On the smallest maximal partial ovoids and spreads of the generalized quadrangles $W(q)$ and $Q(4, q)$

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Available online 12 December 2006

Abstract

We present results on the size of the smallest maximal partial ovoids and on the size of the smallest maximal partial spreads of the generalized quadrangles $W(q)$ and $Q(4, q)$.

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1. Introduction

A finite generalized quadrangle $GQ(s, t)$ is an incidence structure $S = (P, B, I)$ consisting of two non-empty disjoint sets $P$ and $B$, consisting respectively of points and lines, such that:

1. Every line is incident with $s + 1$ points and every point is incident with $t + 1$ lines,
2. Two distinct points are incident with at most one common line, and two distinct lines are incident with at most one common point, and
3. For every non-incident point-line pair $(r, L)$, there exists a unique line $M$ and a unique point $r'$ such that $rIMr'L$.

We call the pair $(s, t)$ the order of this $GQ(s, t)$. We denote collinear points $x$ and $y$ by $x \sim y$, and concurrent lines $L$ and $M$ by $L \sim M$.

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The thick classical finite generalized quadrangles are respectively the non-singular 4-dimensional parabolic quadrics $Q(4, q)$ of order $(q, q)$, the non-singular 5-dimensional elliptic quadrics $Q^-(5, q)$ of order $(q, q^2)$, the non-singular 3- and 4-dimensional Hermitian varieties $H(3, q^2)$ and $H(4, q^2)$ of respective orders $(q^2, q)$ and $(q^2, q^3)$, and the non-singular finite generalized quadrangle $W(q)$ of order $(q, q)$ consisting of the points of PG$(3, q)$ and of the totally isotropic lines of a symplectic polarity $\eta$.

A spread of a GQ$(s, t)$ is a set of lines partitioning the point set of this generalized quadrangle. A partial spread of a GQ$(s, t)$ is a set of pairwise disjoint lines of this generalized quadrangle. A partial spread is called maximal when it is not contained in a larger partial spread. An ovoid $O$ of a GQ$(s, t)$ is a set of points such that every line of this generalized quadrangle shares exactly one point with $O$. A partial ovoid $O$ of a GQ$(s, t)$ is a set of points such that every line of this generalized quadrangle shares at most one point with $O$. A partial ovoid is called maximal when it is not contained in a larger partial ovoid.

A spread and an ovoid of a GQ$(s, t)$ have size $st + 1$.

A lot of attention has been paid to the (non-)existence of spreads and ovoids in finite generalized quadrangles [17,18]. Similarly, a lot of research has already been done on partial spreads and partial ovoids of size $st + 1 - d$, with small deficiency $d$, with special emphasis on the extendability of such partial spreads and partial ovoids to spreads and ovoids [4,12].

Recently, special attention has been paid to the smallest maximal partial ovoids and to the smallest maximal partial spreads of finite generalized quadrangles.

A maximal partial ovoid in a GQ$(s, t)$ must always have size greater than or equal to $s + 1$, and a maximal partial spread in a GQ$(s, t)$ must have size greater than or equal to $t + 1$.

In [1], Aguglia, Ebert and Luyckx studied the smallest maximal partial spreads of $Q^-(5, q) = \text{GQ}(q, q^2)$. They prove that the minimal size for such a maximal partial spread is equal to $t + 1 = q^2 + 1$ if and only if $q$ is even, and in this case, this maximal partial spread is equal to a spread of a subquadrangle $Q(4, q)$. For $q$ odd, they prove that a maximal partial spread of $Q^-(5, q)$ must have size larger than $q^2 + 2$.

Since $Q^-(5, q)$ is dual to the generalized quadrangle $H(3, q^2)$, the analogous results on maximal partial ovoids for $H(3, q^2)$ are valid.

Ebert and Hirschfeld studied the smallest maximal partial spreads of $H(3, q^2)$ [10]. They prove that every maximal partial spread has size at least $2q + 1$, and for $q \geq 4$, at least size $2q + 2$. Their results translate into results on the smallest maximal partial ovoids of $Q^-(5, q)$.

In [6], Cimráková and Fack present computer results obtained for the spectra of sizes of maximal partial ovoids in $Q^-(5, q)$ and $H(3, q^2)$, including values for small sizes.

We contribute to this study by providing results on the two thick finite classical generalized quadrangles $W(q)$ and $Q(4, q)$. We note that $W(q)$ is dual to $Q(4, q)$, and that $Q(4, q)$ and $W(q)$ are self-dual if and only if $q$ is even [14].

In [4,13], a (large) maximal partial ovoid of size $q^2 - q + 1$ in $W(q)$, $q$ even, is constructed and it is proven that no partial ovoids with sizes larger than $q^2 - q + 1$ and smaller than $q^2 + 1$ exist. We present in this article a maximal partial ovoid of size $q^2 - 2q + 3$ of $W(q)$, $q$ even. The motivation for paying special attention to maximal partial ovoids of size $q^2 - 2q + 3$ follows from the fact that computer searches seem to indicate that no maximal partial ovoids of size larger than $q^2 - 2q + 3$ and smaller than $q^2 - q + 1$ exist in $W(q)$, $q$ even; see also Table 1.

A blocking set of PG$(n, q)$ is a set of points having a non-empty intersection with every hyperplane of PG$(n, q)$. A blocking set is called trivial when it contains a line of PG$(n, q)$. A blocking set is called minimal when none of its proper subsets still is a blocking set.

In our study, blocking sets in PG$(2, q)$ and in PG$(3, q)$ will play an important role.
In a generalized quadrangle, for a set $A$ of points, the notation $A^\perp$ denotes the set of points collinear with every point of $A$. For two non-collinear points $x$ and $y$ of a generalized quadrangle, the set $\{x, y\}^\perp$ is called the hyperbolic line defined by $x$ and $y$. We note that for the generalized quadrangle $W(q)$, the hyperbolic lines $\{x, y\}^\perp$ coincide with the projective lines $xy$ of $\text{PG}(3, q)$, which are not totally isotropic with respect to the symplectic polarity $\eta$.

2. Small maximal partial ovoids in $W(q)$

**Theorem 2.1.** The smallest maximal partial ovoids of $W(q)$ have size $q + 1$ and consist of the point sets of the hyperbolic lines of $W(q)$.

**Proof.** Consider $W(q)$ in its natural representation in $\text{PG}(3, q)$ described by the symplectic polarity $\eta$; then it follows that every maximal partial ovoid $O$ of $W(q)$ must be a blocking set of $\text{PG}(3, q)$ with respect to the planes of $\text{PG}(3, q)$. In other words, if there is a plane $\pi$ skew to $O$, then the point $\pi^\eta$ extends $O$ to a larger partial ovoid, which contradicts the maximality of $O$. Since, from the result of Bose and Burton [3], the smallest blocking set of this type consists of the $q + 1$ points of a line, the theorem follows. □

**Corollary 2.2.**

1. The smallest maximal partial spreads of $Q(4, q)$ have size $q + 1$ and consist of the lines of a regulus of $\text{PG}(3, q)$.
2. The smallest maximal partial spreads of $W(q)$, $q$ even, have size $q + 1$ and consist of the lines of a regulus of $\text{PG}(3, q)$.
3. The smallest maximal partial ovoids of $Q(4, q)$, $q$ even, have size $q + 1$ and consist of the point sets of conics having the nucleus of $Q(4, q)$ as their nucleus.

Now that we have classified the smallest maximal partial ovoids of $W(q)$, we focus on results about the second smallest maximal partial ovoids of $W(q)$. Since the preceding proof shows that such a maximal partial ovoid must be a blocking set with respect to the planes of $\text{PG}(3, q)$, the planar non-trivial blocking sets are obvious candidates for such maximal partial ovoids. However, these are easily excluded.

**Theorem 2.3.** A maximal partial ovoid $O$ of $W(q)$, different from a hyperbolic line, cannot be a planar blocking set.

**Proof.** Suppose that $O$ is a planar blocking set, lying in the plane $\pi$ of $\text{PG}(3, q)$. Let $r = \pi^\eta$. Then $r \notin O$. But since $|O| > q + 1$, there is at least one totally isotropic line through $r$ in $\pi$ containing more than one point of $O$; we have a contradiction. □

**Lemma 2.4.** A maximal partial ovoid $O$ of $W(q)$ is a minimal blocking set with respect to the planes of $\text{PG}(3, q)$.

**Proof.** It follows from the preceding proofs that $O$ is a blocking set with respect to the planes of $\text{PG}(3, q)$. Assume that it is not minimal. Suppose that the point $r$ of $O$ is not essential as a point of $O$, considered as a blocking set with respect to the planes of $\text{PG}(3, q)$. Then every plane through $r$ contains a second point of $O$. So also the plane $r^\eta$ contains a second point $r'$ of $O$. Then the totally isotropic line $rr'$ contains at least two points of $O$. This is impossible. □

We now use results on the minimal blocking sets with respect to planes of $\text{PG}(3, q)$. The first result is due to Bruen.

**Theorem 2.5** (Bruen [5]). The smallest non-trivial blocking sets with respect to planes of $\text{PG}(3, q)$ are equal to the smallest planar non-trivial blocking sets of $\text{PG}(2, q)$.
Theorem 2.3 shows us that the second smallest maximal partial ovoids of $W(q)$ cannot be equal to the smallest non-trivial minimal blocking sets with respect to planes of $\text{PG}(3, q)$. So for the second smallest maximal partial ovoids of $W(q)$, we need to focus on the second smallest non-trivial minimal blocking sets with respect to planes of $\text{PG}(3, q)$. This allows us to obtain a considerably stronger result in some specific cases. We will first use the following two theorems from [15].

Let $s(q^2)$ denote the cardinality of the second smallest non-trivial minimal blocking sets in $\text{PG}(2, q)$. Theorem 2.6 (Storme and Weiner [15, Theorem 4.9]). Let $K$ be a blocking set of $\text{PG}(3, q^2)$, $q = p^h$, $p > 3$ prime, $h \geq 1$, of cardinality smaller than or equal to $s(q^2)$. Then $K$ contains a line or a planar blocking set of $\text{PG}(3, q^2)$.

Theorem 2.7 (Storme and Weiner [15, Theorem 5.9 and 5.10]). A minimal blocking set of $\text{PG}(3, q^3)$, $q = p^h$, $p \geq 7$ prime, $h \geq 1$, of size at most $q^3 + q^2 + q + 1$, is one of the following:

- a line,
- a Baer-subplane if $q$ is a square,
- a minimal planar blocking set of size $q^3 + q^2 + 1$,
- a minimal planar blocking set of size $q^3 + q^2 + q + 1$,
- a subgeometry $\text{PG}(3, q)$.

The possibility that a subgeometry $\text{PG}(3, q)$ of $\text{PG}(3, q^3)$ is a partial ovoid of $W(q^3)$ was eliminated in [9].

Theorem 2.8 (De Winter and Thas [9]). The GQ $W(q^3)$, $q = p^h$, $p \geq 7$ prime, does not admit a maximal partial ovoid of size $q^3 + q^2 + q + 1$.

This leads to the following corollaries.

Corollary 2.9. The second smallest maximal partial ovoids $O$ of $W(q^2)$, $q = p^h$, $p > 3$ prime, $h \geq 1$, contain at least $s(q^2) + 1$ points. If $q = p > 2$, then $O$ contains at least $3(p^2 + 1)/2 + 1$ points.

Proof. This follows immediately from Theorem 2.6 and the fact that $s(p^2) = 3(p^2 + 1)/2$ if $p > 2$ (see e.g. [16]).

Corollary 2.10. The second smallest maximal partial ovoids $O$ of $W(p^3)$, $p \geq 7$ prime, contain at least $\frac{3(p^3 + 1)}{2}$ points.

Proof. The minimal blocking sets in $\text{PG}(3, p^3)$, $p \geq 7$ prime, of size smaller than $\frac{3(p^3 + 1)}{2}$ have been classified in [15, Theorem 5.9 and 5.10]. See Theorem 2.7 for the complete list, with the exception of the Baer-subplane. The preceding results show that only a line can define a partial ovoid of $W(p^3)$.

Finally in the case when $q = p$ prime, we can use the result of Blokhuis [2], which states that every non-trivial planar blocking set of $\text{PG}(2, p)$ contains at least $3(p + 1)/2$ points.

Corollary 2.11. Let $O$ be a second smallest maximal partial ovoid of $W(p)$, $p$ prime. Then $|O| \geq 3(p + 1)/2 + 1$. 

Remark 2.12. (1) The preceding results can be translated into results on maximal partial spreads of $Q(4, q)$, on maximal partial spreads of $W(q)$, $q$ even, and on maximal partial ovoids of $Q(4, q)$, $q$ even.

To conclude this section on the size of the second smallest maximal partial ovoids of $W(q)$, we note that an example of a maximal partial ovoid of size $2q + 1$ can be obtained by taking all points except one point $r$ on a hyperbolic line $L$ in $PG(3, q)$, together with one arbitrary point (not collinear with one of the remaining points of $L$) from each of the $q + 1$ lines of $W(q)$ through $r$.

3. Small maximal partial spreads in $W(q)$

The only cases we have not yet discussed are the smallest maximal partial ovoids of $Q(4, q)$, $q$ odd, and the smallest maximal partial spreads of $W(q)$, $q$ odd. Since $W(q)$ is dual to $Q(4, q)$, we concentrate on maximal partial spreads of $W(q)$, $q$ odd.

Recall that when $q$ is an odd prime power, $|\{L_1, L_2, L_3\}| \in \{0, 2\}$ for every triad of skew lines of $W(q)$ (since in $Q(4, q)$ the perp of a conic is a $Q^\perp(1, q)$ if $q$ is odd). We will use a counting technique from [11] to prove the following theorem. In the following theorem, $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

**Theorem 3.1.** Suppose that $S$ is a maximal partial spread of $W(q)$, $q$ odd. Then $|S| \geq \lceil 1.419q \rceil$.

**Proof.** Suppose that $|S| = x$. Then there are exactly $D := q^3 + q^2 + q + 1 - x$ lines of $W(q)$ not belonging to $S$. Let $n_i, i = 1, \ldots, q + 1$, denote the number of such lines intersecting exactly $i$ lines of the partial spread $S$. Since $S$ is a maximal partial spread, $\sum_i n_i = D$. By counting in two ways the pairs $(L, M)$, where $L$ is a line not belonging to $S$, where $M$ is a line belonging to $S$, and where $L \sim M$, we obtain

$$\sum_i in_i = x(q + 1)q.$$  

For the triples $(L_1, L_2, M)$, where $L_1 \neq L_2$ are lines belonging to $S$, where $M$ is a line not belonging to $S$ and where $L_1 \sim M \sim L_2$, we obtain

$$\sum_i \binom{i}{2} n_i = \binom{x}{2} (q + 1),$$

and for the quadruples $(L_1, L_2, L_3, M)$, where $L_1, L_2, L_3$ are distinct lines belonging to $S$, where $M$ is a line not belonging to $S$, and where $M \sim L_m, m = 1, 2, 3$, we obtain

$$\sum_i \binom{i}{3} n_i \leq \binom{x}{3} 2$$

(recall that $|\{L_1, L_2, L_3\}| \in \{0, 2\}$ for every triad of skew lines of $W(q)$). Consider the polynomial $P(i) := (i - r_1)(i - r_2)(i - r_3)$ and the coefficients $a_0, a_1, a_2, a_3$ such that $P(i) = a_3 \binom{i}{3} + a_2 \binom{i}{2} + a_1 i + a_0$. We see that $a_3 = 6, a_2 = -2(r_1 + r_2 + r_3) + 6, a_1 = r_1 r_2 + r_1 r_3 + r_2 r_3 - (r_1 + r_2 + r_3) + 1,$ and $a_0 = -r_1 r_2 r_3$. Henceforth,

$$\sum_i P(i)n_i = a_3 \sum_i \binom{i}{3} n_i + a_2 \sum_i \binom{i}{2} n_i + a_1 \sum_i n_i + a_0 \sum_i n_i.$$
From this, using $a_3 > 0$, it follows that

$$
\sum_i P(i)n_i \leq 2a_3 \left(\frac{x}{3}\right) + (q + 1)a_2 \left(\frac{x}{2}\right) + q(q + 1)a_1x + a_0(q^3 + q^2 + q + 1 - x). \tag{1}
$$

If we choose coefficients $r_1, r_2, r_3$ in such a way that $P(i)n_i \geq 0$ for every $i \in \{1, \ldots, q + 1\}$, then $\sum_i P(i)n_i \geq 0$, and consequently $x$ has to be such that the right hand side of Eq. (1) is greater than or equal to 0. The expansion of $\sum_i (i - 1)(i - 4)(i - 5)n_i$ gives

$$
0 \leq 2x^3 - 13x^2 + 31x - 7x^2q + 27xq + 20xq^2 - 20q^3 - 20q^2 - 20q - 20,
$$

from which we deduce that $x > 1.419q$. \hfill \square

**Remark 3.2.** The result of the previous theorem can be slightly improved to $x \geq [1.419q + b]$, for certain $b > 0$, by substituting $x = 1.419q + b$ in Eq. (2), and by solving for the greatest $b$ for which the obtained polynomial in $q$ is still negative. The expression for $b$ obtained in this way is a tedious formula in $q$, but its value can easily be obtained by computer for a given $q$. For example, in the cases $q = 7, 9, 11$, this increases the smallest theoretical value of $x$ by one to 11, 14 and 17, respectively. It should however be noted that $b$ is extremely small with respect to $q$.

4. **Computer results**

In this section, we present results obtained by computer searches implementing the exhaustive and heuristic search techniques described in [7]. All programs are written in Java, and the results are obtained on a 1.6 GHz Pentium processor running Linux.

4.1. **Maximal partial ovoids in $W(q)$**

In Table 1, we give results for maximal partial ovoids in $W(q)$. For each value of $q$, we list the sizes for which the heuristic search found maximal partial ovoids of that given size. The notation $a..b$ means that a maximal partial ovoid of that size has been found for all values in the interval $[a, b]$.

For $q = 2, 3, 4, 5$, exhaustive search confirmed that the spectrum found by the heuristic is complete. Note that the largest value found for $W(5)$ and $W(7)$ is indeed the size of the largest maximal partial ovoid—this was confirmed by exhaustive search.

The results in Table 1 confirm the result from Theorem 2.1 that the smallest maximal partial ovoids have size $q + 1$. For the cases presented here, we also observe that maximal partial ovoids of size $2q + 1$ were always found, while no maximal partial ovoids with sizes between $q + 1$ and $2q + 1$ were found. As indicated in Remark 2.12, an example of a maximal partial ovoid of size $2q + 1$ can be obtained by taking all points except one point $r$ on a hyperbolic line $L$ in $W(q)$, together with one arbitrary point (not collinear with one of the remaining points of $L$) from each of the $q + 1$ lines of $W(q)$ through $r$.

Moreover, our results show the existence of a maximal partial ovoid of size $3q - 1$, for all values of $q$ considered. Such a maximal partial ovoid can be constructed in the following way if $q \geq 4$.

Let $X$ and $Y$ be two skew totally isotropic lines. Choose distinct points $x_1, x_2, x_3$ and $x$ on the line $X$ and let $y_i$ be $x_i \perp \cap Y$, $i = 1, 2, 3$. Finally, choose a point $y$ on $Y$ distinct from $y_1, y_2, y_3$ and $x \perp \cap Y$ (we can choose $y$ since $q \geq 4$). If we put $O_1$ the set of all points of $(x_1y_2 \cup x_2y_3 \cup x_3y_1) \setminus \{x, y_i | i = 1, 2, 3\}$, then $O := O_1 \cup \{x, y\}$ is a maximal partial...
Table 1
Spectrum of sizes for maximal partial ovoids of $W(q)$, for small values of $q$

<table>
<thead>
<tr>
<th>$q$</th>
<th>Spectrum found</th>
</tr>
</thead>
<tbody>
<tr>
<td>2*</td>
<td>3.5</td>
</tr>
<tr>
<td>3*</td>
<td>4.7</td>
</tr>
<tr>
<td>4*</td>
<td>5.9, 11, 12.14.18</td>
</tr>
<tr>
<td>5*</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.15, 17.20.33</td>
</tr>
<tr>
<td>8</td>
<td>9.17, 21.23.47.49, 51.57.65</td>
</tr>
<tr>
<td>9</td>
<td>10.19, 25.26.51</td>
</tr>
<tr>
<td>11</td>
<td>12.23, 28.32.70</td>
</tr>
<tr>
<td>13</td>
<td>14.27, 38.92</td>
</tr>
<tr>
<td>16</td>
<td>17.33, 47.49.51..163.165, 227.241.257</td>
</tr>
<tr>
<td>17</td>
<td>18.35, 50..129</td>
</tr>
<tr>
<td>19</td>
<td>20.39, 56..150</td>
</tr>
<tr>
<td>23</td>
<td>24.47, 68..70.72..190</td>
</tr>
<tr>
<td>25</td>
<td>26.51, 74.76.78.80..203</td>
</tr>
<tr>
<td>27</td>
<td>28.55, 80.236</td>
</tr>
</tbody>
</table>

For $q = 2, 3, 4, 5$, the complete spectrum was obtained by exhaustive search. For larger values of $q$, the results are obtained by a heuristic search. For $q = 5, 7$, the size of the largest partial ovoid was determined by exhaustive search.

A partial ovoid of size $3q - 1$. To prove that $O$ is indeed a partial ovoid we check here that no point of $x_1y_2 \setminus \{x_1, y_2\}$ can be collinear with a point of $x_2y_3 \setminus \{x_2, y_3\}$ (the other cases are treated analogously). By way of contradiction, we assume that a point $u$ of $x_1y_2 \setminus \{x_1, y_2\}$ is collinear with a point $v$ of $x_2y_3 \setminus \{x_2, y_3\}$. Since $u$ is also collinear with $x_2$, it follows that $u$ is collinear with $y_3$. Hence, as $y_3$ is also collinear with $y_2$, we see that $y_3$ is collinear with $x_1$, a contradiction. We now check for the maximality. Assume that a point $z$ would extend $O$ to a larger partial ovoid. Clearly $z$ does not belong to $X$ or $Y$. The point of $x_1y_2$ collinear with $z$ has to be either $x_1$ or $y_2$. Suppose, without loss of generality, that $z$ is collinear with $x_1$. The point of $x_2y_3$ collinear with $z$ has to be either $x_2$ or $y_3$, but cannot be $x_2$, as $z$ is already collinear with $x_1$ on $X$. Consequently $z$ is collinear with $y_3$. Finally, the point of $x_3y_1$ collinear with $z$ has to be either $x_3$ or $y_1$. However it cannot be either of these points, since $z$ would then be collinear with two points on $X$ or $Y$. We conclude that $O$ is maximal.

For $q$ even, our computer searches also find a maximal partial ovoid of size $q^2 - q + 1$ and no maximal partial ovoids with sizes larger than $q^2 - q + 1$ and smaller than $q^2 + 1$, as the results of [4] and [13] show. We also observed the existence of a maximal partial ovoid with size $q^2 - q + 1 - (q - 2) = q^2 - 2q + 3$, and we found no maximal partial ovoids with size larger than $q^2 - 2q + 3$ and smaller than $q^2 - q + 1$.

We can describe in a compact way a geometric construction for maximal partial ovoids of sizes $q^2 - q + 1$ and $q^2 - 2q + 3$ of $W(q)$, $q$ even. We explain the construction on $Q(4, q)$ (recall that $q$ is even and so $Q(4, q) \cong W(q)$). First, note that $|C^\perp| \in \{1, q + 1\}$ for any conic $C$ in $Q(4, q)$. From this we see that if we consider a conic $C$ in an elliptic quadric $O := Q^-(3, q) \subset Q(4, q)$, then necessarily $C^\perp$ is a unique point $c$. It is easily seen that $(O \cup \{c\}) \setminus C$ is a maximal partial ovoid of size $q^2 - q + 1$. Now let $O$ be an elliptic quadric of $Q(4, q)$, and suppose that $C_1$ and $C_2$ are two conics of $O$, with $|C_1 \cap C_2| = 2$. Clearly the points $c_1 := C_1^\perp$ and $c_2 := C_2^\perp$ are not collinear (since $|C_1 \cap C_2| = 2$). If $q > 2$, it follows easily that $(O \cup \{c_1, c_2\}) \setminus (C_1 \cup C_2)$ is a maximal partial ovoid of size $q^2 - 2q + 3$. 
Table 2
Spectrum of sizes for maximal partial ovoids of $Q(4, q)$, for small values of $q$

<table>
<thead>
<tr>
<th>$q$</th>
<th>LB</th>
<th>Spectrum found (by heuristics)</th>
<th>Non-existence (exhaustive search)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3^*$</td>
<td>5</td>
<td>5, 8, 10</td>
<td>All other values</td>
</tr>
<tr>
<td>$5^*$</td>
<td>8</td>
<td>13, 20, 22, 24, 26</td>
<td>All other values</td>
</tr>
<tr>
<td>7</td>
<td>11</td>
<td>14, 17, 42, 44, 48, 50</td>
<td>10, 11, 43, 45, 46, 47, 49 (still open: 12, 13, 15, 16)</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>22, 68, 70, 73, 74, 82</td>
<td>79, 80</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
<td>28, 30, 106, 109, 110, 112, 120, 122</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>19</td>
<td>41, 42, 44, 136, 138, 140, 146, 148, 158, 170</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>25</td>
<td>67, 218, 220, 224, 226, 228, 230, 232, 238, 240, 244, 246, 248, 258, 260, 274, 290</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>27</td>
<td>84, 118, 122, 275, 278, 280, 282, 286, 294, 296, 298, 300, 310, 312, 326, 328, 344, 362</td>
<td></td>
</tr>
</tbody>
</table>

For $q = 3, 5$, the complete spectrum was obtained by exhaustive search. For larger values of $q$, the results are obtained by heuristic search. For $q = 7, 9$, the non-existence of maximal partial ovoids of certain sizes was confirmed by exhaustive search.

4.2. Maximal partial ovoids in $Q(4, q)$, $q$ odd

In Table 2, we give results for maximal partial ovoids in $Q(4, q)$, $q$ odd. For each value of $q$, we list the value of the lower bound (LB) from Theorem 3.1 and Remark 3.2, and the sizes for which our program found maximal partial ovoids of that given size. The notation $a..b$ means that for all values in the interval $[a, b]$, a maximal partial ovoid of that size has been found.

For $q = 3, 5$, we confirmed by exhaustive search that the spectrum found is complete. For $q = 7, 9$, we confirmed by exhaustive search for some sizes (also given in the table) that no maximal partial ovoid of that size exists.

In spite of the fact that the theoretical lower bounds are linear in $q$, these results rather seem to indicate a quadratic lower bound.

In all cases our heuristic finds an ovoid (of size $q^2 + 1$). For $q = 3, 5, 7, 11$, a maximal partial ovoid of size $q^2 - 1$ is found; for $q = 9$, it is confirmed by exhaustive search that no such maximal partial ovoid exists; for larger values of $q$, no such maximal partial ovoids were found by our heuristic.

Recently, De Beule and Gács proved the non-existence of maximal partial ovoids of size $q^2 - 1$ of $Q(4, q)$, $q = p^h$, $p$ an odd prime, $h > 1$.

**Theorem 4.1** (De Beule and Gács [8]). The quadric $Q(4, q)$, $q = p^h$, $p$ an odd prime, $h > 1$, does not contain maximal partial ovoids of size $q^2 - 1$.

For all values of $q$ considered, the largest (second largest, for the cases $q = 3, 5, 7, 11$) size for a maximal (strictly) partial ovoid found by the heuristic search is $q^2 - q + 2$.

Acknowledgement

L. Storme thanks the Fund for Scientific Research—Flanders (Belgium) for a Research Grant.

References


