On the finite differences schemes for the numerical solution of two-dimensional moving boundary problem

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Abstract

In this paper, two different finite differences schemes are presented for numerical solution of two-dimensional moving boundary problem. The methods are illustrated by solving sample problem in two-dimensional solidification of the square prism. The finite difference schemes developed for this purpose are based on the method of lines using interpolation polynomials. The fully explicit method developed here which has reasonable accuracy, and Peaceman and Rachford alternating direction implicit (ADI) formula illustrated by solving sample problem.

The accuracies of the resulting methods are verified by numerical testing. The numerical results are obtained by present methods are compared with earlier authors. © 2004 Elsevier Inc. All rights reserved.

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1. Introduction

During the freezing of liquid on a cooled surface, a solidified layer is formed which increases in thickness as time passes. Thus the heat conduction across the solid form the liquid–solid interface to the cooled surface takes places in a domain whose size and shape change with time [1]. Only a limited number of moving boundary problems admits an analytic solution. These include, in the main the classical one-dimensional Stefan and Neumann problems [2] and their variants. Also, the vast majority of theoretical work in this area has been limited to analysis of one-dimensional moving boundary problems.

The description of various numerical and other methods with useful bibliography may be found in the surveys of Furzeland [3] and Crank [4].

In an earlier but interesting method for dealing with one-dimensional moving boundary problems, Landau [5] fixed the moving boundary by making some suitable transformation. The use of coordinate transformation for immobilizing the boundary in case of two-dimensional moving boundary problems has also been reported by some authors. For example Furzeland [6] uses body-fitted curvilinear co-ordinate transformation curved-shaped region into a fixed rectangular domain, Sparrow and Hsu [1] also use co-ordinate transformation for their control volume formulation; The extension of Boardway's transformation technique were proposed by Ozis [7,8] and, Gülkac and Öziş [9] and, Öziş and Gülkac [10].

We will consider the two-dimensional moving boundary problem.

\[
\frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}, \quad 0 \leq x, \ y \leq 1, \quad 0 < t \leq T
\]  

(1.1)

with an initial condition

\[
U(x,y,t) = f(x,y), \quad 0 \leq x, \ y \leq 1
\]  

(1.2)

and boundary conditions

\[
U(0,y,t) = f_0(y,t), \quad 0 < t \leq T, \quad 0 \leq y \leq 1,
\]  

(1.3)

\[
U(1,y,t) = f_1(y,t), \quad 0 < t \leq T, \quad 0 \leq y \leq 1,
\]  

(1.4)

\[
U(x,0,t) = f_2(x,t), \quad 0 < t \leq T, \quad 0 \leq x \leq 1,
\]  

(1.5)

\[
U(x,1,t) = f_3(x,t), \quad 0 < t \leq T, \quad 0 \leq x \leq 1
\]  

(1.6)

another well-known condition to be satisfied on the moving interface is

\[
\frac{\partial U}{\partial \eta} = -\beta v_n \quad \text{on} \ u(x,y,t) = 0,
\]  

(1.7)
where $n$ is the outward normal to $\phi(x, y, t) = 0$; $v_n$ is the velocity of the interface in the direction of $n$ and $\beta$ is constant depending on thermal properties of the material undergoing the phase change.

The condition (1.7) at the interface transform to [4].

\[ s_t = \frac{1}{\beta} \left[ (u_x)^2 + (u_y)^2 \right] / u_y. \]  

(1.8)

2. Numerical techniques

The domain $[0, 1]^2 \times [0, T]$ is divided into an $M \times N$ mesh with the spatial step $h = 1/M$ in both $x$ and $y$ direction and the time step size $k = T/N$, respectively. Grid points $(x_i, y_j, t_n)$ are defined by

\[
\begin{align*}
  x_i &= ih, \quad i = 0, 1, 2, \ldots, M, \\
  y_j &= jh, \quad j = 0, 1, 2, \ldots, M, \\
  t_n &= nh, \quad n = 0, 1, 2, \ldots, N,
\end{align*}
\]

where $M$ and $N$ are integers. The notation $U_{i,j}^n$ is used for the finite difference approximation of $u(x_i, y_j, t_n)$. Using the initial condition (1.2), Eqs. (1.1) and (1.8) are solved approximately at the spatial points $(x_i, y_j)$ commencing with initial values $U_{i,j}^n = f(x_i, y_j)$ $i, j = 0, 1, \ldots, M$ and boundary values

\[
\begin{align*}
  U_{i,0}^n &= f_0(y_j, t_n), \\
  U_{i,M}^n &= f_1(y_j, t_n), \\
  U_{0,j}^n &= f_2(x_i, t_n), \\
  U_{i,N}^n &= f_3(x_i, t_n),
\end{align*}
\]

(2.1)  

(2.2)  

(2.3)  

(2.4)

For $n = 1, 2, \ldots, N$. And we have selected $\beta = 1$.

2.1. LOD method

Partial differential equation (1.1) can be solved by splitting it into two one-dimensional equation

\[
\begin{align*}
  \frac{1}{2} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \\
  \frac{1}{2} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial y^2},
\end{align*}
\]
rather than discretising the complete two-dimensional diffusion equation to give an approximating equation based on two-dimensional computational molecule. Each of these equations is then solved over half of the time step used for the complete two-dimensional equation using techniques for the one-dimensional diffusion are much easier to develop and use than single step methods for two-dimensional diffusion equation.

2.1.1. The method of lines using interpolation polynomials with the fully explicit methods

In this method, we replace all spatial derivatives with average of their values at the \( n \) and \( n + 1/2 \) time levels and then substitute centered-difference forms for all derivatives.

\[
\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{k} = \frac{1}{12h^2} \left\{ -u_{i-2,j}^n + 14u_{i-1,j}^n - 24u_{i,j}^n + 10u_{i+1,j}^n + u_{i+2,j}^n \right\}
\]

for \( \forall i = 2, 3, \ldots, M - 3 \) and \( \forall j = 0, 1, 2, \ldots, M \) or

\[
u_{i,j}^{n+1/2} = \frac{1}{12} r_x \left\{ -u_{i-2,j}^n + 14u_{i-1,j}^n + 10u_{i+1,j}^n + u_{i+2,j}^n \right\} + (1 - 2r_x)u_{i,j}^n,
\]

\[
r_x = \frac{\Delta x}{\Delta t^2},
\]

\[
u_{i,j}^{n+1/2} = 14u_{i,j}^{n+1/2} + 10u_{i,j}^{n+1/2} + u_{i,j}^{n+1/2} - \frac{12(2r_x)}{r_x} u_{i,j}^{n+1/2} + \frac{12}{r_x} u_{i,j}^{n+1/2}
\]

and

\[
u_{i,j}^{n+1/2} = 14u_{i,j}^{n+1/2} - 10u_{i,j}^{n+1/2} + \frac{12(2r_y)}{r_y} u_{i,j}^{n+1/2} - \frac{12}{r_y} u_{i,j}^{n+1/2}
\]

then \( t_{n+1/2} \rightarrow t_{n+1} \)

\[
u_{i,j}^{n+1} = \frac{1}{12} r_y \left\{ -u_{i,j-2}^{n+1} + 14u_{i,j-1}^{n+1} + 10u_{i,j+1}^{n+1} + u_{i,j+2}^{n+1} \right\} + (1 - 2r_y)u_{i,j}^{n+1/2}
\]

for \( \forall i = 1, 2, \ldots, M - 1 \) and \( \forall j = 2, 3, \ldots, M - 2 \)

\[
u_{i,j}^{n+1} = \frac{1}{14} u_{i,j}^{n+1} - \frac{5}{7} u_{i,j-1}^{n+1} - \frac{1}{14} u_{i,j+1}^{n+1} + \frac{12(1 + 2r_y)}{14r_y} u_{i,j}^{n+1} - \frac{12}{14r_y} u_{i,j}^{n+1/2}
\]

and

\[
u_{i,j}^{n+1} = \nu_{i,j}^{n+1} - 14u_{i,j-4}^{n+1} - 10u_{i,j-2}^{n+1} + \frac{12(1 + 2r_y)}{r_y} u_{i,j}^{n+1} - \frac{12}{r_y} u_{i,j}^{n+1/2},
\]

\[
r_y = \frac{\Delta y}{\Delta t^2}.
\]
2.1.2. The Paceman–Rachford (ADI) method

The ADI method is used in the following. The test problem (1.1) is solved numerically over the half time step $t_n$ to $t_{n+1/2}$ using

$$-r_x u_{i,j-1}^{n+1/2} + 2(1 + r_x)u_{i,j}^{n+1/2} - r_x u_{i,j+1}^{n+1/2} = r_x u_{i,j-1}^{n} + 2(1 - r_x)u_{i,j}^{n} - r_x u_{i,j+1}^{n}$$

for $j = 2, 3, \ldots J - 2$ for each $i = 0, 1, \ldots, I$.

This procedure is unconditionally von-Neumann stable and solvable for all $r > 0$.

$$u_{i,1}^{n+1/2} = r_x u_{i,0}^{n} + (1 - 2r_x)u_{i,1}^{n} + r_x u_{i,2}^{n},$$  \hspace{1cm} (2.5)

while values at points adjacent to the boundary $y = 1$ are calculated using

$$u_{i,M-1}^{n+1/2} = r_x u_{i,M}^{n} + (1 - 2r_x)u_{i,M-1}^{n} + r_x u_{i,M-2}^{n}.$$  \hspace{1cm} (2.6)

Both (3.2) and (3.3) are stable only for $0 < r \leq 1/2$.

And then the time interval $t_{n+1/2}$ to $t_{n+1}$ as follows. For $i = 1, 2, \ldots, I - 1$ and each $j = 1, 2, \ldots, J - 1$ use.

$$-r_y u_{i-1,j} + 2(1 + r_y)u_{i,j}^{n+1} - r_y u_{i+1,j}^{n+1} = r_y u_{i-1,j}^{n+1/2} + 2(1 - r_y)u_{i,j}^{n+1/2} + r_y u_{i+1,j}^{n+1/2}.$$  \hspace{1cm} (3.2)

This procedure is unconditionally von-Neumann stable, and solvable for all $r > 0$ when the absolute value of the error;

$$e_{i,j}^n = u(ih, jk, nk) - u_{i,j}^n.$$  \hspace{1cm} (3.3)

3. Numerical experiment

In this section, we will test methods described above with a test problem. The comparison of present results with those of earlier authors shows an extremely good agreement.

Let us assume that the square cross-section of the prism extends between $-1 \leq x, y \leq 1$.

If $u(x, y, t)$ denotes temperature at a point $(x, y)$ at some time $t$, governing equation for heat conduction may be written as

$$u_t = u_{xx} + u_{yy} \text{ in } D,$$  \hspace{1cm} (3.1)

where $D$ is the domain enclosed by fixed boundary.

$$f(x, y) \equiv (x^2 - 1)(y^2 - 1) = 0, \quad \text{where } u = 0$$  \hspace{1cm} (3.2)

and the

$$\phi(x, y, t) = 0, \quad \text{where } u = 1 \text{ for } t > 0.$$  \hspace{1cm} (3.3)
Table 1
Comparison of the $x$-co-ordinate of solid–liquid interface on the $x$-axis and diagonal

<table>
<thead>
<tr>
<th>Time</th>
<th>On the $x$-axis</th>
<th></th>
<th></th>
<th>On the diagonal</th>
<th></th>
<th></th>
</tr>
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<tbody>
<tr>
<td>0.05</td>
<td>0.8125</td>
<td>0.8125</td>
<td>0.8152</td>
<td>0.7445</td>
<td>0.6483</td>
<td>0.6476</td>
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<td>0.10</td>
<td>0.6979</td>
<td>0.6982</td>
<td>0.6982</td>
<td>0.4312</td>
<td>0.5812</td>
<td>0.5642</td>
</tr>
<tr>
<td>0.15</td>
<td>0.6157</td>
<td>0.6156</td>
<td>0.6156</td>
<td>0.2495</td>
<td>0.5103</td>
<td>0.4935</td>
</tr>
<tr>
<td>0.20</td>
<td>0.5473</td>
<td>0.5463</td>
<td>0.5465</td>
<td>0.1686</td>
<td>0.4428</td>
<td>0.4264</td>
</tr>
<tr>
<td>0.25</td>
<td>0.4565</td>
<td>0.4837</td>
<td>0.4849</td>
<td>0.1243</td>
<td>0.3948</td>
<td>0.3642</td>
</tr>
<tr>
<td>0.30</td>
<td>0.4302</td>
<td>0.4244</td>
<td>0.4245</td>
<td>0.0965</td>
<td>0.3351</td>
<td>0.3130</td>
</tr>
<tr>
<td>0.35</td>
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<td>0.3663</td>
<td>0.3665</td>
<td>0.0776</td>
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<tr>
<td>0.40</td>
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<td>0.3078</td>
<td>0.3081</td>
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<td>0.2332</td>
<td>0.2176</td>
</tr>
<tr>
<td>0.45</td>
<td>0.2816</td>
<td>0.2495</td>
<td>0.2495</td>
<td>0.0537</td>
<td>0.1947</td>
<td>0.1764</td>
</tr>
<tr>
<td>0.50</td>
<td>–</td>
<td>0.1894</td>
<td>0.1892</td>
<td>0.0456</td>
<td>–</td>
<td>0.1339</td>
</tr>
<tr>
<td>0.55</td>
<td>–</td>
<td>0.1271</td>
<td>0.1251</td>
<td>0.0392</td>
<td>–</td>
<td>0.0899</td>
</tr>
<tr>
<td>0.60</td>
<td>–</td>
<td>0.0562</td>
<td>0.0556</td>
<td>0.0340</td>
<td>–</td>
<td>0.0398</td>
</tr>
<tr>
<td>0.65</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0296</td>
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<td>–</td>
</tr>
<tr>
<td>0.70</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0260</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>0.75</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0229</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>0.80</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0203</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>0.85</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0180</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>0.90</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0161</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>0.95</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0143</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>1.0</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.0115</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
As the nature of curve $\phi(x, y, t) = 0$ representing the solid–liquid interface at any time, $t$, is known, it has to be determined as part of the solution. Obviously we have,

$$\phi(x, y, t) \equiv f(x, y) = 0 \text{ at } t = 0. \quad (3.4)$$

In addition to (3), another well known condition to be satisfied on the moving interface is

$$\partial u / \partial \eta = -\beta v_n \text{ on } \phi(x, y, t) = 0, \quad (3.5)$$

where $n$ is the outward normal to $\phi(x, y, t) = 0$; $v_n$ is the velocity of interface in the direction of $n$ and $\beta$ is a constant depending on the thermal properties of the material undergoing the phase change [9].

The CPU time for explicit method is 60.16 s and for the P–R ADI method is 95.50 s. It is seen that the values of the temperature is the same accurate when the same problem is solved with values $h = 0.1$ and $\Delta t = 0.0001$ given Table 1.

4. Numerical results and discussion

In this paper two-methods namely the explicit method and second order ADI method are used to solve moving boundary problem with boundary condition through a LOD procedure with employed those one-dimensional schemes to apply them in each direction. Using the explicit method for one-dimensional moving boundary problem in a LOD procedure with special treatment on the boundaries at the intermediate time levels gave fourth-order accuracy. Without the special boundary treatment at the intermediate time levels high-order methods used at interior grid points in an LOD procedure only procedure low-order methods.

A comparison with the fully implicit method for the model problem clearly demonstrates that the new techniques use less CPU time. The only disadvantage of these methods was their limited range of stability. This was because of avoiding the use of the boundary values at the intermediate time levels as this makes these procedures to be dependent to some other conditional schemes to evaluate the values near the boundaries.

Also ADI method produced second order results. It used more CPU time than the fourth-order LOD procedure to get results of the same accuracy.

The most general property of the fully explicit finite difference method is the restriction of the size of the time step due to stability requirements. This restriction necessitates extremely small values for $t$. However, when the fully implicit finite difference scheme is used, this limitation is removed. But, a disadvantage of these techniques is the extensive amount of CPU times used in determining the numerical solution compared to the fully explicit method for the same selection of the values $h, k$. 
References