Quasigroup based crypto-algorithms

Victor Shcherbacov

January 17, 2012

Abstract

Modifications of Markovski quasigroup based crypto-algorithm have been proposed. Some of these modifications are based on the systems of orthogonal \(n\)-ary groupoids. \(T\)-quasigroups based stream ciphers have been constructed.

2000 Mathematics Subject Classification: 94A60, 20N05, 20N15

Key words and phrases: \(n\)-ary groupoid, \(n\)-ary quasigroup, \(T\)-quasigroup, cipher, cryptographic primitive, system of orthogonal \(n\)-ary groupoids

Contents

1 Introduction 2

1.1 Preliminaries 2

1.2 Basic definitions 2

1.3 Quasigroup based cryptosystem 3

1.4 Modifications and generalizations 4

1.5 A modification of Algorithm \(\mathbb{1}\) 5

1.6 \(n\)-ary analogs of binary algorithms 6

2 Ciphers based on orthogonal \(n\)-ary groupoids 8

2.1 Some definitions 8

2.2 Construction of orthogonal \(n\)-ary groupoids 9

2.3 Ciphers on base of orthogonal systems of \(n\)-ary operation 10

3 Combined algorithms 10

3.1 Modifications of Algorithm \(\mathbb{8}\) 10

3.2 Stream cipher on base of orthogonal system of binary parastrrophic quasigroups 12

3.3 \(T\)-quasigroup based stream code 14

3.4 Some generalization of functions of Algorithm \(\mathbb{8}\) 15

3.5 On quasigroup based cryptcode 16

3.6 A comparison of the ”power” of proposed algorithms 19
1 Introduction

1.1 Preliminaries

This paper is an extended variant and a prolongation of the paper [19]. Information on quasigroups and \( n \)-ary quasigroups it is possible to find in [10, 13, 14, 52], on ciphers in [43, 53]. Some applications of quasigroups in cryptology are described in [20, 21, 49, 30, 56].

Two main elementary methods of ciphering the information are known.

(i). Symbols in a plaintext (or in its piece (its bit)) are permuted by some law. One of the first known ciphers of such kind is cipher ”Scital” (Sparta, 2500 years ago).

(ii). All symbols in a fixed alphabet are changed by a law on other letters of this alphabet. One of the first ciphers of such kind was Cezar’s cipher \((x \rightarrow x + 3\) for any letter of Latin alphabet, for example \(a \rightarrow d, b \rightarrow e\) and so on).

In many contemporary ciphers (DES, old Russian GOST, Blowfish [44, 23]) the methods (i) and (ii) are used with some modifications. Therefore, permutations and substitutions are main elementary cryptographical procedures.

What does the use of quasigroups in cryptography give us? It gives the same permutations and substitutions but easy generated, requiring not very big volume of a device memory, acting ”locally” on only one block of a plain-text.

"Stream ciphers are an important class of encryption algorithms. They encrypt individual characters (usually binary digits) of a plaintext message one at a time, using an encryption transformation which varies with time.

By contrast, block ciphers tend to simultaneously encrypt groups of characters of a plaintext message using a fixed encryption transformation. Stream ciphers are generally faster than block ciphers in hardware, and have less complex hardware circuitry. They are also more appropriate, and in some cases mandatory (e.g., in some telecommunications applications), when buffering is limited or when characters must be individually processed as they are received. Because they have limited or no error propagation, stream ciphers may also be advantageous in situations where transmission errors are highly probable" [43].

Stream-ciphers based on quasigroups and their parastrophes were discovered in the end of the XX-th century [37, 38, 41].

Often by enciphering a block (a letter) \(B_i\) of a plaintext the previous ciphered block \(C_{i-1}\) is used. Notice that Horst Feistel was one of the first who proposed such method of encryption (Feistel net) [28].

It is clear that by the construction of a stream cipher it is impossible to use method (i) (see above). But it is possible to use method (ii) and Feistel schema. Of course these methods cannot be unique.

1.2 Basic definitions

We give some definitions. A sequence \(x_m, x_{m+1}, \ldots, x_n\), where \(m, n\) are natural numbers and \(m \leq n\), will be denoted by \(x_m^n\). If \(m > n\), then \(x_m^n\) will be considered empty. The sequence \(x, \ldots, x\) (k times) will be denoted by \(x^k\). The expression \(1, n\) designates the set \(\{1, 2, \ldots, n\}\) of natural numbers [13].

A non-empty set \(Q\) together with an \(n\)-ary operation \(A : Q^n \rightarrow Q\), \(n \geq 2\) is called \(n\)-groupoid and it is denoted by \((Q, A)\).

It is convenient to define \(n\)-ary quasigroup in the following manner.
Definition 1. An \( n \)-ary groupoid \((Q, A)\) with \( n \)-ary operation \( A \) such that in the equality \( A(x_1, x_2, \ldots, x_n) = x_{n+1} \) the knowledge of any \( n \) elements from the elements \( x_1, x_2, \ldots, x_n, x_{n+1} \) uniquely specifies the remaining one is called \( n \)-ary quasigroup \([13]\).

From Definition 1 follows \([10, 52, 53]\) that any quasigroup \((Q, A)\) defines else \((n + 1)! - 1\) \( n \)-quasigroups, so-called parastrophes of quasigroup \((Q, A)\).

In binary case any quasigroup \((Q, A)\) defines else five quasigroups namely \((Q, (13)A), (Q, (23)A), (Q, (12)A), (Q, (123)A), (Q, (132)A)\). See \([10, 52, 55]\) for details.

We give classical equational definition of binary quasigroup \([26]\).

Definition 2. A binary groupoid \((Q, A)\) is called a binary quasigroup if on the set \( Q \) there exist operations \((13)A\) and \((23)A\) such that in the algebra \((Q, A, (13)A, (23)A)\) the following identities are fulfilled:

\[
\begin{align*}
A^{(13)}(x, y, y) &= x, \\
A^{(13)}(A(x, y), y) &= x, \\
A(x, (23)A(x, y)) &= y, \\
A(x, A(x, y)) &= y.
\end{align*}
\]

By tradition the operation \( A \) is denoted by \( \cdot \), \((23)A\) by \( \setminus \) and \((13)A\) by \( / \).

It is possible to give equational definition of \( n \)-ary quasigroup as a generalization of Definition 2.

We follow \([13, 50]\).

Definition 3. An \( n \)-ary groupoid \((Q, A)\) is called an \( n \)-ary quasigroup if on the set \( Q \) there exist operations \((1, n+1)A, (2, n+1)A, \ldots, (n, n+1)A\) such that in the algebra \((Q, A, (1, n+1)A, \ldots, (n, n+1)A)\) the following identities are fulfilled for all \( i \in 1, n \):

\[
\begin{align*}
A^{(i, n+1)}(x^i_1, A^{(i, n+1)}(x^n_1), x^n_{i+1}) &= x_i, \\
A^{(i, n+1)}(A^{(i, n+1)}(x^i_1), x^n_{i+1}) &= x_i.
\end{align*}
\]

In \([29]\) it is proved that any \( n \)-ary quasigroup of order \( k \geq 7 \) is a special kind composition of binary quasigroups isotopic to a fixed quasigroup \([1]\).

Definition 4. Let \((G, \cdot)\) be a groupoid and let \( a \) be a fixed element in \( G \). Translation maps \( L_a \) (left) and \( R_a \) (right) are defined by the following equalities \( L_a x = a \cdot x, R_a x = x \cdot a \) for all \( x \in G \). For quasigroups it is possible to define a third kind of translation, namely, middle translations. If \( P_a \) is a middle translation of a quasigroup \((Q, \cdot)\), then \( x \cdot P_a x = a \) for all \( x \in Q \) \([12]\).

It is well known that in a quasigroup \((Q, \cdot)\) any left and right translation is a bijective map of the set \( Q \) \([10, 52]\).

1.3 Quasigroup based cryptosystem

We give based on binary quasigroup encoding algorithm. We use \([53]\).

A quasigroup \((Q, \cdot)\) and its \((23)\)-parastrophe \((Q, \setminus)\) satisfy the following identities \( x \cdot (x \setminus y) = y, x \setminus (x \cdot y) = y \). These are identities \([33]\) and \([1]\), respectively.

The authors \([37, 38]\) propose to use this quasigroup property to construct the following stream cipher.

\footnote{The author thanks Prof. F.M. Sokhatsky that informed his about this result of M.M. Glukhov.}
Algorithm 1. Let $Q$ be a non-empty finite alphabet, $k$ be a natural number, $u_i, v_i \in Q$, $i \in \{1, \ldots, k\}$. Define a quasigroup $(Q, A)$. It is clear that the quasigroup $(Q, (23)A)$ is defined in a unique way.

Take a fixed element $l$ ($l \in Q$), which is called a leader. Let $u_1, u_2, \ldots, u_k$ be a $k$-tuple of letters from $Q$. It is proposed the following ciphering procedure

$$v_1 = A(l, u_1),$$
$$v_i = A(v_{i-1}, u_i), \quad i = 2, \ldots, k.$$ 

Therefore we obtain the following cipher-text $v_1v_2\ldots v_k$.

The deciphering algorithm is constructed in the following way: $u_1 = (23)A(l, v_1)$, $u_i = (23)A(v_{i-1}, v_i)$, $i = 2, \ldots, k$.

Indeed $(23)A(v_{i-1}, v_i) = (23)A(v_{i-1}, A(v_{i-1}, u_i)) \equiv u_i$.

Notice, the equality $A = (23)A$ is fulfilled if and only if $A(x, A(x, y)) = y$ for all $x, y \in Q$.

1.4 Modifications and generalizations

The improvements and researches of Algorithm 1 were carried out intensively. Some information on this process is given in [53]. We thank our colleagues A. Krapez, V. Bakeva, V. Dimitrova and A. Popovska-Mitrovikj for the following new information.

Remark 1. In article [5], the authors find the distribution of $k$-tuples of letters after $n$ applications of quasigroup transformation ($k > n$) (i.e. Algorithm 1) and give an algorithm for statistical attack in order to discover the original message. Also, they give some conclusions on how to protect the original messages.

In work [34], Krapez defines parastrophic quasigroup transformation. In [6], the authors propose a modification of this transformation and give a new classification of quasigroups of order 4. Finally, in [17] the authors presented this transformation and gave relationship between the new classification and the symmetries of quasigroups.

Notice, parastrophic transformations from [34, 22] are promising for further applications and researches.

In Algorithm 1 it is possible to use also a quasigroup $(Q, A)$ and its (13)-, (123)-, (132)-parastrophe since quasigroup $(Q, A)$ and these parastrophes fulfill the following identities, namely, identities (2), (7), and (8), respectively [55, 34, 22].

$$A(x, y) = y$$

$$A(y, A(x, y)) = x$$

More details in this direction are in [34].

In [38], the authors claimed that this cipher is resistant to the brute force attack (exhaustive search) and to the statistical attack (in many languages some letters meet more frequently, than other letters). Later similar results were presented in [19].

In dissertation of Milan Vojvoda [62] has been proved that this cipher is not resistant to chosen ciphertext attack and chosen plaintext attack. It is claimed that this cipher is not resistant to special kind of statistical attack (Slovak language) [62].

---

2The author thanks his colleagues A. Krapez, V. Bakeva, V. Dimitrova and A. Popovska-Mitrovikj for this information (private letter).
There exist a few other ways to generalize Algorithm 1. The most obvious way is to increase the arity of a quasigroup, i.e., instead of binary to apply $n$-ary ($n \geq 3$) quasigroups. This way was proposed in [53, 54] and was realized in [51, 50]. See below Algorithm 4. Notice Prof. A. Petrescu writes that he found this $n$-ary generalization independently.

In [19], the authors proved that cipher based on Algorithm 4 is not resistant to chosen ciphertext attack and chosen plaintext attack.

Some modifications in order to make Algorithm 1 more resistant against known attacks can be found in [34, 22]. One of these attempts, taking into consideration Vojvoda results [62], was proposed in [56]. Namely instead of a binary quasigroup and its parastrophe it was proposed to use a system of $n$ $n$-ary orthogonal operations (groupoids).

Also it was proposed to use these two crypto-primitives together in one cryptographical procedure.

### 1.5 A modification of Algorithm 1

Sometimes only the use of other record of a mathematical fact leads to a generalization.

We re-write Algorithm 1 using concept of translation in the following way:

**Algorithm 2.** Let $Q$ be a non-empty finite alphabet. Define a quasigroup $(Q, \cdot)$. It is clear that the quasigroup $(Q, (23))$ is defined in a unique way. Take a fixed element $l$ ($l \in Q$), which is called a leader.

Let $u_1 u_2 \ldots u_k$ be a $k$-tuple of letters from $Q$.

It is proposed the following ciphering procedure:

\[
v_1 = l \cdot u_1 = L_l u_1,
\]

\[
v_2 = v_1 \cdot u_2 = L_{v_1} u_2.
\]

\[
v_i = v_{i-1} \cdot u_i = L_{v_{i-1}} u_i, \quad i = 3, \ldots, k.
\]

Therefore we obtain the following cipher-text $v_1 v_2 \ldots v_k$.

The deciphering algorithm is constructed in the following way. We have the following cipher-text: $v_1 v_2 \ldots v_k$. Recall $L_a^{(23)} = (L_a)^{-1}$ for any $a \in Q$ [53]. Below we shall denote translation $L_a^{(23)}$ as $L_a^*$, translation $L_a$ as $L_a$ for any $a \in Q$. Then

\[
u_1 = l^{(23)} \cdot v_1 = L_l^* (v_1) = L_l^* (L_l u_1) = L_{L_l^{-1} (L_l u_1)} u_1;
\]

\[
u_i = v_{i-1}^{(23)} \cdot v_i = L_{v_{i-1}}^* (v_i) = L_{v_{i-1}}^* (L_{v_{i-1}} u_i) = L_{L_{v_{i-1}}^{-1} (L_{v_{i-1}} u_i)} u_i \quad (9)
\]

for all $i \in \overline{2,k}$.

From this form of Algorithm 1 we can obtain easily the following generalization. Instead of translations $L_x, x \in Q$, we propose to use in the enciphering part of this algorithm powers of these translations, i.e., to use permutations of the form $L_x^k, k \in \mathbb{Z}$, instead of permutations of the form $L_x$.

The proposed modification forces us to use permutations of the form $L_x^k, k \in \mathbb{Z}$, also in the decryption procedure.

**Algorithm 3.** Let $Q$ be a non-empty finite alphabet. Define a quasigroup $(Q, \cdot)$. It is clear that the quasigroup $(Q, (23))$ is defined in a unique way.

Take a fixed element $l$ ($l \in Q$), which is called a leader.
Let \( u_1 u_2 \ldots u_k \) be a \( k \)-tuple of letters from \( Q \).

It is proposed the following ciphering procedure

\[
\begin{align*}
v_1 &= L^a_i u_1, a \in \mathbb{Z}, \\
v_2 &= L^b_v u_2, b \in \mathbb{Z}, \\
v_i &= L^c_{v_{i-1}} u_i, i \in \overline{3, k}, c \in \mathbb{Z}.
\end{align*}
\] (10)

Therefore we obtain the following cipher-text \( v_1 v_2 \ldots v_k \). The deciphering algorithm is constructed in the following way. We use notations of Algorithm 2. Recall \((L^*_x)^a = L_x^{-a}\) for all \( x \in Q \). Then

\[
\begin{align*}
(L^*_x)^a (v_1) &= (L^*_x)^a (L^a_x u_1) = u_1, \\
(L^*_x)^b (v_2) &= (L^*_x)^b (L^b_x u_2) = u_2, \\
(L^*_x)^c (v_i) &= (L^*_x)^c (L^c_x u_i) = u_i, i \in \overline{3, k}.
\end{align*}
\] (11)

Notice, the elements \( a, b, c \) in equalities (10) should be vary from step to step in order to protect this Algorithm against chosen plain-text and chosen cipher-text attack. It is clear that the right and middle translations are also possible to use in Algorithm 3 instead of the left translations. See below.

### 1.6 \( n \)-ary analogs of binary algorithms

We give \( n \)-ary analog of Algorithm 4 [51, 19].

**Algorithm 4.** Let \( Q \) be a non-empty finite alphabet, \( k \) be a natural number, \( u_i, v_i \in Q, i \in \{1, \ldots, k\} \). Define an \( n \)-ary quasigroup \((Q, f)\). It is clear that any quasigroup \((Q, (i, n+1) f)\) for any fixed value \( i \) is defined in a unique way. Below for simplicity we put \( i = n \).

Take fixed elements \( l_i^{(n-1)(n-1)} (l_i \in Q) \), which are called leaders.

Let \( u_1 u_2 \ldots u_k \) be a \( k \)-tuple of letters from \( Q \).

It is proposed the following ciphering (encryption) procedure

\[
\begin{align*}
v_1 &= f(l_1^{n-1}, u_1), \\
v_2 &= f(l_2^{n-2}, u_2), \\

\ldots \\
v_{n-1} &= f(l_{n-2}^{n-1}, u_{n-1}), \\
v_n &= f(l_n^{n-1}, u_n), \\
v_{n+1} &= f(v_n^n, u_{n+1}), \\
v_{n+2} &= f(v_{n+1}^n, u_{n+2}), \\
\ldots
\end{align*}
\] (12)

Therefore we obtain the following cipher-text \( v_1 v_2 \ldots, v_{n-1}, v_n, v_{n+1}, \ldots \)
The deciphering algorithm also is constructed similarly with binary case:

\[
\begin{align*}
    u_1 &= (n,n+1) f(t_1^{n-1}, v_1), \\
    u_2 &= (n,n+1) f(t_2^{n-2}, v_2), \\
    & \vdots \\
    u_{n-1} &= (n,n+1) f(t_1^{(n-1)(n-1)}, v_{n-1}), \\
    u_n &= (n,n+1) f(t_1^{n-1}, v_n), \\
    u_{n+1} &= (n,n+1) f(t_2^3, v_{n+1}), \\
    u_{n+2} &= (n,n+1) f(t_3^{n+1}, v_{n+2}), \\
    & \vdots
\end{align*}
\]

(13)

Indeed, for example, \((n,n+1) f(v_1^{n-1}, v_n) = (n,n+1) f(v_1^{n-1}, f(v_1^{n-1}, u_n)) = u_n.

Remark 2. It is easy to see that in encryption procedure (equalities (12)) and, therefore, in decryption procedure (equalities (13)) it is possible to use more than one fixed \(n\)-quasigroup operation \(f\).

Below we shall denote this encryption algorithm as \(G(u)\), because on any step it is enciphered only one element of a plaintext. Probably it makes sense to use in Algorithm 3 irreducible 3-ary or 4-ary finite quasigroup \([13, \text{p. 115}]\). We give an example of 3-ary irreducible quasigroup \((Q, A)\) of order 4 \([13, \text{p. 115}]\).

Example 1.

\[
\begin{array}{ccc|ccc|ccc|ccc}
A_0 & 0 & 1 & 2 & 3 & A_1 & 0 & 1 & 2 & 3 & A_2 & 0 & 1 & 2 & 3 & A_3 & 0 & 1 & 2 & 3 \\
0 & 1 & 0 & 2 & 3 & 1 & 0 & 1 & 2 & 3 & 1 & 0 & 1 & 2 & 3 & 1 & 0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 2 & 1 & 3 & 0 & 1 & 2 & 1 & 3 & 0 & 1 \\
2 & 1 & 3 & 0 & 1 & 2 & 1 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 1 & 3 & 0 & 1 \\
3 & 3 & 2 & 0 & 1 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 3 & 0 & 1 \\
\end{array}
\]

Notice \(A(0,1,2) = A_0(1,2) = 3, A(2,3,2) = A_2(3,2) = 3\). Moreover \(A(0,1,x) = A(2,3,x)\) for any \(x \in Q\). Then translations \(T(0,1,-)\) and \(T(2,3,-)\) are equal, pairs of leaders \((0,1)\), \((2,3)\) are equal from cryptographical point of view.

Recall there exist two groups of order 4, namely cyclic group \(Z_4\) and Klein group \(Z_2 \times Z_2\). Any binary quasigroup of order 4 is a group isotope \([3, \text{p. 4}]\).

Lemma 1. Quasigroup from Example 4 is not an isotope of a 3-ary group \((Q, f)\) with the form \(f(x_1^2) = x_1 + x_2 + x_3\) where \((Q, +)\) is a binary group of order 4.

Proof. If a quasigroup is an isotope of a 3-ary group \((Q, f)\) with the form \(f(x_1^2) = x_1 + x_2 + x_3\) where \((Q, +)\) is a binary group, then this quasigroup is reducible \([13, \text{Corollary, p. 115}]\).
Lemma 2. If \( fT(a_1, \ldots, a_{n-1}, -) \) is a translation of a quasigroup \((Q, f)\), then
\[
fT^{-1}(a_1, \ldots, a_{n-1}, -) = (n, n+1)fT(a_1, \ldots, a_{n-1}, -)
\]

Proof. In the proof we omit the symbol \( f \) in the notation of translations of quasigroup \((Q, f)\). We have
\[
T^{-1}(a_1, \ldots, a_{n-1}, -)(T(a_1, \ldots, a_{n-1}, -)x) = \\
T^{-1}(a_1, \ldots, a_{n-1}, -)f(a_1, \ldots, a_{n-1}, x) = \\
(n, n+1)f(a_1, \ldots, a_{n-1}, f(a_1, \ldots, a_{n-1}, x)) \tag{14}
\]

We propose an \( n \)-ary analogue of Algorithm 3.

Algorithm 5. Let \( Q \) be a non-empty finite alphabet. Define an \( n \)-ary quasigroup \((Q, f)\). It is clear that the quasigroup \((Q, (n, n+1)f)\) is defined in a unique way.

Take fixed elements \((l_i)_{i=1}^{n-1} (l_i \in Q)\), which are called leaders.
Let \( u_1u_2\ldots u_k \) be a \( k \)-tuple of letters from \( Q \).

It is proposed the following ciphering (encryption) procedure
\[
v_1 = T^a(l_1, l_2, \ldots, l_{n-1}, u_1), \\
v_2 = T^b(l_2, l_3, \ldots, l_{n-2}, u_2), \\
\vdots \\
v_{n-1} = T^c(l_{n-3}, \ldots, l_{n-1}, u_{n-1}), \\
v_n = T^d(v_1, \ldots, v_{n-1}, u_n), \\
v_{n+1} = T^e(v_2, \ldots, v_n, u_{n+1}), \\
v_{n+2} = T^f(v_3, \ldots, v_{n+1}, u_{n+2}), \\
\vdots 
\]

Therefore we obtain the following cipher-text \( v_1v_2\ldots v_k \).

Taking into consideration Lemma 2 we can say that deciphering algorithm is possible, it is constructed similarly with the deciphering in Algorithm 3.

Remark 3. It is easy to see that in Algorithm 5 it is possible to use various quasigroup translations and to take quasigroups of various arity.

2 Ciphers based on orthogonal \( n \)-ary groupoids

2.1 Some definitions

We give classical definition of orthogonality of \( n \)-ary operations [9, 15].

Definition 5. \( n \)-ary groupoids \((Q, f_1), (Q, f_2), \ldots, (Q, f_n)\) are called orthogonal, if for any fixed \( n \)-tuple \( a_1, a_2, \ldots, a_n \) the following system of equations
\[
\begin{aligned}
&f_1(x_1, x_2, \ldots, x_n) = a_1 \\
f_2(x_1, x_2, \ldots, x_n) = a_2 \\
&\ldots \\
f_n(x_1, x_2, \ldots, x_n) = a_n
\end{aligned}
\]

has a unique solution.

If the set \( Q \) is finite, then any system of \( n \) orthogonal \( n \)-ary groupoids \((Q, f_i)\) \( i \in 1, n \), defines a permutation of the set \( Q^n \) and vice versa [11, 15, 9]. Therefore if \(|Q| = q\), then there exist \((q^n)!\) systems of \( n \)-ary orthogonal groupoids defined on the set \( Q \).

There exist various generalizations of definition of orthogonality of \( n \)-ary operations. Fresh generalizations are in [57, 58].

**Definition 6.** \( n \)-ary groupoids \((Q, f_1), (Q, f_2), \ldots, (Q, f_k)\) \((2 \leq k \leq n)\) given on a set \( Q \) of order \( m \) are called orthogonal if the system of equations (16) has exactly \( m^{n-k} \) solutions for any k-tuple \( a_1, a_2, \ldots, a_k \), where \( a_1, a_2, \ldots, a_k \in Q \) (see [14]).

If \( k = n \), then from Definition 6 we obtain standard Definition 5. Definition of orthogonality of binary systems has rich and long history [20]. About \( n \)-ary case, for example, see [27].

### 2.2 Construction of orthogonal \( n \)-ary groupoids

In the following example sufficiently convenient and general way for the construction of systems of orthogonal \( n \)-ary groupoids is given.

**Example 2.** Define operations \( A_1(x_1, x_2, x_3), A_2(x_1, x_2, x_3), A_3(x_1, x_2, x_3) \) over the set \( M = \{0, 1, 2\} \) in the following way. Take all 27 triplets \( K = \{(R_i, S_i, T_i) \mid R_i, S_i, T_i \in M, i \in 1, 27\} \) in any fixed order and put

\[
\begin{aligned}
A_1(0, 0, 0) &= R_1, \quad A_1(0, 0, 1) = R_2, \quad A_1(0, 0, 2) = R_3, \ldots, \quad A_1(2, 2, 2) = R_{27}, \\
A_2(0, 0, 0) &= S_1, \quad A_2(0, 0, 1) = S_2, \quad A_2(0, 0, 2) = S_3, \ldots, \quad A_2(2, 2, 2) = S_{27}, \\
A_3(0, 0, 0) &= T_1, \quad A_3(0, 0, 1) = T_2, \quad A_3(0, 0, 2) = T_3, \ldots, \quad A_3(2, 2, 2) = T_{27}.
\end{aligned}
\]

The operations \( A_1, A_2 \) and \( A_3 \) form a system of orthogonal operations. If we take this 27 triplets in other order, then we obtain other system of orthogonal 3-ary groupoids.

This way gives a possibility to construct easily inverse system \( B \) of orthogonal \( n \)-ary operations to a fixed system \( A \) of orthogonal \( n \)-ary operations. Recall inverse system means that \( B(A(x^n_i)) = x^n_i \), \( x_i \in Q \).

**Example 3.** [19]. We give example of three orthogonal ternary groupoids that are defined on four-element set \( \{0, 1, 2, 3\} \). Multiplication table of the first groupoid (in fact, of a quasigroup) is given in Example 1. Below we give multiplication tables of other two 3-ary groupoids.

<table>
<thead>
<tr>
<th>( B_0 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( B_1 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( B_2 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( B_3 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>


From formula \((q^n)!\) follows that there exist \((4^3)! = 64!\) orthogonal systems of 3-ary groupoids over a set of order 4.

2.3 Ciphers on base of orthogonal systems of \(n\)-ary operation

Here we propose to use a system of orthogonal \(n\)-ary groupoids as additional procedure in order to construct almost-stream cipher [56].

Orthogonal systems of \(n\)-ary quasigroups were studied in [59, 60, 25]. Such systems have more uniform distribution of elements of base set and therefore such systems may be more preferable in protection against statistical cryptanalytic attacks.

Algorithm 6. [19]. Let \(A\) be a non-empty finite alphabet, \(k\) be a natural number, \(x^1\) be a plaintext. Take a system of \(n\) \(n\)-ary orthogonal operations \((A,f_i), i = 1,2,\ldots,n\). This system defines a permutation \(F\) of the set \(A^n\). We propose the following enciphering procedure.

- **Step 1:** \(y^n = F^l(x^n), \) where \(l \geq 1, l\) is a natural number, \(l\) is vary from one enciphering round to other. If \(t < n\), then we can add to plaintext some ”neutral” symbols.

- **On the Steps \( \geq 2 \) it is possible to use Feistel schema [28, 43].** For example, we can do the following enciphering procedure \(z^n = F^s(y_2, y_3, \ldots, y_n, x_{n+1}),\) if arity \(n \geq 2,\) or \(z^n = F^s(y_3, y_4, \ldots, y_n, x_{n+1}, x_{n+2}),\) if \(n \geq 3.\) And so on.

The deciphering algorithm is based on the fact that orthogonal system of \(n\) \(n\)-ary operations (16) has a unique solution for any tuple of elements \(a_1, \ldots, a_n.\)

Algorithm 6 is sufficiently safe relative to chosen ciphertext and plaintext attack since the key is a non-periodic sequence of applications of permutation \(F\), i.e. sequence of powers of permutation \(F.\) Therefore any permutation of the group \(\langle F \rangle\) can be used by ciphering information using Algorithm 6.

Recall application of only one step Algorithm 6 is not very safe since this procedure is not resistant relatively chosen ciphertext attack and chosen plaintext attack.

3 Combined algorithms

3.1 Modifications of Algorithm 6

By our opinion some modifications of this algorithm are desirable. Following ”vector ideas” [45] we propose as the first step to write any letter \(u_i\) of a plaintext as \(n\)-tuple \((n\)-vector\) and after that to apply Algorithm 6. For example it is possible to use a binary representation of characters of the alphabet \(A.\)

It is possible to divide plain text \(u_1, \ldots, u_n\) on parts and to use Algorithm 6 to some parts, to a text a part of which has been ciphered by Algorithm 6 on a previous ciphering round.
It is possible to change in Algorithm 6 variables \(x_1, \ldots, x_k\) \((1 \leq k \leq (n-1))\) by some fixed elements of the set \(Q\) and name these elements as leaders. Notice, if \(k = n-1\), then we obtain \(n\) chiphering images from any plaintext letter \(u\).

If in a system of orthogonal \(n\)-ary operations there is at least one \(n\)-ary quasigroup, then we can apply by ciphering of information Algorithm 4 and Algorithm 6 together with some non-periodical frequency, i.e., for example, we can apply four times Algorithm 4 and after this we can apply five times Algorithm 6 and so on.

It is possible to use as a period sequence decimal representation of an irrational or transcendent number. In this case we can take as a key the sequence of application of Algorithm 4 and Algorithm 6.

Proposed modifications make realization of chosen plaintext attack and chosen ciphertext attack more complicate.

Taking into consideration that in binary case one application of Algorithm 6 generates from one plaintext symbol \(u\) two cipher symbols, say \(v_1, v_2\), we may propose to apply Algorithm 6 for two plaintext symbols (or to one cipher symbol and one plain symbol, else to two cipher symbols) simultaneously.

We propose to use Algorithm 4 and Algorithm 6 simultaneously.

Algorithm 7. Suppose that we have a plaintext \(x_1^t, t \geq n\).

1. Divide plaintext on \(n\)-tuples.
2. We apply to any \(n\)-tuple of plaintext \(n\)-ary permutation \(F^1(x_1^n) = y_1^n\).
3. To \(n\)-tuple \(y_1^n\) we apply Algorithm 6 (its binary or \(k\)-ary variant) \(G(y_1^n) = z_1^n\). Probably it will be better, if \(k < n\).
4. We apply to \(n\)-tuple \(z_1^n\) \(n\)-ary permutation \(F^s(z_1^n) = t_1^n\).

Deciphering algorithm is clear.

Below we denote the action of the left (right, middle) translation in the power \(a\) of a binary quasigroup \((Q, g_1)\) on the element \(u_1\) by the symbol \(g_1 T_{u_1}^a\). And so on.

Algorithm 8. Enciphering. Initially we have plaintext \(u_1, u_2, \ldots, u_6\).

\[
\begin{align*}
\text{Step 1.} & \quad g_1 T_{u_1}^a(u_1) = v_1 \\
g_2 T_{v_1}^b(u_2) & = v_2 \\
F_1^c(v_1, v_2) & = (v_1', v_2') \\
\text{Step 2.} & \quad g_3 T_{v_1}^d(u_3) = v_3 \\
g_4 T_{v_2}^e(u_4) & = v_4 \\
F_2^f(v_3, v_4) & = (v_3', v_4') \\
\text{Step 3.} & \quad g_5 T_{v_3}^g(u_5) = v_5 \\
g_6 T_{v_4}^h(u_6) & = v_6 \\
F_3^i(v_5, v_6) & = (v_5', v_6')
\end{align*}
\]
And so on. We obtain ciphertext $v'_1, v'_2, \ldots, v'_6$.

Deciphering. Initially we have ciphertext $v'_1, v'_2, \ldots, v'_6$.

Step 1.

\[ F_1^{-c}(v'_1, v'_2) = (v_1, v_2) \]
\[ g_1 T_1^{-a}(v_1) = u_1 \]
\[ g_2 T_2^{-b}(v_2) = u_2 \]

Step 2.

\[ F_2^{-f}(v'_3, v'_4) = (v_3, v_4) \]
\[ g_3 T_3^{-d}(v_3) = u_3 \]
\[ g_4 T_4^{-e}(v_4) = u_4 \]

Step 3.

\[ F_3^{-i}(v'_5, v'_6) = (v_5, v_6) \]
\[ g_5 T_5^{-g}(v_5) = u_5 \]
\[ g_6 T_6^{-h}(v_6) = u_6 \]

We obtain plaintext $u_1, u_2, \ldots, u_6$.

It is clear that Algorithm 8 is a partial case of Algorithm 8.

As in Algorithm 8 in Algorithm 8 the elements $a, b, c, \ldots, h$ should be vary in order to protect this algorithm against chosen plain-text and chosen cipher-text attack.

Algorithm 8 allows to obtain almost "natural" stream cipher, i.e. stream cipher that encode a pair of elements of a plaintext on any step. It is easy to see that Algorithm 8 can be generalized on $n$-ary ($n \geq 3$) case. One of the possible generalizations is realized in Algorithm 10.

Additional researches are necessary for the proposed in this subsection modifications.

### 3.2 Stream cipher on base of orthogonal system of binary parastrophic quasigroups

This subsection is more of algebraic than cryptographical character. For the construction of Algorithms 8 and 8 we propose the use of orthogonal systems of binary parastrophic quasigroups.

We start from the following theorem [47]. Here expression $A \perp A$ means that quasigroups $(Q, A)$ and $(Q, A)$ are orthogonal.

**Theorem 1.** For a finite quasigroup $(Q, A)$ the following equivalences are fulfilled:

(i) $A \perp A \iff (x \cdot z) \cdot x = (y \cdot z) \cdot y \iff x = y$;
(ii) $A \perp A \iff (z \cdot x = zy \cdot y \implies x = y)$;
(iii) $A \perp A \iff (x \cdot xz = y \cdot yz \implies x = y)$;
(iv) $A \perp A \iff (x \cdot zx = y \cdot zy \implies x = y)$;
(v) $A \perp A \iff (x \cdot x = yz \cdot y \implies x = y)$

for all $x, y, z \in Q$.

In order to construct quasigroups mentioned in Theorem 8 probably computer search is preferable. It is possible to use GAP and Prover [42].

12
**Definition 7.** A $T$-quasigroup $(Q, A)$ is a quasigroup of the form $A(x, y) = \varphi x + \psi y + c$, where $(Q, +)$ is an abelian group, $\varphi, \psi$ are some fixed automorphisms of this group, $c$ is a fixed element of the set $Q$ [48, 33].

If $(Q, \cdot)$ is a $T$-quasigroup of the form $x \cdot y = \varphi x + \psi y + c$, then its parastrophes have the following forms, respectively:

\[
egin{align*}
x^{(12)} & \cdot y = \psi x + \varphi y + c, \\
x^{(13)} & \cdot y = \varphi^{-1} x - \varphi^{-1} \psi y - \varphi^{-1} c, \\
x^{(23)} & \cdot y = -\psi^{-1} \varphi x + \psi^{-1} y - \psi^{-1} c, \\
x^{(123)} & \cdot y = -\varphi^{-1} \psi x + \varphi^{-1} y - \varphi^{-1} c, \\
x^{(132)} & \cdot y = \psi^{-1} x - \psi^{-1} \varphi y - \psi^{-1} c.
\end{align*}
\]

See, for example, [47].

In order to construct a quasigroup $(Q, A)$ that is orthogonal with its parastrophe in more theoretical way it is possible to use the following theorem [47].

**Theorem 2.** For a $T$-quasigroup $(Q, A)$ of the form $A(x, y) = \varphi x + \psi y + c$ over an abelian group $(Q, +)$ the following equivalences are fulfilled:

(i) $A \perp_{12} A \iff (\varphi - \psi), (\varphi + \psi)$ are permutations of the set $Q$;

(ii) $A \perp_{13} A \iff (\varepsilon + \varphi)$ is a permutation of the set $Q$;

(iii) $A \perp_{23} A \iff (\varepsilon + \psi)$ is a permutation of the set $Q$;

(iv) $A \perp_{123} A \iff (\varphi + \psi^2)$ is a permutation of the set $Q$;

(v) $A \perp_{132} A \iff (\varphi^2 + \psi)$ is a permutation of the set $Q$.

**Corollary 1.** $T$-quasigroup $(Z_p, \circ)$ of the form $x \circ y = k \cdot x + m \cdot y + c$, where $(Z_p, +)$ is the cyclic group of a prime order $p$, $k, m, c \in Z_p$; $k, m, k + m, k - m, k + 1, m + 1, k^2 + m, k + m^2 \neq 0 \pmod p$, where the operation $\cdot$ is multiplication modulo $p$, is orthogonal to any of its parastrophes.

Quasigroups from Corollary 1 are suitable objects to construct above mentioned Algorithms (binary case).

The following table contains connections between different kinds of translations in different parastrophes of a binary quasigroup $(Q, \cdot)$ [53, 55].

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon$</th>
<th>(12)</th>
<th>(13)</th>
<th>(23)</th>
<th>(123)</th>
<th>(132)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>$R$</td>
<td>$L$</td>
<td>$R^{-1}$</td>
<td>$P$</td>
<td>$P^{-1}$</td>
<td>$L^{-1}$</td>
</tr>
<tr>
<td>$L$</td>
<td>$L$</td>
<td>$R$</td>
<td>$P^{-1}$</td>
<td>$L^{-1}$</td>
<td>$R^{-1}$</td>
<td>$P$</td>
</tr>
<tr>
<td>$P$</td>
<td>$P$</td>
<td>$P^{-1}$</td>
<td>$L^{-1}$</td>
<td>$R$</td>
<td>$L$</td>
<td>$R^{-1}$</td>
</tr>
<tr>
<td>$R^{-1}$</td>
<td>$R^{-1}$</td>
<td>$L^{-1}$</td>
<td>$R$</td>
<td>$P^{-1}$</td>
<td>$P$</td>
<td>$L$</td>
</tr>
<tr>
<td>$L^{-1}$</td>
<td>$L^{-1}$</td>
<td>$R^{-1}$</td>
<td>$P$</td>
<td>$L$</td>
<td>$R$</td>
<td>$P^{-1}$</td>
</tr>
<tr>
<td>$P^{-1}$</td>
<td>$P^{-1}$</td>
<td>$P$</td>
<td>$L$</td>
<td>$R^{-1}$</td>
<td>$L^{-1}$</td>
<td>$R$</td>
</tr>
</tbody>
</table>

From Table 1 it follows, for example, that $R^{(13)} = R^{-1}$. 

13
3.3 T-quasigroup based stream code

We give a numerical example of encryption Algorithm 8 based on T-quasigroups. Notice the number 257 is prime.

Example 4. Take the cyclic group \((\mathbb{Z}_{257}, +) = (A, +)\).

1. Define T-quasigroup \((A, *)\) with the form \(x * y = 2 \cdot x + 131 \cdot y + 3\) with a leader element \(l_1\), say, \(l_1 = 17\). Denote the mapping \(x \mapsto x * l_1\) by the letter \(R_{l_1}\), i.e. \(R_{l_1}(x) = x * l_1\) for all \(x \in A\).

In order to find the mapping \(R_{l_1}^{-1}\) taking into consideration Table 1 we find the form of operation \(*^{(13)}\) using formula (19). We have \(x^{(13)} * y = 129 \cdot x + 63 \cdot y + 127\), \(R_{l_1}^{-1}(x) = x * l_1 = R_{l_1}^{(13)} x\).

In some sense quasigroup \((A, *)^{(13)}\) is the "right inverse quasigroup" to quasigroup \((A, *)\). From identity (8) follows that quasigroup \((A, \cdot^{(13)}\) is the "left inverse" quasigroup to quasigroup \((A, \cdot)\). Notice from Corollary 1 follows that \((A, \cdot) \perp (A, *)^{(13)}\).

2. Define T-quasigroup \((A, \circ)\) with the form \(x \circ y = 10 \cdot x + 81 \cdot y + 53\) with a leader element \(l_2\), say, \(l_2 = 71\). Denote the mapping \(x \mapsto l_2 * x\) by the letter \(L_{l_2}\), i.e. \(L_{l_2}(x) = l_2 \circ x\) for all \(x \in A\).

In order to find the mapping \(L_{l_2}^{-1}\) we use Table 1 and find the form of operation \(\circ^{(23)}\) by formula (19). We have \(x^{(23)} \circ y = 149 \cdot x + 165 \cdot y + 250\).

3. Define a system of two parastroph orthogonal T-quasigroups \((A, \cdot)\) and \((A, \circ^{(23)})\) in the following way

\[
\begin{align*}
  x \cdot y &= 3 \cdot x + 5 \cdot y + 6 \\
  x^{(23)} \circ y &= 205 \cdot x + 103 \cdot y + 153
\end{align*}
\]

Denote quasigroup system \((A, \cdot, \circ^{(23)})\) by \(F(x, y)\), since this system is a function of two variables. In order to find the mapping \(F^{-1}(x, y)\) we solve the system of linear equations

\[
\begin{align*}
  3 \cdot x + 5 \cdot y + 6 &= a \\
  205 \cdot x + 103 \cdot y + 153 &= b
\end{align*}
\]

We have \(\Delta = 55, 1/\Delta = 243, x = 100 \cdot a + 70 \cdot b + 255, y = 43 \cdot a + 215 \cdot b\). Therefore we have, if \(F(x, y) = (a, b)\), then \(F^{-1}(a, b) = (100 \cdot a + 70 \cdot b + 255, 43 \cdot a + 215 \cdot b)\), i.e.

\[
\begin{align*}
  x &= 100 \cdot a + 70 \cdot b + 255 \\
  y &= 43 \cdot a + 215 \cdot b
\end{align*}
\]

We have defined the mappings \(g_1 = R_{l_1}, g_2 = L_{l_2}, F\) and now we can use them in Algorithm 8. Let 212; 17; 65; 117 be a plaintext. We take the following values in formula (17): \(a = b = d = e = f = 1; c = 2\). Below we use Gothic font to distinguish leader elements, i.e. 17, 71 are leader elements. Then

Step 1.
\[
\begin{align*}
  g_1(212) &= 212 * 17 = 2 \cdot 212 + 131 \cdot 17 + 3 = 84 \\
  g_2(17) &= 71 \circ 17 = 10 \cdot 71 + 81 \cdot 17 + 53 = 84
\end{align*}
\]

14
Step 2.
\[ g_1(65) = 65 \cdot 67 + 131 = 2 \cdot 65 + 131 \cdot 67 + 3 = 172 \]
\[ g_2(117) = 171 \cdot 117 + 81 = 10 \cdot 171 + 81 \cdot 117 + 53 = 189 \]
\[ F(172; 189) = (3 \cdot 172 + 5 \cdot 189 + 6; 205 \cdot 172 + 103 \cdot 189 + 153) = (182; 139) \]

We obtain the following ciphertext 67; 171; 182; 139.

For deciphering we use formula \( F^{-1} \).

Step 1.
\[ F^{-1}(67; 171) = (100 \cdot 67 + 70 \cdot 171 + 255, 43 \cdot 67 + 215 \cdot 171) = (164; 84) \]
\[ g_1^{-1}(84) = 84 \cdot 17 = 129 \cdot 84 + 63 \cdot 17 + 127 = 212 \]
\[ g_2^{-1}(84) = 71 \cdot 84 = 149 \cdot 71 + 165 \cdot 84 + 250 = 17 \]

Step 2.
\[ F^{-1}(182; 139) = (100 \cdot 182 + 70 \cdot 139 + 255, 43 \cdot 182 + 215 \cdot 139) = (172; 198) \]
\[ g_1^{-1}(172) = 172 \cdot 67 = 129 \cdot 172 + 63 \cdot 67 + 137 = 65 \]
\[ g_2^{-1}(189) = 171 \cdot 189 = 149 \cdot 171 + 165 \cdot 189 + 250 = 117 \]

A little program using freeware version of programming language Pascal was developed. First little experiments demonstrate that encoding-decoding is executed sufficiently fast.\(^3\)

Remark 4. Proper binary groupoids are more preferable than linear quasigroups by construction of the mapping \( F(x, y) \) in order to make encryption more safe, but in this case decryption may be slower than in linear quasigroup case and definition of these groupoids needs more computer (or some other device) memory. The same remark is true for the choice of the function \( g \). Maybe a golden mean in this choice problem is to use linear quasigroups over non-abelian, especially simple, groups.

Remark 5. In this cipher there exists a possibility of protection against standard statistical attack. For this scope it is possible to denote more often used letters or pair of letters by more than one integer or by more than one pair of integers.

3.4 Some generalization of functions of Algorithm \( ^8 \)

We give a method for the construction of functions that it is possible to use in cryptographical procedures. Suppose that all functions are defined on a set \( Q \). Functions \( F(x^n) \) and \( g(x^n) \) are functions of \( n \) variables.

Function \( F \) (\( n \) orthogonal groupoids, a permutation of the set \( Q^n \)) has inverse function of \( n \) variables \( F^{-1}(x^n) \) such that \( F(F^{-1}(x^n)) = x^n \).

We recall, if \( g \) is \( n \)-ary quasigroup operation, then, in general, we cannot decode values \( x, y \), for example, from equality \( g(\pi^{n-2}, x, y) = b \), but we can easy solve equation \( g(\pi^{n-1}, x) = b \) of one variable, i.e. we can decode value of variable \( x \).

Taking into consideration this quasigroup feature, we describe the set (clone) of functions that it is possible to use in cryptology on base of these two kinds of functions, namely, functions \( F \) and

\(^3\)The author thanks D.I. Pushkashu and A.V. Shcherbacov for their help by the writing of this program.
We shall use concept of term \([63]\) to define cryptographical terms (cryptographical functions) inductively.

Cryptographical function (cryptographical term) below in Case 3 means that encoding and decoding of a text using this function (this term) is performed uniquely.

Algorithm 9. 1. Any individual constant is a cryptographical term.

2. Any individual variable is a cryptographical term.

3. (a) If \(g\) is an \(n\)-ary quasigroup functional constant (\((Q, g)\) is an \(n\)-ary quasigroup) and \(t\) is a term, \(b_1^n\) are individual constant, then \(g^a(b_1^{i-1}; t, b_{i+1}^n)\), \(i \in 1, n\), where \(a \in \mathbb{Z}\), is a cryptographical term.

(b) If \(F\) is a permutation of a set \(Q^n\) which is constructed using \(n\) orthogonal \(n\)-ary groupoids and \(t_1, t_2, \ldots, t_n\) are quasigroup cryptographical terms, then \(F^a(t_1, \ldots, t_n)\), where \(a \in \mathbb{Z}\), is a cryptographical term.

Example 5. Let \(Q = B \times B\) be a non-empty set, \(F\) be a pair of orthogonal groupoids every of which is defined on the set \(B\), and \((Q, g)\) be a ternary quasigroup. Then \(g(q_1, q_2, F)\), where \(q_1, q_2\) are fixed elements of \(Q\), is a cryptographical term constructed following Rule 3, (a) of Algorithm 9.

In Example 4 cryptographical term \(F^a(g_1, g_2)\) is constructed following Rule 3, (b) of Algorithm 9. Indeed, the function \(F\) is a pair of parastrophic orthogonal \(T\)-quasigroups that are defined on the set \(Z_{257}\), i.e. \(F\) is a permutation of the set \(Z_{257} \times Z_{257}\); \((Z_{257}, g_1)\), \((Z_{257}, g_2)\) are binary \(T\)-quasigroups, and \(a = 1; 2\).

Algorithm 10. Suppose that we have \(n\)-ary permutation \(F\), \(n\) procedures \(G_j\) (they may be of various arity and it is supposed that leader elements are used) and plaintext \(x_1^m\).

By the letter \(y\) with an index we denote an element of enciphered text or a leader element. We propose the following enciphering procedure.

The \(i\)-th step of this procedure can have the following form

\[ iF^x(y_1^m, x_i), \ldots, G_n(y_1^m, x_i) = i y_1^n \]  

(20)

Deciphering algorithm is executed "from the top to the bottom" in general and "from the bottom to the top" on any step. See more details in Algorithm 8.

3.5 On quasigroup based cryptcode

Using possibilities that give us Algorithms 9 and 10 we give an example of a quasigroup based hybrid\(^4\) of a code and a cypher. Following Markovski, Gligoroski, and Kocarev \([40, 39]\), we name such hybrid as a cryptcode.

We shall use Klein group \(Z_2 \oplus Z_2\), its automorphism group and the system of three ternary orthogonal groupoids (Example 3).

Denote elements of the group \(Z_2 \oplus Z_2\) as follows: \([(0; 0), (1; 0), (0; 1), (1; 1)]\). The group \(\text{Aut}(Z_2 \oplus Z_2)\) consists of the following automorphisms:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}
\]

\(^4\)Hybrid idea is sufficiently known. For example, see [54] page 2, [53] page 65.
Denote these automorphisms by the letters \( \varepsilon, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \), respectively.

Notice \( \varphi_2^2 = \varphi_3^2 = \varphi_4^2 = \varepsilon, \varphi_5^2 = \varphi_6, \varphi_6^2 = \varphi_5 \). It is known that \( \text{Aut}(Z_2 \oplus Z_2) \cong S_3 \) \[31\ \[32\].

For convenience we give Cayley table of the group \( \text{Aut}(Z_2 \oplus Z_2) \).

\[
\begin{array}{cccccc}
\cdot & \varepsilon & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 \\
\varepsilon & \varepsilon & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 \\
\varphi_2 & \varphi_2 & \varepsilon & \varphi_5 & \varphi_6 & \varphi_3 & \varphi_4 \\
\varphi_3 & \varphi_3 & \varphi_6 & \varepsilon & \varphi_5 & \varphi_4 & \varphi_2 \\
\varphi_4 & \varphi_4 & \varphi_5 & \varphi_6 & \varepsilon & \varphi_2 & \varphi_3 \\
\varphi_5 & \varphi_5 & \varphi_4 & \varphi_2 & \varphi_3 & \varepsilon & \varphi_6 \\
\varphi_6 & \varphi_6 & \varphi_3 & \varphi_4 & \varphi_2 & \varphi_6 & \varepsilon \\
\end{array}
\]

Information on codes is in \[17\]. We shall use a code that is given in \[46\ Example 19\]. Suppose that the symbols \( x, y \) are informational symbols and the symbol \( z \) is a check symbol. Remember, \( x, y, z \in (Z_2 \oplus Z_2) \). We propose the following check equation \( x + \varphi_5 y + \varphi_6 z = (0; 0) \), i.e., we propose the following formula to find the element \( z \):

\[
z = \varphi_5 x + \varphi_6 y \tag{21}
\]

Recall, statistical investigations of J. Verhoeff \[61\] and D.F. Beckley \[8\] have shown that the most frequent errors made by human operators during transmission of data are single errors (i.e. errors in exactly one component), adjacent transpositions (in other words errors made by interchanging adjacent digits, i.e. errors of the form \( ab \rightarrow ba \)), and insertion or deletion errors. We note, if all codewords are of equal length, insertion and deletion errors can be detected easily.

Proposed code detects any single, transposition, and twin \( (aa \rightarrow bb) \) errors \[46\].

Further we construct three \( T \)-quasigroups over the group \( Z_2 \oplus Z_2 \):

\( (Z_2 \oplus Z_2, D) \) with the form \( D(x, y) = \varphi_3 x + \varphi_6 y + a_1 \);

\( (Z_2 \oplus Z_2, E) \) with the form \( E(x, y) = \varphi_2 x + \varphi_5 y + a_2 \);

\( (Z_2 \oplus Z_2, F) \) with the form \( F(x, y) = \varphi_3 x + \varphi_5 y + a_3 \).

We use the following

**Theorem 3.** A \( T \)-quasigroup \( (Q, \cdot) \) of the form \( x \cdot y = \alpha x + \beta y + c \) and a \( T \)-quasigroup \( (Q, \circ) \) of the form \( x \circ y = \gamma x + \delta y + d \), both over a group \( (Q, +) \) are orthogonal if and only if the map \( \alpha^{-1} \beta - \gamma^{-1} \delta \) is an automorphism of the group \( (Q, +) \) \[37\].

**Lemma 3.** The quasigroups \((Z_2 \oplus Z_2, D), (Z_2 \oplus Z_2, E), \) and \((Z_2 \oplus Z_2, F)\) are orthogonal in pairs.

**Proof.** We can use Theorem 3 and Cayley table of the group \( \text{Aut}(Z_2 \oplus Z_2) \).

Define three ternary operations in the following way: \( K_1(D(x, y), z) = D(x, y) + z, K_2(E(x, y), z) = E(x, y) + z, K_3(F(x, y), z) = F(x, y) + z \).

**Lemma 4.** The triple of ternary operations \( K_1(x, y, z), K_2(x, y, z), K_3(x, y, z) \) forms orthogonal system of operation.

**Proof.** We solve the following system of equations

\[
\begin{align*}
\varphi_3 x + \varphi_6 y + a_1 + z &= b_1 \\
\varphi_2 x + \varphi_5 y + a_2 + z &= b_2 \\
\varphi_3 x + \varphi_5 y + a_3 + z &= b_3
\end{align*}
\tag{22}
\]
where \( b_1, b_2, b_3 \) are fixed elements of the set \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

We use properties of the groups \((\mathbb{Z}_2 \oplus \mathbb{Z}_2)\) and \(\text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)\).

\[
\begin{align*}
\varphi_3 x + \varphi_6 y + z & = b_1 + a_1 \\
\varphi_2 x + \varphi_5 y + z & = b_2 + a_2 \\
\varphi_3 x + \varphi_5 y + z & = b_3 + a_3
\end{align*}
\] (23)

We are doing the following transformations of the system (23): (first row + third row) \(\rightarrow\) first row; (second row + third row) \(\rightarrow\) second row; and obtain the following system:

\[
\begin{align*}
y & = b_1 + a_1 + b_3 + a_3 \\
x & = \varphi_4 (b_2 + a_2 + b_3 + b_4) \\
\varphi_3 x + \varphi_5 y + z & = b_3 + a_3
\end{align*}
\] (24)

If in the system (24) in the third equation we replace \(x\) by \(\varphi_4 (b_2 + a_2 + b_3 + b_4)\) and \(y\) by \(b_1 + a_1 + b_3 + a_3\), then we obtain

\[
\begin{align*}
x & = \varphi_4 (b_2 + a_2 + b_3 + a_3) \\
y & = b_1 + a_1 + b_3 + a_3 \\
z & = b_3 + a_3 + \varphi_5 (b_1 + a_1 + b_2 + a_2)
\end{align*}
\] (25)

Therefore the system (22) has a unique solution for any fixed elements \( b_1, b_2, b_3 \in (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \), operations \(K_1(x, y, z), K_2(x, y, z), K_3(x, y, z)\) are orthogonal.

Triple of orthogonal operations \(K_1(x, y, z), K_2(x, y, z), K_3(x, y, z)\) defines on the set \(Q^3\) a permutation. Denote this permutation by the letter \(K\).

We shall use the system of three ternary orthogonal groupoids \((Q, A), (Q, B), (Q, C)\) of order 4 from Example 3. See also [19]. Denote permutation that defines this system of three ternary orthogonal groupoids by the letter \(M\).

In order to use the system of orthogonal groupoids and the system of orthogonal \(T\)-quasigroups simultaneously we redefine the basic set of the \(T\)-quasigroups in the following (non-unique) way \((0; 0) \rightarrow 0, (1; 0) \rightarrow 1, (0; 1) \rightarrow 2, (1; 1) \rightarrow 3\).

We propose the following cryptographical term (a cryptographical primitive):

\[
H(x, y, z) = M^k (K^l (x, y, z)), k, l \in \mathbb{Z}
\]

Transformation \(H\) is a permutation of the set \(Q^3\). Indeed, this transformation is a composition of two permutations: \(K^l\) and \(M^k\).

Therefore we propose the following

**Algorithm 11.**

1. Take a pair of information symbols \(a, b \in (\mathbb{Z}_2 \oplus \mathbb{Z}_2)\);
2. by formula (27) find value of the check symbol \(c\);
3. apply cryptographical term \(H\) to the triple \((a, b, c)\);
4. take a pair of information symbols \(d, e \in (\mathbb{Z}_2 \oplus \mathbb{Z}_2)\);
5. by formula (27) find value of the check symbol \(f\);
6. change values of the numbers \(k, l\) in the cryptographical term \(H\); also it is possible to change the term \(H\) by some other term of such or other type;

7. apply cryptographical term \(H\) to the triple \((d, e, f)\);

8. and so on.

Procedure of decoding in Algorithm \([11]\) is clear.

Recall, the number \(N(n)\) of mutually (in pairs) orthogonal Latin squares of order \(n\) fulfills the following inequality \(N(n) \leq (n - 1)^3\) \([35]\). Then for \(n = 4\) we have \(N(4) \leq 3\). Therefore, for real applications an analog of Algorithm \([11]\) should be constructed over a set of order more than 4 and, probably, with more powerful code \([7]\).

### 3.6 A comparison of the ”power” of proposed algorithms

We shall compare how many permutations and of what length can be generated and can be used by the working of some above mentioned algorithms.

**Algorithm 1.** If we shall use only one quasigroup \((Q, \cdot), |Q| = n\), then we can obtain by encoding not more than \(n\) permutations of the group \(S_n\).

**Algorithm 3.** If we shall use only one quasigroup \((Q, \cdot), |Q| = n\), then we shall use by encoding the set \(S = \bigcup_{i=1}^{n} \langle L_{a_i} \rangle\) of permutations which is a subset of the left multiplication group \(LM\) of quasigroup \((Q, \cdot)\). We recall \(LM(Q, \cdot) = \langle L_x | x \in Q \rangle\) \([10, 52, 53]\).

It is possible to construct a quasigroup \((Q, \cdot)\) such that \(LM(Q, \cdot) = S_Q\). Notice, it is proved \([24]\) that there exist quasigroups with the property \(LM(Q, \cdot) = A_Q\), where \(A_Q\) is the alternating group defined on the set \(Q\) \([32, 31]\).

Therefore by encoding using Algorithm \([3]\) we can obtain not more than \(|S_n| = n!\) permutations.

Situation with Algorithm \([4]\) is similar to the situation with Algorithm \([11]\). Since by encoding translations of an \(m\)-ary quasigroup \((Q, f)\) are used, we can obtain not more than \(|S_n| = n!\) permutations. The properties of multiplication group (more exactly, multiplication groups) of \(n\)-ary quasigroups are not researched well.

Information on the multiplication groups of linear \(n\)-ary quasigroups is in \([36]\). These quasigroups are used in \([51, 50]\) by construction of some ciphers (see above).

Algorithm \([5]\) is a synthesis of Algorithms \([3]\) and \([1]\). Here by the symbol \(T_i\) we denote translations of an \(n\)-ary quasigroup \((Q, f)\). It is clear that the order of the set \(S = \bigcup \langle T_i \rangle\) can be large but cannot be more than \(|S_n| = n!\).

In Algorithm \([5]\) elements of the cyclic group \(\langle F \rangle \subset S_{n^m}\), where \(|Q| = n, m\) is the arity of orthogonal groupoids, can appear. In the above-mentioned inclusion cannot be equality even theoretically, since the minimal number of generators of the symmetric group is equal to two \([32, 31]\).

It is well known that a cycle of order \(n\) and a cycle of order two generate the symmetric group \(S_n\) \([32, 31]\).

The group \(S_{n^m}\) is an upper bound of the sets of permutations that can be generated during the work of Algorithms \([4, 10]\). For Algorithm \([8]\) the group \(S_{n^2}\) is such upper bound. It is clear that in Algorithm \([8]\) by the encryption any permutation of the group \(S_{n^2}\) may be realized. But it also is clear that this is not necessary from the cryptographical point of view.

The possible number of permutation generated during the work of the algorithm from Example \([4]\) is bounded by the number \((257^2)! = 66049!\) and during the work of Algorithm \([11]\) is bounded by the number \((64)!\).
Acknowledgement. The author started this project together with Professor Piroska Cs"org"o [19]. Unfortunately Prof. Cs"org"o has informed the author that she cannot continue this project. The author is grateful to Prof. Cs"org"o for useful discussions and the help by writing this paper.

References


[34] A. Krapez. An application of quasigroups in cryptology. Accepted for publication in Math. Maced.


