ON 2 × 2 CONSERVATION LAWS AT A JUNCTION

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Abstract. This paper deals with 2 × 2 conservation laws at a junction. For the Cauchy problem, existence, uniqueness and Lipschitz continuous dependence of the solution from the initial data as well as from the conditions at the junction are proved. The present construction comprehends the case of the p-system used to describe gas flow in networks and hereby unifies different approaches present in the literature. Furthermore, different models for water networks are considered.

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1. Introduction. This paper studies the initial value problem consisting of

\[ \partial_t u_l + \partial_x f(u_l) = 0 \quad \text{with } l = 1, \ldots, n, \ t \in [0, +\infty], \ x \in [0, +\infty]. \]  

along n pipes together with

\[ \Psi \left( u_1(t,0^+), u_2(t,0^+), \ldots, u_n(t,0^+) \right) = 0 \]

at a junction. In other words, we deal with n initial – boundary value problems for systems of 2 × 2 conservation laws, coupled through nonlinear boundary conditions. In this general setting, extending the results in [10], we derive conditions under which the Cauchy problem for (1.1) has a unique solution. Furthermore, the Lipschitz continuous dependence of the solution on initial data and coupling conditions is proved.

We model a junction connecting n rectilinear pipes by n pairwise distinct vectors \( \nu_1, \ldots, \nu_n \) parallel to the pipes and such that \( \| \nu_l \| \) equals to the cross section of the \( l \)-th pipe. Furthermore, we assign a space coordinate \( x_l > 0 \) to each pipe. The transport of a specific quantity in the \( l \)-th pipe is given by (1.1), where \( u_l \) is the vector of variables along the \( l \)-th duct and \( f \) is a general nonlinear flux function. Condition (1.2) describes the interaction of the transported quantities at the intersection of the pipes, see [2, 3, 9, 10, 11, 19]. The standard situation of the Cauchy problem on a line is recovered in the case \( n = 2, \nu_1 + \nu_2 = 0 \) and \( \Psi(u_1, u_2) = f(u_1) - f(u_2) \), see Paragraph 4.1. To simplify the notation, below we denote by \( x \) all the coordinates \( x_l \) along the various pipes.

Applications of the theoretical results are in the field of fluid flow in networks and in particular in high–pressure gas pipelines open canals. In recent years, there has been intense research in flow problems on networks, see e.g. the book on gas networks [23] and the publications of the Pipeline Simulation Interest group [26]. Most of the proposed models [12, 14, 21, 23, 24, 25, 27, 22, 18] consider each pipe as a 1D domain and use balance laws to describe the dynamics. The validity of one–dimensional models is the subject of intense discussions, we refer to [26] for more details.

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In this context, the most challenging and interesting point is the coupling condition at pipe–pipe intersections. In an engineering context, this coupling is typically modeled through tables prescribing suitable relations depending on e.g. the geometry of the pipe, the material and flow conditions [12, 21]. Among the first mathematical treatments of this situation are [2, 3, 9, 10, 19]. The current presentation considers the subsonic case as in [2, 3, 9, 10, 11, 19] and typical physical conditions [14]. Extending the presentations in [4, 16, 20], we consider solutions possibly containing shocks.

Our purpose is to present a general framework for coupling conditions and prove well posedness. Indeed, we unify and extend the approaches [2, 3, 9, 10, 16, 19]. First, the present framework includes the currently used one–dimensional isothermal model for gas flow as well as the shallow–water equations for flows in open channels. In particular, we extend the results in [10] covering not only the isentropic Euler equations, but a general $2 \times 2$ system. More important, the present result allows to consider any coupling conditions specified through any (possibly nonlinear) function $\Psi$, the sole constraint being condition (2.2) below. Hence, the present results also extend previous works on open channel flow with gate control or pumps, see [16], as well the model for a kink in a pipe introduced in [19]. Finally, within this general setting, we also prove the Lipschitz continuous dependence of the solutions from the condition $\Psi$ at the junction, see (3.2) in Theorem 3.2. As a consequence, in all the cases mentioned above, it is possible to prove the existence of an optimal control.

Numerically, we show that different 1D coupling conditions lead to qualitatively different solutions. The comparison is in the context of the present theory, i.e. for one–dimensional models. 2D situations are considered, for instance, in [17, 18]. For a comparison with results of the engineering community we refer to the pressure loss tables [12, 21].

The paper is organized as follows. Section 2 is devoted to the Riemann Problem and extends [9]. In Section 3, the well–posedness of the Cauchy problem is stated. Applications of this result to gas flow in pipes as well as flow in open canals are collected in Section 4. Section 5 contains the detailed constructions and proofs.

2. The Riemann Problem at a Junction. Throughout, we refer to [6] for the general theory of hyperbolic systems of conservation laws. Let $\Omega \subseteq \mathbb{R}^2$ be a non empty open set. Fix a flow $f \in C^4(\Omega; \mathbb{R}^2)$ satisfying the following assumption:

(F) There exists a $\tilde{u} \in \Omega$ such that $Df(\tilde{u})$ admits a strictly negative eigenvalue $\lambda_1(\tilde{u})$ and a strictly positive one $\lambda_2(\tilde{u})$, the corresponding eigenvectors are linearly independent and each characteristic field is either genuinely nonlinear or linearly degenerate.

Under this condition, (1.1) generates a Standard Riemann Semigroup, see [6, Chapter 8]. By Riemann Problem at the Junction we mean the problem

\[
\begin{cases}
\partial_t u_l + \partial_x f (u_l) = 0 & t \in \mathbb{R}^+ \quad l \in \{1, \ldots, n\} \\
u_l(0, x) = \tilde{u}_l, & x \in \mathbb{R}^+ \quad u_l \in \Omega
\end{cases}
\]

where $\tilde{u}_1, \ldots, \tilde{u}_n$ are constant states in $\Omega$. For $l = 1, \ldots, n$, $u_l$ has two components, i.e. $u_l = (u_{l,1}, u_{l,2})$ denotes the densities of the conserved quantities in the $l$-th tube.

**Definition 2.1.** Fix a map $\Psi \in C^4(\Omega^n; \mathbb{R}^n)$. A $\Psi$-solution to the Riemann Problem (2.1) is a function $u : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \Omega^n$ such that

(L) For $l = 1, \ldots, n$, the function $(t, x) \mapsto u_l(t, x)$ is self-similar and coincides with
the restriction to \( x \in \mathbb{R}^+ \) of the Lax solution to the standard Riemann Problem

\[
\begin{align*}
\partial_t u_l + \partial_x f(u_l) &= 0 \\
u_l(0, x) &= \begin{cases} 
\bar{u}_l & \text{if } x > 0 \\
u_l(1, 0+) & \text{if } x < 0.
\end{cases}
\end{align*}
\]

(Ψ) The trace \( u(t, 0+) \) of \( u \) at the junction satisfies (1.2) for a.e. \( t > 0 \).

Given an entropy – entropy flux pair \((E, F)\), the \( \Psi \)-solution is entropic at the junction if

(E) At the junction, entropy may not decrease, i.e. for a.e. \( t > 0 \)

\[
\sum_{l=1}^{n} ||\nu_l|| F(u_l(t, 0+)) \leq 0.
\]

For later use, with a slight abuse of notation, we denote by

\[
\begin{align*}
||u|| &= \sum_{l=1}^{n} ||u_l|| & \text{for } u \in \Omega^n, \\
||u||_{L^1} &= \int_{\mathbb{R}^+} ||u(x)|| \, dx & \text{for } u \in L^1(\mathbb{R}^+; \Omega^n), \\
\text{TV} (u) &= \sum_{l=1}^{n} \text{TV} (u_l) & \text{for } u \in BV(\mathbb{R}^+; \Omega^n).
\end{align*}
\]

The following proposition yields the well posedness of the Riemann problem and the continuous dependence of the solution to the Riemann problem from the initial state and from the function \( \Psi \). These results are used in Section 3 to prove well posedness of the Cauchy problem by the wave tracking algorithm. The proofs are deferred to Section 5.

**Proposition 2.2.** Let \( n \in \mathbb{N} \) with \( n \geq 2 \). Fix the pairwise distinct vectors \( \nu_1, \ldots, \nu_n \in \mathbb{R}^3 \setminus \{0\} \) and an \( n \)-tuple of constant states \( \hat{u} \in \Omega^n \) giving a stationary solution to the Riemann Problem (2.1) in the sense of Definition 2.1. Assume that for \( l = 1, \ldots, n \), \( (F) \) holds in \( \hat{u}_l \). If \( \Psi \in C^1(\Omega^n; \mathbb{R}^n) \) satisfies

\[
(2.2) \quad \det \begin{bmatrix} D_1 \Psi(\hat{u}) r_2(\hat{u}_1) & D_2 \Psi(\hat{u}) r_2(\hat{u}_2) & \cdots & D_n \Psi(\hat{u}) r_2(\hat{u}_n) \end{bmatrix} \neq 0
\]

where \( D_l \Psi = D_{u_l} \Psi \), then there exist positive \( \delta, K \) such that

1. For all \( \bar{u} \in \Omega^n \) satisfying \( ||\bar{u} - \hat{u}|| < \delta \), the Riemann Problem (2.1) admits a unique self-similar solution \( (t, x) \mapsto (R^\Psi(\bar{u}))(t, x) \) in the sense of Definition 2.1.
2. If (1.1) admits an entropy – entropy flux pair \((E, F)\), requiring that

\[
(2.3) \quad \sum_{l=1}^{n} ||\nu_l|| F(\hat{u}_l) < 0
\]

ensures that the solution \( (t, x) \mapsto (R^\Psi(\bar{u}))(t, x) \) is also entropic.
3. If \( \bar{u}, \bar{w} \in \Omega^n \) both satisfy \( ||\bar{u} - \hat{u}|| < \delta \) and \( ||\bar{w} - \hat{u}|| < \delta \), then the traces at the junction of the corresponding solutions to (2.1) satisfy

\[
(2.4) \quad \left\| \left( (R^\Psi(\bar{u}))(t, 0+) - (R^\Psi(\bar{w}))(t, 0+) \right) \right\| \leq K \cdot ||\bar{u} - \bar{w}||.
\]
4. For any $\tilde{\Psi} \in C^1(\Omega^n; \mathbb{R}^n)$ with $\|\tilde{\Psi} - \Psi\|_{C^1} < \delta$, $\tilde{\Psi}$ also satisfies (2.2) and for all $\tilde{u} \in \Omega^n$ satisfying $\|\tilde{u} - \hat{u}\| < \delta$,

$$\left\| \left( R_{\tilde{\Psi}}(\tilde{u}) \right)(t) - \left( R_{\Psi}(\tilde{u}) \right)(t) \right\|_{L^1} \leq K \cdot \|\tilde{\Psi} - \Psi\|_{C^1} \cdot t.$$ 

In the previous proposition, $\gamma_2(\hat{u})$ is the right eigenvector of $Df(\hat{u})$ corresponding to the second characteristic field.

We remark that, for subsonic initial data, we obtain here a unique $\Psi$-solution, without any additional condition, such as (E).

3. The Cauchy Problem at an Intersection. Next we consider the Cauchy problem at a junction. First, we give a definition of a solution which naturally extends Definition 2.1 to the Cauchy problem. Then, we prove existence of solutions for initial data with small total variation and the continuous dependence on the coupling condition $\Psi$.

**Definition 3.1.** Fix $\tilde{u} \in \Omega^n$ and $T \in [0, +\infty]$. A weak $\Psi$-solution to

$$\left( \begin{array}{ll}
\partial_t u_l + \partial_x f(u_l) = 0 & t \in \mathbb{R}^+ \\
u(0, x) = u_\nu(x) & l \in \{1, \ldots, n\} \\
x \in \mathbb{R}^+ & u_\alpha \in \tilde{u} + L^1(\mathbb{R}^+; \Omega^n).
\end{array} \right.$$

on $[0, T]$ is a map $u \in C^0([0, T]; \tilde{u} + L^1(\mathbb{R}^+; \Omega^n))$ such that

(W) For all $\varphi \in C_c^\infty([-\infty, T] \times \mathbb{R}^+; \mathbb{R})$ and for $l = 1, \ldots, n$

$$\int_0^T \int_{\mathbb{R}^+} (u_l \partial_t \varphi + f(u_l) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}^+} u_{\alpha,l}(x) \varphi(0, x) \, dx = 0.$$

(Ψ) The condition at the junction is met: for a.e. $t \in \mathbb{R}^+$, $\Psi(u(t, 0+)) = 0$.

If (1.1) admits an entropy – entropy flux pair $(E, F)$, then the weak $\Psi$-solution $u$ is entropic if for all $\varphi \in C_c^\infty([-\infty, T] \times \mathbb{R}^+; \mathbb{R})$

$$\sum_{l=1}^n \left( \int_0^T \int_{\mathbb{R}^+} (E(u_l) \partial_t \varphi + F(u_l) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}^+} E(u_\alpha) \varphi(0, x) \, dx \right) \|\nu_l\| \geq 0.$$

We are now ready to state the main result of this paper, namely the well posedness of the Cauchy Problem (3.1) at the junction.

**Theorem 3.2.** Let $n \in \mathbb{N}$, $n \geq 2$. Fix the pairwise distinct vectors $\nu_1, \ldots, \nu_n$ in $\mathbb{R}^3 \setminus \{0\}$. Fix an n-tuple of states $\hat{u} \in \Omega^n$ such that $f$ satisfies (F) at $\hat{u}$ and the Riemann Problem (2.1) with initial datum $\hat{u}$ admits the stationary solution in the sense of Definition 2.1. Let $\Psi \in C^1(\Omega^n; \mathbb{R}^n)$ satisfy (2.2). Then, there exist positive $\delta, L$ and a map $\mathcal{S} : [0, +\infty] \times \mathcal{D} \to \mathcal{D}$ such that:

1. $\mathcal{D} \supset \{ u \in \hat{u} + L^1(\mathbb{R}^+; \Omega^n); TV(u) \leq \delta \}$;
2. for $u \in \mathcal{D}$, $S_0u = u$ and for $s, t \geq 0$, $S_su = S_{s+t}u$;
3. for $u, w \in \mathcal{D}$ and $s, t \geq 0$, $\|S_su - S_sw\|_{L^1} \leq L \cdot (\|u - w\|_{L^1} + |t - s|)$.
4. If $u \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small, $S_tu$ coincides with the juxtaposition of the solutions to Riemann Problems centered at the points of jumps or at the junction.

Moreover, for every $u \in \mathcal{D}$, the map $t \mapsto S_tu$ is a $\Psi$-solution to the Cauchy Problem (3.1) according to Definition 3.1.
For any \( \tilde{\Psi} \in C^1(\Omega^n; \mathbb{R}^n) \) with \( \| \tilde{\Psi} - \Psi \|_{C^1} < \delta \), \( \tilde{\Psi} \) generates a semigroup of solutions on \( D \) and for \( u \in D \)
\[
\left\| S_t^\Psi \tilde{u} - S_t^\Psi \tilde{u} \right\|_{L^1} \leq L \cdot \left\| \tilde{\Psi} - \Psi \right\|_{C^1} \cdot t.
\]

If (1.1) admits an entropy – entropy flux pair \( (E,F) \) and \( \tilde{u} \) is strictly entropic in the sense of (2.3), then the \( \Psi \)-solution \( t \mapsto S_t u \) is entropic at the junction.

The proof is deferred to Section 5.

4. Gas Networks and Open Channels. A widely used model for gas flow in pipe networks is the system of isothermal Euler equations, see [23] and the references therein. In this section, we discuss the coupling conditions [2, 10] in the context of the presented theory in the case of a general \( p \)-system. By \( p \)-system we mean
\[
\begin{aligned}
&\partial_t \rho_l + \partial_x q_l = 0, & t \in \mathbb{R}^+, \\
&\partial_t q_l + \partial_x \left( \frac{q_l^2}{\rho_l} + p(\rho_l) \right) = 0, & l \in \{1, \ldots, n\}
\end{aligned}
\]

where \( \rho > 0 \) is the mass density of a given fluid, \( q \) its linear momentum density and \( p = p(\rho) \) the pressure law, which we assume to satisfy
\[
(\text{P}) \quad p \in C^2(\mathbb{R}^+; \mathbb{R}^+), \quad p(0) = 0 \quad \text{and for all} \quad \rho \in \mathbb{R}^+, \quad p'(\rho) > 0, \quad p''(\rho) \geq 0.
\]

In the context of gas pipelines, the pressure law typically chosen is \( p(\rho) = a^2 \rho \), where the sound speed \( a \) depends on the gas type and temperature [23]. As is well known, (4.1) is equipped with the (mathematical) entropy – entropy flux pair
\[
\begin{aligned}
E(\rho, q) &= \frac{q^2}{2\rho} + \rho \int_{\rho_*}^{\rho} \frac{p(r)}{r^2} \, dr \quad \text{(total energy)} \\
F(\rho, q) &= \frac{q}{\rho} \cdot (E(\rho, q) + p(\rho)) \quad \text{(flow of the total energy)}
\end{aligned}
\]
for a \( \rho_* > 0 \). Choosing an initial datum \( \tilde{u} = (\tilde{\rho}, \tilde{q}) \) in the \textit{subsonic} region
\[
\Omega = \left\{ (\rho, q) \in \mathbb{R}^+ \times \mathbb{R} : \lambda_1(\rho, q) < 0 < \lambda_2(\rho, q) \right\},
\]
ensures that (F) holds at \( \tilde{u} \). Recall the standard relations
\[
\begin{aligned}
\lambda_1(\rho, q) &= (q/\rho) - \sqrt{p'(\rho)}, & \lambda_2(\rho, q) &= (q/\rho) + \sqrt{p'(\rho)}, \\
r_1(\rho, q) &= \begin{bmatrix}
-1 \\
-\lambda_1(\rho, q)
\end{bmatrix}, & r_2(\rho, q) &= \begin{bmatrix}
1 \\
\lambda_2(\rho, q)
\end{bmatrix}
\end{aligned}
\]

Below we consider different coupling conditions that appeared in the literature. Remark that the geometry of the junction implicitly enters in all the relations below through the choice of the \( x_l \) coordinates. Explicitly, only the area \( \| \nu_l \| \) of the cross section appears.

We consider below the conditions presented in [3, 4, 10, 14, 16, 19, 23]. Any of them prescribes the conservation of mass, so that the first component in (1.2) reads
\[
\sum_{l=1}^{n} \| \nu_l \| \cdot q_l = 0.
\]
With a slight abuse of notation, we use \( \Psi \) to denote the remaining \( n - 1 \) conditions.
We prescribe the equal momentum flow for all connected pipes, see [9] and, in the case open channels [16]:

$$P(\rho_i(t,0+), q_i(t,0+)) = P(\rho_j(t,0+), q_j(t,0+)) \quad \forall i \neq j.$$  

We prescribe a single pressure at the pipe–to–pipe intersection, see [2, 3] and [14] in the engineering literature. For open canals, see [15]:

$$p(\rho_i(t,0+)) = p(\rho_j(t,0+)) \quad \forall i \neq j$$

In case of only two connected pipes, a further coupling condition is proposed in [19] (see (4.7) for the definition of k(θ)):

$$P(\rho_1(t,0+), q_1(t,0+)) + k(\theta) = P(\rho_2(t,0+), q_2(t,0+))$$

Some remarks are in order. Most of the engineering literature uses (b) modified by so–called minor loss factors. These factors are listed in tables and depend on additional information, see [12, 21]. On the other hand, (a) is not so commonly used in the engineering literature but yields the $L^1$ continuity across stationary transonic shocks, see [10, Example 2.3], which does not hold when (b) or (c) are adopted. A numerical study of a two–dimensional situation for the $p$–system can be found in [18] and for the Euler system in [17].

4.1. Equal Linear Momentum Flow for the $p$-System. We consider the setting in [10], i.e. a junction among $n$ pipes, each modeled through the $p$–system (4.1) with a general pressure law and with the corresponding function $\Psi$ in (1.2) given by (4.3) together with

$$\Psi(\rho, q) = \begin{bmatrix} P(\rho_1, q_1) & - P(\rho_2, q_2) \\ \vdots & \vdots & \vdots \\ P(\rho_{n-1}, q_{n-1}) & - P(\rho_n, q_n) \end{bmatrix}$$

where $P(\rho, q) = q^2/\rho + p(\rho)$ is the flow of the linear momentum. The well posedness of the Cauchy Problem for (4.1) at a junction was proved in [9, Theorem 3.3].

Remark that this choice of $\Psi$ allows to consider the problem at the junction as an extension of the standard Cauchy problem. Indeed, setting $n = 2$ and $\nu_1 + \nu_2 = 0$, then (4.1)–(4.3)–(4.4) reduces to the usual situation.

In the general case, the total linear momentum $Q$ varies by

$$Q(t_2) - Q(t_1) = \int_{t_1}^{t_2} \sum_{i=1}^{n} P(\rho_i(t,0+), q_i(t,0+)) \nu_i dt = \left( \int_{t_1}^{t_2} P_*(t) dt \right) \sum_{i=1}^{n} \nu_i$$

see [9, Section 1], where $P_*(t)$ is the trace of $P(\rho_i(t,0+), q_i(t,0+))$, which is independent from $l$ by (1.2)–(4.4). Note that the right hand side above explicitly depends on the geometry of the junction.

4.2. Equal Pressure for the $p$-System. In [3], the condition at the junction amounts to mass conservation and to pressure equality. Here, we extend that approach to the case of $n$ ducts with a general pressure law, so that (1.2) consists of (4.3) with

$$\Psi(\rho, q) = \begin{bmatrix} p(\rho_1) & - p(\rho_2) \\ \vdots & \vdots & \vdots \\ p(\rho_{n-1}) & - p(\rho_n) \end{bmatrix}$$
Using [9, Lemma 4.4], the determinant in condition (2.2) evaluates to

\[
(4.6) \quad (-1)^{n+1} \prod_{i=1}^{n} \lambda_2(\hat{\rho}_i, \hat{q}_i) \cdot \prod_{i=1}^{n} \frac{p'(\hat{\rho}_j)}{\lambda_2(\hat{\rho}_j, \hat{q}_j)}.
\]

Since \((\hat{\rho}, \hat{q}) \in \Omega^n\), we have \(\lambda_2(\hat{\rho}_i, \hat{q}_i) > 0\), while the assumption \((P)\) on the pressure law implies \(p'(\hat{\rho}_i) > 0\). Thus, Theorem 3.2 applies, yielding well posedness in the case of \(n\) pipes with a general pressure law.

### 4.3. Two Pipes with Friction for the \(p\)-System

The case studied in [19] corresponds to (4.1) with \(p(\rho) = \rho, n = 2, \nu_1 = [-1, 0] \) and \(\nu_2 = [\cos \theta, \sin \theta]\). Here, the angular dependence is explicitly modeled and taken into account in the coupling conditions. This situation mimics a kink forming an angle \(\theta\) in a pipe. Due to the kink, the linear momentum is assumed to vary by a factor \(k\) such that

\[
(4.7) \quad k = \sqrt{2(1 - \cos \theta)}
\]

for \(\theta \in [0, \pi/2]\). In the case of pipes with possibly different cross-sections \(\|\nu_i\|\) and a general pressure law satisfying \((P)\), at the junction we obtain condition (4.3) with

\[
(4.8) \quad \Psi(\rho, q) = \left(\frac{\hat{q}_1^2}{\rho_1} + p(\rho_1) + fkq_1\right) - \left(\frac{\hat{q}_2^2}{\rho_2} - p(\rho_2)\right),
\]

which reduces to the case considered in [19] when \(p(\rho) = \rho\) and equal cross sections. The parameter \(f\) denotes a non-negative empirical friction coefficient. The condition (2.2) is

\[
\lambda_2(\hat{\rho}_2, \hat{q}_2) \lambda_2(\hat{\rho}_1, \hat{q}_1) \left(\lambda_2(\hat{\rho}_2, \hat{q}_2) + \lambda_2(\hat{\rho}_1, \hat{q}_1) + fk\right) \neq 0.
\]

Hence, Theorem 3.2 applies yielding well posedness for general pressure laws.

### 4.4. Equal Momentum Flow for Open Canals

The model presented in [20, formulæ (2.3)–(2.7)] for a node among \(n\) open canals reads

\[
(4.9) \quad \begin{cases}
\partial_t A_l + \partial_x (A_l V_l) = 0 & t \in \mathbb{R}^+ \\
\partial_t V_l + \partial_x \left(\frac{1}{2} V_l^2 + g h(A_l)\right) = 0, & l \in \{1, \ldots, n\} \\
(A_l, V_l) & (A_l, V_l) \in \mathbb{R}^+ \times \mathbb{R}.
\end{cases}
\]

where \(V_l\) is the water speed in the \(l\)-th canal, \(A_l\) is the vertical cross section occupied by the water, \(g\) is gravity, \(h\) is the water level. Here, differently from [20], we assume that the canals’ beds are all at the same height above sea level, which is acceptable in a neighborhood of the node among the pipes.

Other descriptions for the dynamics of open canals are found in the literature. In particular, [15, Section 6.1] presents a different model, based on [13], that fits in our framework of Paragraph 4.2.

The coupling condition at the node is given by [20, formulæ (2.9)–(2.16)]:

\[
\Psi(A, V) = \begin{bmatrix}
\sum_{i=1}^{n} A_l V_l \\
\frac{1}{2} V_1^2 + g h(A_1) - \frac{1}{2} V_2^2 - g h(A_2) \\
\vdots \\
\frac{1}{2} V_{n-1}^2 + g h(A_{n-1}) - \frac{1}{2} V_n^2 - g h(A_n)
\end{bmatrix}.
\]
Consider a junction between maximal moduli of the propagation speeds. 

This condition is analogous to the equal linear momentum flow condition (4.4) discussed in Paragraph 4.1.

Choosing an initial datum \( \bar{u} = (\bar{A}, \bar{V}) \) such that \( \bar{V} < \sqrt{Agh'(A)} \), ensures that (F) is fulfilled at \( \bar{u} \). In the present case we have

\[
\lambda_1(A, V) = V - \sqrt{Agh'(A)}, \quad \lambda_2(\rho, q) = V + \sqrt{Agh'(A)},
\]

\[
r_1(A, V) = \left[ -\frac{\sqrt{A}}{\sqrt{gh'(A)}} \right], \quad r_2(A, V) = \left[ \frac{\sqrt{A}}{\sqrt{gh'(A)}} \right].
\]

The determinant in condition (2.2) evaluates therefore to

\[
(-1)^{n+1} \left( \prod_{i=1}^{n} \lambda_2(A_i, V_i) \right) \cdot \sum_{j=1}^{n} \sqrt{A_i} \prod_{j \neq i} \sqrt{gh'(A_j)}.
\]

Here, \( A > 0 \) by assumption, as well as \( \sqrt{gh'(A)} > 0 \) by the monotonicity of \( h(A) \), which ensures \( \lambda_2(A, V) > 0 \). Thus, Theorem 3.2 can be applied, yielding the well posedness for a junction of \( n \) open canals.

**4.5. Numerical Examples for the \( p \)-System.** This paragraph is devoted to comparisons among the different coupling conditions at the junction in the case of the \( p \)-system (4.1). Throughout, we use the \( \gamma \)-law: \( p(\rho) = p_* (\rho / \rho_*)^\gamma \) with \( \gamma = 1.4, p_* = 1 \) and \( p_* = 1 \), which clearly satisfies (P). In the case of the coupling conditions (4.8), we set \( f = 1 \).

In general, stationary solutions for (4.4) fail to be stationary solutions for (4.5) or (4.8). Therefore, we perturb below static solutions, which are stationary for all coupling conditions and allow the comparisons.

Below, the initial data \( u_l(x) = (\rho_l(x), q_l(x)) \) along the \( l \)-th duct attains at most two values, say \( u_l^\infty = \lim_{x \to +\infty} u_l(x) \) and \( u_l^0 = \lim_{x \to -0} u_l(x) \). When waves hit the junction, we solve numerically condition (1.2) using Newton’s method and obtain the traces \( u_l^+ \) of the solution at the junction. The solution to (2.1) is then computed solving a classical Riemann problem between the states \( u_l^+ \) (on the left) and \( u_l^\infty \) (on the right). Due to the chosen directions of the space variables, waves approaching the junction belong to the first family, those exiting it to the second.

We selected three different examples. In the case of the coupling conditions (4.4) and (4.5), only the cross section of the connected pipes appears explicitly. In the case of (4.8), the angular dependence is taken into account explicitly.

Remark that the numerical values provided below are expressed in the coordinates \( x_l \) adapted to the junction. In particular, the column Wave speed is the modulus of the propagation speed of the wave, its \( x \) and \( y \) component depend on the direction of the pipe. In case of rarefactions, the column Wave speed displays the minimal and maximal moduli of the propagation speeds.

**4.5.1. Two Pipes, Different Cross Sections, Possibly Different Angles.** Consider a junction between \( n = 2 \) horizontal pipes having cross sections \( \| \nu_1 \| = 1 \) and \( \| \nu_2 \| = 2 \). We choose the cases \( \theta = 0, \theta = \pi / 4, \theta = \pi / 16 \) and \( \theta = \pi / 32 \). In each of these cases, a shock with right state \( u_1^0 = [1.1000, -0.1253] \) propagating along pipe 1 hits the junction. At first, we compare conditions (4.4) and (4.5). In Table 4.1, the first column refers to the coupling condition, in the case of (4.8), Table 4.2 displays
Table 4.1

Comparison of results obtained by condition (4.4), (4.8), \( \theta = 0 \) and (4.5) for two connected pipes with different cross sections.

<table>
<thead>
<tr>
<th>( \Psi )</th>
<th>( l )</th>
<th>( u_l^\infty )</th>
<th>( u_l^r )</th>
<th>Wave</th>
<th>Wave speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4.4); (4.8) ( \theta \equiv 0 )</td>
<td>1</td>
<td>+1.1000</td>
<td>+1.0553</td>
<td>R</td>
<td>+1.0322</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.1253</td>
<td>-0.1728</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+1.0000</td>
<td>+1.0701</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+0.0000</td>
<td>+0.0864</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4.5)</td>
<td>1</td>
<td>+1.1000</td>
<td>+1.0658</td>
<td>R</td>
<td>+1.0466</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.1253</td>
<td>-0.1618</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+1.0000</td>
<td>+1.0658</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+0.0000</td>
<td>+0.0809</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>+1.1000</td>
<td>+1.0921</td>
<td>S</td>
<td>+1.2324</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.1253</td>
<td>-0.1728</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+1.0000</td>
<td>+1.0701</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+0.0000</td>
<td>+0.0864</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The angle \( \theta \). The type of the resulting wave is in the fifth column, where \( S \) and \( R \) refer to 2-(Lax-)shocks and 2-rarefaction waves, respectively.

These numerical integrations suggest that, according to (4.8), there exists an angle \( \theta^* \) at which no wave is reflected. Clearly, \( \theta^* \) depends on the initial states, on the ducts’ sections and on the empirical factor \( f \).

4.5.2. Three Pipes, Different Cross Sections. Consider \( n = 3 \) pipes identified by \( \nu_1 = [-1 - \sqrt{2} 0]^T \), \( \nu_2 = [1 1]^T \) and \( \nu_3 = [1 0]^T \) and having cross sections \( ||\nu_l|| \). The situation is depicted in Figure 4.3. Again, we assume that the flow is initially at rest, i.e. \( \tilde{q}_l = 0 \) for \( l = 1, 2, 3 \). We consider the case of a shock approaching the junction along pipe 1. Due to the choice of the initial data and to the geometry of the junction, the numerical solutions to both coupling conditions (4.4) and (4.5) yield in fact the same results. The final states and the corresponding waves are in Table 4.3.

4.5.3. Four Pipes, Different Cross Sections. Finally, we consider a junction with \( n = 4 \) pipes defined by \( \nu_1 = [-1 0] \), \( \nu_2 = [0 1] \), \( \nu_3 = [0 -1] \) and \( \nu_4 = [1 0] \). Initially the gas flow is at rest, i.e. \( \tilde{q}_l = 0 \), and \( \tilde{\mu}_l = 1 \) for \( l = 1, \ldots, 4 \). We let three 1-Lax-
Table 4.2
Comparison for the situation of a kink of angle $\theta$. The case $\theta = 0$ can be found in Table 1.

<table>
<thead>
<tr>
<th>$\Psi$</th>
<th>$l$</th>
<th>$u_i^\infty$</th>
<th>$u_i^+$</th>
<th>Wave</th>
<th>Wave speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4.8)$</td>
<td>1</td>
<td>+1.0000</td>
<td>+1.0645</td>
<td>R</td>
<td>+1.0447</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.1253</td>
<td>-0.1633</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta = \frac{\pi}{32}$</td>
<td>2</td>
<td>+1.0000</td>
<td>+0.0816</td>
<td>S</td>
<td>+1.1298</td>
</tr>
<tr>
<td>$(4.8)$</td>
<td>1</td>
<td>+1.1000</td>
<td>+1.0725</td>
<td>R</td>
<td>+1.0556</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.1253</td>
<td>-0.1548</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta = \frac{\pi}{16}$</td>
<td>2</td>
<td>+1.0000</td>
<td>+0.0774</td>
<td>S</td>
<td>+1.2275</td>
</tr>
<tr>
<td>$(4.8)$</td>
<td>1</td>
<td>+1.1000</td>
<td>+1.1056</td>
<td>S</td>
<td>+1.0958</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.1253</td>
<td>-0.1192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta = \frac{\pi}{8}$</td>
<td>2</td>
<td>+1.0000</td>
<td>+0.0596</td>
<td>S</td>
<td>+1.2177</td>
</tr>
</tbody>
</table>

Fig. 4.3. A shock hitting a T-junction.

Fig. 4.4. Three Lax shocks hit simultaneously the junction.

shocks of different strengths collide simultaneously at the junction along the pipes 1, 3 and 4.

It is remarkable that the two coupling conditions yield qualitatively different results. The wave reflected in tube 1 is a rarefaction according to (4.4) and a shock according to (4.5). However, the propagation speeds in the two cases are close to each other, coherently with the $L^1$ continuous dependence.

Table 4.4 displays, for each duct, the type of wave arising after the interaction.
Table 4.3  
Numerical results for Paragraph 4.5.2

<table>
<thead>
<tr>
<th>(4.4) and (4.5)</th>
<th>$l = 1$</th>
<th>$l = 2$</th>
<th>$l = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_l^\infty$</td>
<td>+1.0000</td>
<td>+1.0000</td>
<td>+1.0000</td>
</tr>
<tr>
<td></td>
<td>+0.0000</td>
<td>+0.0000</td>
<td>+0.0000</td>
</tr>
<tr>
<td>$u_l^0$</td>
<td>+1.2000</td>
<td>+1.0000</td>
<td>+1.0000</td>
</tr>
<tr>
<td></td>
<td>-0.2642</td>
<td>+0.0000</td>
<td>+0.0000</td>
</tr>
<tr>
<td></td>
<td>-0.2640</td>
<td>+0.2640</td>
<td>+0.2640</td>
</tr>
</tbody>
</table>

Coupling condition (4.4)

| $u_l^+$         | +1.1396 | +1.1441 | +1.1625 | +1.1611 |
|                 | -0.2039 | +0.1848 | -0.0545 | +0.0736 |
| Wave type       | R      | S      | S      | S      |
| Wave speed      | +1.0357 | +1.2831 | +1.1328 | +1.2113 |

Table 4.4  
Numerical results for Paragraph 4.5.3.

| $u_l^\infty$    | +1.0000 | +1.0000 | +1.0000 | +1.0000 |
|                 | +0.0000 | +0.0000 | +0.0000 | +0.0000 |
| $u_l^0$         | +1.1500 | +1.0000 | +1.1000 | +1.0500 |
|                 | -0.1931 | +0.1253 | -0.0609 |          |
| $u_l^+$         | +1.1520 | +1.1520 | +1.1520 | +1.1520 |
|                 | -0.1909 | +0.1957 | -0.0667 | +0.0620 |
| Wave type       | S      | S      | S      | S      |
| Wave speed      | +1.0501 | +1.2884 | +1.1260 | +1.2053 |

at the junction and its speed is reported below for the two different coupling conditions (4.4) and (4.5).

5. Technical Details. As a general reference on the theory of hyperbolic systems of conservation laws, we refer to [6]. Denote by $\sigma \mapsto \mathcal{L}_i(u_o, \sigma)$ the $i$-th Lax curve through $u_o$, for $i = 1, 2$. As usual, $O(1)$ denotes a sufficiently large constant dependent only on $f$ restricted to a neighborhood of the initial states.

Proof of Proposition 2.2. It is sufficient to show that for all $\bar{u}$ sufficiently near to $\hat{u}$ the nonlinear system $\Psi(L_2(\bar{u}_1, \sigma_1), \ldots, L_2(\bar{u}_n, \sigma_n)) = 0$ admits a unique $n$-tuple of solution $\sigma_1, \ldots, \sigma_n$. Indeed, condition (2.2) allows to use the Implicit Function Theorem.

The Lipschitz estimate (2.4) immediately follows from the regularity of the implicit function.

To prove the latter statement, it is sufficient to consider a single pipe, say the first one. Let $u$, respectively $\tilde{u}$, be the solutions to (2.1) with condition $\Psi$, respectively $\tilde{\Psi}$,
at the junction. If \( u \) and \( \tilde{u} \) contain a single shock, then
\[
\| u - \tilde{u} \| = \int_{0}^{\min\{\Lambda^1, \Lambda^2\} t} \| u(x) - \tilde{u}(x) \| \, dx \\
+ \int_{\min\{\Lambda^1, \Lambda^2\} t}^{\max\{\Lambda^1, \Lambda^2\} t} \| u(x) - \tilde{u}(x) \| \, dx
\]
\[
= \mathcal{O}(1) \cdot \| u(t, 0+) - \tilde{u}(t, 0+) \| \cdot t + \mathcal{O}(1) \cdot \left| \Lambda^2 - \Lambda^1 \right| \cdot t
\]
\[
= \mathcal{O}(1) \cdot \left\| \tilde{\Psi} - \Psi \right\|_{C^1} \cdot t + \mathcal{O}(1) \cdot \| u(t, 0+) - \tilde{u}(t, 0+) \| \cdot t
\]
\[
= \mathcal{O}(1) \cdot \left\| \tilde{\Psi} - \Psi \right\|_{C^1} \cdot t.
\]

The case of one or both the solutions containing a rarefaction is similar, see also the proof of \([5, \text{Corollary 2.5}]\). □

We now pass to the proof of Theorem 3.2. To do this, we first use wave front tracking to construct approximate solutions to the Cauchy problem (3.1) adapting the wave front tracking technique, see \([6, \text{Chapter 7}]\).

Let \( \delta > 0 \) be such that the closed sphere \( B(\hat{u}_l, \delta) \subset \Omega \) for \( l = 1, \ldots, n \) and introduce the compact set \( B = \prod_{l=1}^{n} B(\hat{u}_l, \delta) \). We omit the proof of the following simple estimate.

**Lemma 5.1.** For all \( u \in B(\hat{u}, \delta) \) there exists \( C > 0 \) such that if \( \| u - \hat{u} \| < \delta \) and \( \| L_i(u; \sigma) - u \| \leq \hat{\delta} \), then
\[
\frac{1}{C} \cdot |\sigma| \leq \| L_i(u; \sigma) - u \| \leq C \cdot |\sigma|.
\]

Fix \( \varepsilon > 0 \). Approximate the initial datum \( u_o \) with a sequence \( u_{o,\varepsilon} \) of piecewise constant initial data each having a finite number of discontinuities so that \( \lim_{\varepsilon \to 0} \| u_{o,\varepsilon} - u_o \|_{L^1} = 0 \). Then, at the junction and at each point of jump in the approximate initial datum along the pipes, we solve the corresponding Riemann Problem according to Definition 2.1. If the total variation of the initial datum is sufficiently small, then Proposition 2.2 ensures the existence and uniqueness of solutions to the Riemann Problem. We approximate each rarefaction wave with a rarefaction fan, i.e. by means of (non entropic) shock waves traveling at the characteristic speed of the state to the right of the shock and with size at most \( \varepsilon \).

This construction can be extended up to the first time \( \bar{t}_1 \) at which two waves interact in a pipe or a wave hits the junction. Clearly, at time \( \bar{t}_1 \) the functions so constructed are piecewise constant with a finite number of discontinuities. Hence, at any subsequent interaction or collision with the junction, we repeat the previous construction with the following provisions:

1. no more than 2 waves interact at the same point or at the junction;
2. a rarefaction fan of the \( i \)-th family produced by the interaction between an \( i \)-th rarefaction and any other wave, is not split any further;
3. when the product of the strengths of two interacting waves falls below a threshold \( \varepsilon \), then we let the waves cross each other, their size being unaltered, and introduce a non physical wave with speed \( \hat{\lambda} \), with \( \hat{\lambda} > \sup_u \lambda_2(u) \); see \([6, \text{Chapter 7}]\) and the refinement \([1]\).
In the present case, we have to complete the above algorithm stating how the Riemann Problem at the junction is to be solved. At time $t = 0$ and whenever a physical wave with size greater than $\varepsilon$ hits the junction, the accurate solver is used, i.e. the exact solution as in Definition 2.1 is approximated replacing rarefaction waves with rarefaction fans. When a wave with strength smaller than $\varepsilon$ hits the junction, then we let it be reflected into a non physical wave with speed $\hat{\lambda}$ and no wave in any other pipe is produced.

Repeating recursively this procedure, we construct a wave front tracking sequence of approximate solutions $u_\varepsilon$ in the sense of [6, Definition 7.1].

At interactions of waves in a pipe, we have the following classical result.

**Fig. 5.1.** Notation for the standard interaction estimates in Lemma 5.2.

**Lemma 5.2.** There exists a constant $K$ with the following property.

1. If there is an interaction in a pipe between two waves $\sigma_1^-$ and $\sigma_2^-$, respectively of the first and second family, producing the waves $\sigma_1^+$ and $\sigma_2^+$ (see Figure 5.1, left), then

   \[
   |\sigma_1^+ - \sigma_1^-| + |\sigma_2^+ - \sigma_2^-| \leq K \cdot |\sigma_1^- \sigma_2^-| .
   \]

2. If there is an interaction in a pipe between two waves $\sigma_1'$ and $\sigma_2''$ of the same $i$-th family producing waves of total size $\sigma_1^+$ and $\sigma_2^+$ (see Figure 5.1, right, for the case $i = 2$), then

   \[
   |\sigma_1^+ - (\sigma_2'' + \sigma_1')| + |\sigma_2^+| \leq K \cdot |\sigma_1' \sigma_2''| \quad \text{if } i = 1,
   \]

   \[
   |\sigma_1^+| + |\sigma_2^+ - (\sigma_2'' + \sigma_2')| \leq K \cdot |\sigma_2' \sigma_2''| \quad \text{if } i = 2.
   \]

3. If there is an interaction in a pipe between two physical waves $\sigma_1^-$ and $\sigma_2^-$ producing a non physical wave $\sigma_3^+$ (see Figure 5.2, left), then

   \[
   |\sigma_3^+| \leq K \cdot |\sigma_1^- \sigma_2^-| .
   \]

4. If there is an interaction in a pipe between a physical wave $\sigma$ and a non physical wave $\sigma_3^-$ producing a physical wave $\sigma_3^-$ and a non physical wave $\sigma_3^+$ (see Figure 5.2, right), then

   \[
   |\sigma_3^+| - |\sigma_3^-| \leq K \cdot |\sigma \sigma_3^-| .
   \]

For a proof of this result see [6, Chapter 7]. By construction, non physical waves cannot interact with the junction or with other non physical waves. In the case of the
junction, we have the following result.

**Proposition 5.3.** There exist $\delta_J > 0$ and $K_J \geq 1$ with the following property. For any $\bar{u} \in \mathcal{B}$ that yields a stationary solution to the Riemann Problem (2.1), for any $1$-waves $\sigma_{\bar{l}}^- \in [-\delta_J, \delta_J]$ hitting the junction and producing the $2$-waves $\sigma_i^+$,\n
\[ \sum_{l=1}^n |\sigma_i^+| \leq K_J \cdot |\sigma_{\bar{l}}^-|. \] \hspace{1cm} (5.2)\n
The proof immediately follows from (2.4) and Lemma 5.1.

Define $\hat{K} = 2K_J + 1$. Fix a wave front tracking approximate solution $u_\varepsilon$. For $t > 0$ and $l \in \{1, \ldots, n\}$, we denote with $\{x_{l,\alpha} : \alpha \in \mathcal{J}_l(u)\}$ the set of the positions of the discontinuities of the approximate solution $u$ in the $l$-th pipe and with $\sigma_{l,1,\alpha}$, $\sigma_{l,2,\alpha}$, $\sigma_{l,3,\alpha}$ the strengths of the waves respectively of the first family, of the second family and of the non physical waves at $x_{l,\alpha}$. Introduce the Glimm-type functionals

\[ V(t) = \sum_{l=1}^n \sum_{\alpha \in \mathcal{J}_l} \left( 2K_J \cdot |\sigma_{l,1,\alpha}| + |\sigma_{l,2,\alpha}| + |\sigma_{l,3,\alpha}| \right) \]
\[ Q(t) = \sum_{l=1}^n \sum \left\{ |\sigma_{l,1,\alpha} \sigma_{l,2,\beta}| : (\sigma_{l,1,\alpha}, \sigma_{l,2,\beta}) \in \mathcal{A}_l \right\} \]
\[ \Upsilon(t) = V(t) + \hat{K} \cdot Q(t), \] \hspace{1cm} (5.3)

where $\mathcal{A}_l$ denotes the set of approaching waves in the $l$-th pipe; see [6, Paragraph 7.3].

The functionals above are well defined for every $t > 0$ at which no interaction takes place. Suppose now that at a time $\tau > 0$ there is an interaction between the wave $\sigma_{l,1}$ of the first family and the junction. In general, this interaction produces $n$
Proof of Theorem 3.2.

Let \( \hat{\sigma} \) (see [6, Chapter 8]) and the weights \( x \) solution to (1.1) attaining values in \( \tilde{C} \)

We have thus proved the following basic result.

Suppose now an interaction between two waves \( \sigma_{l,i,o}, \sigma_{l,j,\beta} \) happens in a pipe at time \( \tau \). By Lemma 5.2 we deduce that

\[
\begin{align*}
\Delta V(\tau) & \leq 2K J \left| \sigma_{l,1} \right| \leq -K J \cdot \left| \sigma_{l,1} \right| \\
\Delta Q(\tau) & \leq \sum K J \left( \left( \sigma_{l,1} \right) \sum \left( \sigma_{l,i,o} \right) \right) \leq K J \cdot V(\tau-) \cdot \left| \sigma_{l,1} \right| \\
\Delta \Upsilon(\tau) & \leq K J \cdot V(\tau-) \cdot \left( K J \cdot V(\tau-) - 1 \right) \cdot \left| \sigma_{l,1} \right|.
\end{align*}
\]

We pass now to the Proposition 5.4. Suppose now an interaction between two waves \( \sigma_{l,i,o}, \sigma_{l,j,\beta} \) happens in a pipe at time \( \tau \). By Lemma 5.2 we deduce that

\[
\begin{align*}
\Delta V(\tau) & \leq 2K J \cdot \left| \sigma_{l,i,o} \sigma_{l,j,\beta} \right| \\
\Delta Q(\tau) & \leq K J \cdot \left| \sigma_{l,i,o} \sigma_{l,j,\beta} \right| \cdot V(\tau-) \cdot \left| \sigma_{l,i,o} \sigma_{l,j,\beta} \right| \\
\Delta \Upsilon(\tau) & \leq \left| \sigma_{l,i,o} \sigma_{l,j,\beta} \right| \left( K J \cdot (2K J + K J \cdot V(\tau-)) - K J \right)
\end{align*}
\]

We have thus proved the following basic result.

Proposition 5.4. Let \( \delta = \min \{1/(2K+1), 1/(2K \hat{K} + 1), \hat{\delta}, \delta_J \} \). At any interaction time \( \tau > 0 \), if \( V(\tau-) < \delta \) then \( \Delta \Upsilon(\tau) < 0 \) with \( \Upsilon \) defined in (5.3).

Proof of Theorem 3.2. Let \( \delta \) be as in Proposition 5.4 and define

\[
\tilde{D} = \left\{ u \in \hat{u} + L^1 \left( \mathbb{R}^+; (\mathbb{R}^+ \times \mathbb{R})^n \right); u \in PC \text{ and } \Upsilon(u) \leq \delta \right\},
\]

here, \( PC \) denotes the set of piecewise constant functions with finitely many jumps. It is immediate to prove that there exists a suitable \( C_1 > 0 \) such that \( \frac{1}{C_1} TV(u)(t, \cdot) \leq V(t) \leq C_1 TV(u)(t, \cdot) \) for all \( u \in \tilde{D} \). Any initial data in \( \tilde{D} \) yields an approximate solution to (1.1) attaining values in \( \tilde{D} \) by Proposition 5.4.

We pass now to the \( L^1 \)-Lipschitz continuous dependence of the approximate solutions from the initial datum. Consider two wave front tracking approximate solutions \( u_1 \) and \( u_2 \). Define the functional

\[
(5.4) \quad \Phi(u_1, u_2) = \sum_{l=1}^{n} \sum_{i=1}^{2} \int_{0}^{+\infty} \left| s_{l,i}(x) \right| W_{l,i}(x) \, dx,
\]

where \( s_{l,i}(x) \) measures the strengths of the \( i \)-th shock wave in the \( l \)-th pipe at point \( x \) (see [6, Chapter 8]) and the weights \( W_{l,i} \) are defined by

\[
\begin{align*}
W_{l,1}(x) &= \hat{K} \cdot \left( 1 + \kappa_1 A_{l,i}(x) + \kappa_1 \kappa_2 \left( \Upsilon(u_1) + \Upsilon(u_2) \right) \right) \\
W_{l,2}(x) &= 1 + \kappa_1 A_{l,i}(x) + \kappa_1 \kappa_2 \left( \Upsilon(u_1) + \Upsilon(u_2) \right)
\end{align*}
\]

for suitable positive constants \( \kappa_1, \kappa_2 \) chosen as in [6, formula (8.7)] and \( \hat{K} = 1 + (\max W_{l,2}) \frac{\lambda}{\inf \lambda_1} \). Here \( \Upsilon \) is the functional defined in (5.3), while the \( A_{l,i} \) are defined by

\[
A_{l,i}(x) = \sum \left\{ \left| \sigma_{l,i,o} \sigma_{l,j,\beta} \right| : \begin{array}{l} x_{\alpha} < x, \ 1 \leq k_{\alpha} < i \\ x_{\alpha} > x, \ 1 \leq k_{\alpha} < i \end{array} \right\}
\]
for a positive constant $C_2$. The same calculations as in [6, Chapter 8] show that, at any time $t > 0$ when an interaction happens neither in $u_1$ nor in $u_2$,

$$
\frac{d}{dt} \Phi (u_1(t), u_2(t)) \\
\leq C_3 \varepsilon + \sum_l \sum_i |s_{l,i}(0+)| W_{l,i} \lambda_{l,i}(0+) \\
\leq C_3 \varepsilon - \left( \sum_l |s_{l,1}(0+)| \right) \hat{K} \inf \lambda_{l,1} + K_J \left( \sum_l |s_{l,1}(0+)| \right) \max_l W_{l,2} \hat{\lambda} \\
\leq C_3 \varepsilon + \sum_l |s_{l,1}(0+)| \left( K_J \max W_{l,2} \hat{\lambda} - \hat{K} \inf \lambda_{l,1} \right) \\
\leq C_3 \varepsilon ,
$$

where $C_3$ is a suitable positive constant depending only on a bound on the total variation of the initial data. Above, we used the analog of Proposition 5.3 for shock curves, i.e. if $\Psi \left( S_2 \left( S_1(u, q_{1,1}), q_{2,1} \right) \right) = 0$, then $\sum_l |q_{l,1}| \leq K_J \sum_l |q_{l,1}|$, for a suitable $K_J$.

If $t > 0$ is an interaction time for $u_1$ or $u_2$, then, by Proposition 5.4, $\Delta \left[ \Upsilon (u_1(t)) + \Upsilon (u_2(t)) \right] < 0$ and, choosing $\kappa_2$ large enough, we obtain

$$
\Delta \Phi (u_1(t), u_2(t)) < 0 .
$$

Thus, $\Phi (u_1(t), u_2(t)) - \Phi (u_1(s), u_2(s)) \leq C_2 \varepsilon (t - s)$ for every $0 \leq s \leq t$. The proof is now completed using the standard arguments in [6, Chapter 8].

The proof that the semigroup trajectory does indeed yield a solution to (3.1) and, in particular, that $(\Psi)$ is satisfied on the traces, is exactly as that of [7, Proposition 5.3].

We now pass to the stability estimate (3.2). Its proof is similar to those of [5, Theorem 2.1] or [8, Theorem 3.1] and is based on [6, Theorem 2.9], which we recall for convenience: for every Lipschitz map $w: [0, T] \to D$ and every Lipschitz semigroup $S: [0, +\infty[ \times D \to D$, the following estimate holds:

$$
\| w(T) - S_T w(0) \| \leq L \cdot \int_0^T \left( \liminf_{h \to 0+} \frac{1}{h} \| w(t + h) - S_h w(t) \| \right) dt ,
$$

$L$ being the Lipschitz constant of $S$. In the present case, we are led to

$$
\| S_t^\Psi u - S_t^\Psi u \| \leq L^\Psi \cdot \int_0^t \left( \liminf_{h \to 0+} \frac{1}{h} \left\| S_h^\Psi S_t^\Psi u - S_h^\Psi S_t^\Psi u \right\| \right) dt .
$$
It remains to estimate \( \| S^\Psi_h v - S^\Psi h u \|_{L^1} \) for \( v = S^\Psi_r u \). We use the wave front tracking approximations \( v^{\Psi,\varepsilon}(h, \cdot) = S^\Psi h u \) and \( v^{\tilde{\Psi},\varepsilon}(h, \cdot) = S^\tilde{\Psi} h u \). For \( h > 0 \) sufficiently small, we can assume that there is at most a single interaction of the waves of \( u \in D \) with the intersection. Then \( S^\Psi h u \) coincides with the Riemann solver of Proposition (2.2) and estimate (2.4) can be applied:

\[
\| S^\Psi_h u - S^\Psi h u \|_{L^1} \leq L^\Psi \| \Psi_1 - \Psi_2 \|_{C^1} h.
\]

Since the right-hand side is independent of \( \varepsilon \) and since \( S^\Psi h u \) converges in \( L^1 \) to \( S^\Psi u \), we obtain

\[
\| S^\Psi_h u - S^\Psi_h u \|_{L^1} \leq \int_0^t \left( \liminf_{h \to 0^+} \frac{1}{h} \| \Psi - \tilde{\Psi} \|_{C^1} h \right) d\tau \leq L^\Psi \cdot \| \Psi - \tilde{\Psi} \|_{C^1} \cdot t
\]

completing the proof. □

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