \{\text{Sm}_{\nu} : W \rightarrow \mathcal{F}(\mathbb{R}) | \text{Sm}_{\nu, w}(x) = \nu(L_w(x)), \nu \in \mathcal{H}_f\}
\]

(ii) M-intensions:

\[
\text{Me} = \{\text{Me}_{\nu} : W \rightarrow \mathcal{F}(\mathbb{R}) | \text{Me}_{\nu, w}(x) = \nu(M_w(x)), \nu \in \mathcal{H}_f\}
\]

(iii) B-intensions:

\[
\text{Bi} = \{\text{Bi}_{\nu} : W \rightarrow \mathcal{F}(\mathbb{R}) | \text{Bi}_{\nu, w}(x) = \nu(R_w(x)), \nu \in \mathcal{H}_f\}
\]
Let \( A \) be an evaluating expression. Then its intension \( \text{Int}(A) \) is a function from one of the above classes, that is,

\[
\text{Int}(A) \in \text{Sm} \cup \text{Me} \cup \text{Bi}
\]  

(5)

Let \( w \in W \) be a context. Then the extension of \( A \) in the context \( w \) is

\[
\text{Ext}_w(A) = \text{Int}(A)(w) \subseteq [\nu_L, \nu_R]
\]  

(6)

The construction of extensions of evaluating expressions is depicted in Figure 1 where

\[
\begin{align*}
 c_{\text{Sm}} &= L_w^{-1}(c) & a_{\text{Sm}} &= L_w^{-1}(a) \\
 c_{\text{Me}}^1 &= (\neg L_w)^{-1}(c) & a_{\text{Me}}^1 &= (\neg L_w)^{-1}(a) \\
 c_{\text{Me}}^2 &= (\neg R_w)^{-1}(c) & a_{\text{Me}}^2 &= (\neg R_w)^{-1}(a) \\
 c_{\text{Bi}} &= R_w^{-1}(c) & a_{\text{Bi}} &= R_w^{-1}(a)
\end{align*}
\]

(for the discussion of the resulting shape see Ref. 20).

In the following, we will often use a general variable \( \text{Ev} \), which represents intension of some evaluating linguistic expression, that is,

\[
\text{Ev} \in \text{Sm} \cup \text{Me} \cup \text{Bi}
\]  

(7)

Similarly as above, we will usually write \( \text{Ev}_w \) for the extension of \( \text{Ev} \) in a context \( w \), instead of \( \text{Ev}(w) \). By abuse of language, we will also quite often blur the distinction between evaluating expression and its intension, that is, we may say “the evaluating expression \( \text{Ev} \)” having in mind either the expression together with its intension or only its intension. We hope that this will not cause a misunderstanding.

Let us remark that the meaning of fuzzy quantities is modeled similarly as that of the expressions of type “medium.” For simplicity, we have omitted details in this article.

\[\text{Figure 1. Scheme of the construction of extensions of evaluating linguistic expressions.}\]
2.4. Specificity Ordering of Evaluating Expressions

In the perception-based logical deduction described further, an important role is played by natural ordering of evaluating expressions, which we call the specificity ordering. This is determined by the fact that we can distinguish hedges with narrowing and widening effects. The latter make the meaning of the atomic expression before which they stand more precise whereas the former make it the opposite.

In applications described in Section 5, we work with several concrete hedges, namely “extremely (Ex), significantly (Si), very (Ve), empty hedge, more or less (ML), roughly (Ro), quite roughly (QR), very roughly (VR)”. Among them, the hedges Ex, Si, and Ve have a narrowing effect and ML, Ro, QR, and VR have a widening effect. In Ref. 14, we have defined empirical values of the parameters \(a, b, c\) of these hedges. We have chosen these hedges because they are very common in ordinary speech. However, our theory is general enough to include many other concrete examples of hedges.

The distinction between hedges with narrowing and widening effects induces an ordering \(\leq\) of specificity between them:

\[
\text{Ex} \leq \text{Si} \leq \text{Ve} \leq (\text{empty hedge}) \leq \text{ML} \leq \text{Ro} \leq \text{QR} \leq \text{VR} \quad (8)
\]

Definition 8 means that all values in some context, that are extremely small (or big), are also significantly small (or big) and so forth.

We can generalize the definition of specificity ordering to all hedges. Of course, there exist also hedges that have neither a narrowing nor a widening effect (e.g., rather), and so the ordering \(\leq\) is, in general, only partial.

On the basis of that, we can extend the specificity ordering to all evaluating expressions, namely, the expressions of the form “(linguistic hedge) small” are the most specific, and “(linguistic hedge) big” are the least specific. If two expressions differ only in hedges, then they are ordered by the specificity ordering of hedges introduced above. Consequently, with respect to definition 6, extensions of the evaluating expressions containing hedges from definition 8 make in each possible world a nested sequence of fuzzy sets.

In the following, we will suppose that the specificity ordering between evaluating expressions of concern is fixed.

3. LINGUISTIC DESCRIPTION AND LOGICAL DEDUCTION

3.1. Linguistic Description and Its Meaning

A linguistic description is a set of fuzzy IF–THEN rules

\[
\mathcal{R} := \{ \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m \} \quad (9)
\]

where each \(\mathcal{R}_i\) has the form 1.

Let us now consider one fuzzy IF–THEN rule \(\mathcal{R}\) and let

\[
\text{Int}(X \text{ is } A) = \text{Ev}^A \\
\text{Int}(Y \text{ is } B) = \text{Ev}^S
\]
where \( \text{Ev}^A, \text{Ev}^S \) are intensions of some evaluating expressions. Then intension of \( \mathcal{R} \) is a function \( \text{Int}(\mathcal{R}): W \times W \to \mathcal{F}(\mathbb{R} \times \mathbb{R}) \) given by

\[
\text{Int}(\mathcal{R})(w, w') = \text{Ev}^A_w \rightarrow \text{Ev}^S_{w'}, \quad w, w' \in W
\]  

(10)

where the formula on the right-hand side of Equation 10 represents a function (fuzzy relation) composed of the functions \( \text{Ev}^A, \text{Ev}^S \), and the Łukasiewicz implication \( \rightarrow \).

Formula 10 gives at the same time rules for computation of extension of \( \mathcal{R} \) in the contexts \( w, w' \). Indeed, let extensions of the evaluating predications in the antecedent and succedent in the respective contexts \( w, w' \) be fuzzy sets

\[
\begin{align*}
\text{Ext}_w(X \text{ is } A) &= \text{Ev}^A_w \\
\text{Ext}_w(Y \text{ is } B) &= \text{Ev}^S_w.
\end{align*}
\]  

(11, 12)

Then the extension of \( \mathcal{R} \) in a couple of contexts \( w, w' \in W \) is a fuzzy relation defined by

\[
\text{Ext}_{(w, w')}(\mathcal{R})(v, v') = \text{Ev}^A_v \rightarrow \text{Ev}^S_{v'}, \quad v \in w, v' \in w'.
\]  

(13)

### 3.2. Defuzzification

For dealing with evaluating expressions, a special defuzzification operation \( \text{DEE} \) (Defuzzification of Evaluating Expressions) is necessary. This operation is a realization of the so-called description operator described in Ref. 21. In general, it leads to a very simple conclusion that the result of defuzzification can be any element from the kernel of the given fuzzy set. For applications, however, we need a systematic result. Therefore, we will define the defuzzification as follows.

Let \( \text{Ev} \) be intension of some evaluating expression and \( w \in W \) be a context. Let \( c \) be the parameter of the corresponding linguistic hedge \( \nu_{a,b,c} \) and \( \sigma \in (0,1] \). Then we put

\[
\text{DEE}_\sigma(\text{Ev}_w) = \begin{cases} 
  v_L + \sigma(1-c)(v_S - v_L), & \text{if } \text{Ev} \in \text{Sm} \\
  v_S + \frac{\sigma(1-c)}{2}(v_L + v_R - 2v_S), & \text{if } \text{Ev} \in \text{Me} \\
  v_R + \sigma(1-c)(v_S - v_R), & \text{if } \text{Ev} \in \text{Bi}
\end{cases}
\]  

(14)

The parameter \( \sigma \) is a global characteristic of the defuzzification and it should be set close to 1. A schematic picture demonstrating the behavior of the DEE defuzzification method is given in Figure 2. It is easy to see that this defuzzification is a generalization of three methods, namely \textit{Last of Maxima} for “small,” \textit{First of Maxima} for “big,” and \textit{Center of Gravity} for “medium” values.

Let us remark that, similar to all the other defuzzification methods, the general theoretical frame of the \( \text{DEE}_\sigma \) method is fuzzy logic, but the concrete formula has been derived on the basis of practical experience. Our conclusion that its result should be an arbitrary element from the kernel of the fuzzy set is in accordance with the general result in the fuzzy approximation theory (cf. Ref. 7,
Chap. 5) where the concrete defuzzification has no influence on the precision of approximation. Practical testing of the DEE method demonstrates that it gives results that are in accordance with the human way of reasoning with evaluating expressions.

3.3. Perception-Based Logical Deduction

The way people make inferences on the basis of linguistic description can be explained by an example. Let us consider a linguistic description, which consists of two rules:

\[ R_1 : \text{IF } X \text{ is } \text{small} \text{ THEN } Y \text{ is } \text{big} \]
\[ R_2 : \text{IF } X \text{ is } \text{big} \text{ THEN } Y \text{ is } \text{small} \]

Furthermore, let the contexts for the respective variables \( X, Y \) be \( w = w' = (0, 0.5, 1) \). Then small values are some values around 0.3 (and smaller) and big ones some values around 0.7 (and bigger). We know from the linguistic description that small input values correspond to big output ones and vice versa. Therefore, given an input, for example, \( X = 0.3 \), we expect the result \( Y \approx 0.7 \) due to the rule \( R_1 \). The reason is that with respect to the above linguistic description, our perception of 0.3 (in the given context) is “small,” and thus, in this case, the output value of \( Y \) should be “big.” Similarly, for \( X = 0.75 \) we expect the result \( Y \approx 0.25 \) due to the rule \( R_2 \).

This intuitive procedure can be characterized using precise formal means of fuzzy logic. This has been done in the frame of fuzzy intensional logic in Ref. 8, and in the frame of fuzzy logic with evaluated syntax in Refs. 6 and 7. We refer the reader to these papers because as mentioned in the Introduction, in this article we focus on semantics only and so we have not developed here appropriate formal means.

Less formally, we recall that the meaning of linguistic description is represented by a set of intensions

\[ \text{Int}(R_1), \ldots, \text{Int}(R_m) \]
where each Int(\(\mathcal{R}_i\)) is defined in Equation 10. The logical deduction may proceed, if we learn an observation leading to an intension, which is equal to some of the antecedents Ev\(_i^A\) occurring in the given linguistic description. This is done as follows.

The observation is some value \(u \in w\) in a context \(w = (u_L, u_S, u_R)\). We must first transform it into a suitable perception using a function

\[
\text{Suit} : \mathbb{R} \times W \rightarrow \text{Sm} \cup \text{Me} \cup \text{Bi}
\]  

(16)

The result of \(\text{Suit}(u, w)\) is (extension of) such an evaluating expression \(Ev\) that the observation \(u \in w\) is the most specific and typical for its extension \(Ev_w\). To be typical means that the membership degree \(Ev_u(u)\) is nonzero and greater than some reasonable threshold \(a^0\) (we usually put \(a^0 = 0.9\) or even \(a^0 = 1\)). To be most specific means that the evaluating expression \(Ev\) is the most specific (sharpest) one in the sense of the natural ordering \(\preceq\) mentioned in Section 2 (cf. definition 8). This definition of \(\text{Suit}\) can be justified by the empirical finding that in the given context, each value can be classified by some evaluating expression. Because the expressions are more or less specific, the most specific one gives the most precise information. If there is no evaluating expression being most specific and typical, then \(\text{Suit}\) gives nothing.

Let us now fix an observation \(u_0 \in w\). If the linguistic description 15 is given, then we expect \(\text{Suit}\) to give an (extension of) evaluating expression

\[
\text{Suit}(w, u_0) = Ev_{i,w}^A
\]

(17)

(provided that it exists) where \(1 \leq i \leq m\). This, together with description 15 is used in the deduction. We say that the corresponding rule \(\mathcal{R}_i\), fired. Note that if the most specific and typical extension \(Ev_{i,w}^A\) is found, then \(\text{Suit}\) extends it to the whole intension \(Ev_i^A\).

Let \(u_0 \in w\) be an observation such that the rule

\[
\mathcal{R}_i = \text{IF } X \text{ is } A_i \text{ THEN } Y \text{ is } B_i
\]

fired where Int(\(A_i\)) = Ev\(_i^A\) and Int(\(B_i\)) = Ev\(_i^S\). Then \(Ev_{i,w}^A(u_0) \in [0,1]\) is a nonzero truth value. Let us denote it by \(b\) (i.e., \(b = Ev_{i,w}^A(u_0)\)). Then for each context \(w' \in W\), it can be demonstrated (cf. Ref. 8) that the perception-based deduction gives a fuzzy set

\[
(\text{Ev}_{i}^S)'(w') = b \rightarrow \text{Ev}_{i}^S(w')
\]

(18)

Each element \(v \in w'\) such that \((\text{Ev}_{i}^S)'(w',v) = 1\) surely belongs to the extension \(Ev_{i,w'}^S\), and can be taken as the result of perception-based deduction. The appropriate element is obtained by the DEE\(_{\text{\#}}\) defuzzification.

If we repeat the just described procedure for all elements \(u \in w\), then we conclude that in the case of one fuzzy IF–THEN rule, \(\mathcal{R}_i\), it determines in each couple of contexts \(w,w'\) a function \(f_{\mathcal{R}_i} : w \rightarrow w'\) given by

\[
f_{\mathcal{R}_i}(u) = \text{DEE}_{\text{\#}}(Ev_{i,w}^A(u) \rightarrow Ev_{i,w'}^S), \quad u \in w
\]

(19)
If a linguistic description consists of more fuzzy IF–THEN rules, then the perception-based logical deduction provides a function \( f_R \) that is piecewise continuous and its pieces consist of parts of the functions 19. The points of discontinuity depend on the perceptions determined by the Suit function 17.

4. FUZZY TRANSFORM AND SMOOTH PERCEPTION-BASED LOGICAL DEDUCTION

4.1. Fuzzy Transform

The fuzzy transform (F-transform) is a technique developed by I. Perfilieva\(^{11,12}\) that can be ranked among fuzzy approximation techniques. It works with a continuous function \( f \) defined on an interval of real numbers \( w = [v_L, v_R] \subset \mathbb{R} \). There are several purposes for which F-transform can be used. The purpose important in this article is to use it for approximation of \( f \) with sufficient precision and to filter its possible noise.

Let us choose some points \( p_1, \ldots, p_N \in w \) in which the function \( f \) is computed. Furthermore, let the interval \( w \) be divided into a set of equidistant nodes \( x_k = v_L + h(k - 1), k = 1, \ldots, n \) where \( N > n \) and \( h = (v_R - v_L)/(n - 1) \) is the fixed length. Obviously, \( x_1 = v_L \) and \( x_n = v_R \). The F-transform has two phases.

4.1.1. Direct F-Transform

We define \( n \) basic functions \( A_1, \ldots, A_n \), which cover \( w \) and divide it into \( n \) vague areas. The basic functions must fulfill the following conditions \((k = 1, \ldots, n)\):

1. \( A_k: w \to [0, 1], \quad A_k(x_k) = 1 \)
2. \( A_k(x) = 0 \) if \( x \not\in (x_{k-1}, x_{k+1}) \) where we formally put \( x_0 = x_1 = v_L, \quad x_{n+1} = x_n = v_R \).
3. \( A_k(x) \) is continuous.
4. \( A_k(x) \) monotonously increases on \( [x_k, x_{k+1}] \) and monotonously decreases on \( [x_{k-1}, x_k] \).
5. \( \sum_{k=1}^n A_k(x) = 1 \), for all \( x \in w \).

Using the basic functions, we transform the given function \( f \) into \( n \)-tuple of real numbers \( [F_1, \ldots, F_n] \) defined by

\[
F_k = \frac{\sum_{j=1}^N f(p_j) A_k(p_j)}{\sum_{j=1}^N A_k(p_j)}, \quad k = 1, \ldots, n
\]

4.1.2. Inverse F-Transform

The result of the direct F-transform is a vector of numbers \( [F_1, \ldots, F_n] \). This set contains information about the original function \( f \) and can be used to obtain a function...
Function 21 is called an inverse F-transform. It can be proved that if \( n \) increases, then \( f_{F,n}(p_j) \) converges to \( f(p_j), j = 1, \ldots, N \). It is clear that the function \( f_{F,n} \) is continuous.

The F-transform has (besides others) the following properties important for its use in this article:

(a) It has nice filtering properties.
(b) It is easy to compute.
(c) Let the function \( f \) be given. Then the F-transform is stable with respect to the choice of the points \( p_1, \ldots, p_N \), provided that the number of nodes is fixed. This means that when choosing other points \( p_k \) (and possibly changing their number \( N \)), the resulting function \( f_{F,n} \) does not significantly change. Note that this is not true for many classical numerical methods.

The detailed formal description of the F-transform including theorems characterizing its behavior can be found in Refs. 11 and 12.

4.2. Smoothing Perception-Based Logical Deduction

Recall that logical deduction gives a piecewise continuous function \( f_R \) consisting of pieces defined in Equation 19. We can make it continuous by joining perception-based deduction with the F-transform. Of course, we could use also other (classical) numerical techniques, for example, splines or the least squares method. In our case, F-transform surpasses these techniques for the following reasons: We need a sufficiently simple technique that keeps the main properties of the given function and its main role is to make it smooth. For example, the least squares would be very difficult to use because the course of \( f_R \) can vary significantly, and so it is not clear in advance which kind of the polynomial should be used, and, of course, it is far from being simple. Important for us is the stability mentioned at item (c) above. Also significant is the fact that F-transform is a fuzzy technique, and thus it is integral with the perception-based deduction of concern.

Let a linguistic description be \( R \). Then the function \( f_R \) takes the role of the function to be filtered using the F-transform. We choose some smoothing number, which is the number of nodes \( n \).

Clearly, there are infinitely many possibilities for choosing the basic functions considered in the direct F-transform. In our case, we want to follow the theory of linguistic evaluating expressions, among which we rank also fuzzy numbers that are expressions of the form

\[
\langle \text{linguistic hedge} \rangle [\text{approximately}] x_0
\]

Extension of the fuzzy number is a fuzzy set \( F_{\nu_{a,b,c}} \) where \( \nu_{a,b,c} \) (see Equation 4) is a linguistic hedge and \( x_0 \) is the central point around which the fuzzy number is
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defined. Without going into details, which are not important for our purpose, we only remark that it is analogous to the membership function of the extension $M_{\nu, w}$. We explicitly set

$$F_{n_{\nu}, x_0}(x) = \begin{cases} 1, & x \in [c_{n_0}^L, c_{n_0}^R], \quad c_{n_0}^L = x_0 - (1 - c)h \\ 1 - \frac{(c_{n_0}^R - x)^2}{K_1 h^2}, & x \in [b_{n_0}^L, c_{n_0}^R], \quad b_{n_0}^L = x_0 - (1 - b)h \\ 1 - \frac{(x - c_{n_0}^L)^2}{K_1 h^2}, & x \in (c_{n_0}^R, b_{n_0}^R], \quad b_{n_0}^R = x_0 + (1 - b)h \\ \frac{(x - a_{n_0}^L)^2}{K_2 h^2}, & x \in (a_{n_0}^L, b_{n_0}^R), \quad a_{n_0}^L = x_0 - (1 - a)h \\ \frac{(a_{n_0}^R - x)^2}{K_2 h^2}, & x \in (b_{n_0}^L, a_{n_0}^R), \quad a_{n_0}^R = x_0 + (1 - a)h \\ 0, & x \leq a_{n_0}^L, \quad x \geq a_{n_0}^R \end{cases}$$

(22)

Note that $F_{n_{\nu}, x_0}(x_0) = 1$ and $F_{n_{\nu}, x_0}(x_0 \pm h) = 0$. Thus, one fuzzy number is spread over three neighboring nodes $x_0 - h, x_0, x_0 + h$. Furthermore,

$$\sum_{k=1}^w F_{n_{\nu}, x_0}(x) = 1$$

holds for each $x \in w$. Consequently, each $x \in w$ is covered by exactly two neighboring fuzzy numbers $F_{n_{\nu}, x_{0k}}, F_{n_{\nu}, x_{0k+1}}$. This means that $x_{0k} \leq x \leq x_{0k+1}$ and $F_{n_{\nu}, x_{0k}}(x) + F_{n_{\nu}, x_{0k+1}}(x) = 1$. This property is used below for smooth logical deduction.

Let an observation $u_0 \in w$ be given. Because only one data item is to be filtered, namely $(u_0, f_{R}(u_0))$, only two numbers $F_k, F_{k+1}$ in Equation 20 are needed for each two nodes $x_{0k}, x_{0k+1}$ such that $x_{0k} \leq u_0 \leq x_{0k+1}$. To do it we have to choose a step $r > 0$, compute a sequence of values $u_1 = x_{0k-1}, u_2 = u_1 + r, u_3 = u_1 + 2r, \ldots, u_M = x_{0k+2}$ lying between the nodes $x_{0k+1}$ and $x_{0k+2}$ and generate a sequence of auxiliary data

$$(u_1, f_{R}(u_1)), \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..
Now, the smooth logical deduction consists of two steps. First we compute two numbers $F_k, F_{k+1}$ according to Equation 20:

$$F_k = \frac{\sum_{j=1}^{M} f_R(u_j) \cdot F_{n, x_0}(u_j)}{\sum_{j=1}^{M} F_{n, x_0}(u_j)}$$  \hspace{1cm} (24)$$

Second, compute the resulting smoothed output using the formula

$$f_{R,n}(u_0) = F_k \cdot F_{n, x_0}(u_0) + F_{k+1} \cdot F_{n, x_{k+1}}(u_0)$$  \hspace{1cm} (25)$$

The principle of smooth logical deduction is depicted in Figure 3.

From the point of view of the perception-based logical deduction, the above-described smoothing behaves as a new special defuzzification operation. This will be demonstrated in the next section.

5. DEMONSTRATION OF PERCEPTION-BASED LOGICAL DEDUCTION

In this section, we will demonstrate the power of perception-based logical deduction and compare its original behavior with the smooth one.

Demonstration 1. First, let us consider a simple linguistic description
We learn from this description that for very small input values, the output should be roughly big, for small it should be big (i.e., bigger than for very small), medium for medium input values, and very small for big input values. This behavior is clearly seen from Figure 4 for the case of the linguistic context, which is the same for both variables, namely, $w = w' = \langle 0.0, 0.4, 1 \rangle$. Then indeed, Rule 1 is fired if the input values are very small, that is, around 0.1–0.2. Then the output is roughly big (around 0.5–0.6). If they are greater, that is, small, then Rule 2 is fired and the output is big (around 0.7). If the input values are medium (around 0.4–0.5), the output is also medium, and if they are big (around 0.7 and greater), then the output is very small (around 0.2 and smaller). Let us stress that this behavior is independent of the chosen context, that is, when changing it, the general behavior will be the same, which means that the output values will be different but again corresponding to perceptions of big, roughly big, medium, and very small in the new context.

Note also that the output values decrease. This is caused by the evaluation because, for example, if Rule 1 fires, then the truth of the perception very small for the input value is computed. The greater is the input value, the less is true that it is

![Graph showing output values](image)

**Figure 4.** Result of original (nonsmooth) perception-based deduction. In the upper part, the output fuzzy set (extension in the context $w'$) with the result of the DEE defuzzification and the corresponding fired rule for the input $X = 0.1$ are depicted.
indeed “very small” and thus, the less it is true that the output is roughly big. Of course, analogous but opposite behavior would be obtained in the case of Rule 4.

It should be stressed that different values (i.e., small, very small, medium, big) are distinguished and the output is appropriate. However, all values that are very small are at the same time also small (but not vice versa). Hence, if Rule 1 is deleted, then the general behavior is not changed but the output provided by Rule 1 is now taken over by Rule 2. This is well seen from Figure 6.

The result of smooth perception-based deduction on the basis of the same linguistic description is depicted in Figure 5. It is important that the essential shape of the output function has not changed, that is, the above expected behavior is preserved. On the other hand, it is continuous.

Let us delete Rule 1. Then its role is taken by Rule 2, that is, the output is big but not roughly big; this means that the output values are bigger than in the previous case. The difference is depicted in Figure 6. The smooth deduction on the same description is in Figure 7. Again, the essential behavior is the same.

Demonstration 2. The following problem has been discussed in the literature on fuzzy logic applications. The task is to avoid some obstacle given instructions that if the obstacle is very near then we should turn to the left, if it is near then turn to
the right, otherwise do nothing. The perception-based deduction is able to cope with this problem by means of the linguistic description

\[
\text{Rule 1: } X \rightarrow Y
\]

where \( X \) is the distance of the obstacle and \( Y \) is the turn of the steering wheel.

The result is in Figure 8. One can see that the obstacle is avoided to the left if the perception of the distance is extremely or very small, otherwise it is avoided to the right. Figure 9 demonstrates the same in the case of smooth perception-based deduction. The position of the steering wheel is in both cases changed rapidly (as expected depending of the realized perception), but in smooth deduction, the change is not abrupt.
The behavior of the perception-based deduction does not significantly change also when Rule 1 is deleted, as can be seen from Figures 10 and 11. This is due to the fact that extremely small values are also very small. Let us stress that this behavior is very important (cf. also Demonstration 1). It clearly shows that the perception-based logical deduction obeys the information contained in the linguistic description, both in the original (nonsmooth) case as well as in the smooth case.

Discussion. We have demonstrated behavior of the perception-based logical deduction on two different linguistic descriptions. The reader can see that this method is able to mimic the method of human reasoning when dealing with conditional statements consisting of evaluating expressions. The disadvantage of piecewise, smooth, and continuous output can be overcome by using the F-transform so that the obtained output is finally continuous and smooth.

Let us note that the demonstrated behavior in avoiding the obstacle is not possible when applying the usual Mamdani method with COG defuzzification (which, of course, is continuous). If the shapes of fuzzy sets from Figure 1 are used, then the result of the Mamdani method necessarily leads to striking the obstacle. To avoid it, we must use symmetrical and little overlapping membership functions, as depicted in Figure 12. However, these membership functions cannot be
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Figure 8. Avoiding of an obstacle (nonsmooth perception-based deduction, input $X = 6$).

Figure 9. Avoiding of an obstacle (smooth perception-based deduction, input $X = 6$).
extensions of the corresponding evaluating expressions for, at least, two reasons: first, these functions break the property that each extremely small value is at the same time also very small, and thus also small. Second, it follows from these shapes that there are values that are very small with the high membership degree and then even smaller values that are very small with smaller membership degree. A similar counterintuitive conclusion follows for small, and also for comparison of the expressions extremely small, very small, and small. We conclude that symmetric fuzzy sets cannot be used as extensions of evaluating linguistic expressions from the fundamental linguistic trichotomy. Therefore, the Mamdani method is not suitable for modeling of human reasoning based on the use of evaluating expressions and of linguistic descriptions containing them.

We conclude that there are two essential approaches to elaboration of fuzzy IF–THEN rules. The first is the Mamdani method, which is suitable for approximation of some function and has many applications (e.g., in fuzzy control it proved to be very effective). The second one is the perception-based logical deduction. Let us remark that it has great application potential in various decision problems (such as the discussed avoiding the obstacle) but it has been used for fuzzy control, too (see Ref. 10). The possibility for smoothing the output makes it even more attractive.
6. CONCLUSION

In this article, we described a method for the derivation of a conclusion on the basis of information provided using genuine linguistically characterized fuzzy IF–THEN rules. We call it perception-based logical deduction. Its advantage is the possibility to use linguistic expressions, which are interpreted in accordance with

![Diagram](image-url)

Figure 11. (Smooth) avoiding of an obstacle when Rule 1 is deleted (input \( X = 6 \)).

![Diagram](image-url)

Figure 12. Shapes of extensions of evaluating expressions for the obstacle-avoiding problem using the Mamdani method.
the human way of understanding them. Thus, the perception-based deduction mimics the human way of reasoning.

The disadvantage of this method is that it provides, in general, only piecewise-continuous function. This disadvantage can be overcome when joining this method with a special technique of fuzzy approximation called F-transform. The result is a function that keeps the mentioned advantages and, moreover, is continuous and smooth.

The perception-based logical deduction has been implemented in the LFLC 2000 software package developed at the University of Ostrava and many times successfully applied to control and decision-making problems (see Ref. 22).

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