The research presented in this paper was motivated by our aim to study a problem due to J. Bourgain [3]. The problem in question concerns the uniform boundedness of the classical separation rank of the elements of a separable compact set of the first Baire class. In the sequel we shall refer to these sets (separable or non-separable) as Rosenthal compacta and we shall denote by \( \alpha(f) \) the separation rank of a real-valued function \( f \) in \( \mathcal{B}_1(X) \), with \( X \) a Polish space. Notice that in [3], Bourgain has provided a positive answer to this problem in the case of \( K \) satisfying \( K = K \cap C(X) \) with \( X \) a compact metric space. The key ingredient in Bourgain’s approach is that whenever a sequence of continuous functions pointwise converges to a function \( f \), then the possible discontinuities of the limit function reflect a local \( \ell^1 \)-structure to the sequence \((f_n)_n\). More precisely the complexity of this \( \ell^1 \)-structure increases as the complexity of the discontinuities of \( f \) does. This fruitful idea was extensively studied by several authors (c.f. [5], [7], [8]) and for an exposition of the related results we refer to [1]. It is worth mentioning that A. S. Kechris and A. Louveau have invented the rank \( r_{ND}(f) \) which permits the link between the \( c_0 \)-structure of a sequence \((f_n)_n\) of uniformly bounded continuous functions and the discontinuities of its pointwise limit. Rosenthal’s \( c_0 \)-theorem [11] and the \( c_0 \)-index theorem [2] are consequences of this interaction.

Passing to the case where either \((f_n)_n\) are not continuous or \( X \) is a non-compact Polish space, this nice interaction is completely lost. Easy examples show that there exist sequences of continuous functions on \( \mathbb{R} \) pointwise convergent to zero and in the same time they are equivalent to the \( \ell^1 \) basis. Also there are sequences \((f_n)_n\) of Baire-1 functions, equivalent to the summing basis of \( c_0 \), pointwise convergent to a Baire-2 function. Thus if we wish to preserve the main scheme, invented by Bourgain, namely to pass from the elements of the separable Rosenthal compactum to a well-founded tree related to the dense sequence \((f_n)_n\), this has to take into account not only the finite subsets of \((f_n)_n\) but also the points of the Polish space \( X \). This is the key observation on which we have based our approach. Thus for every \( D \) subset of \( \mathbb{R}^X \) we associate a tree \( T((f_\xi)_{\xi<\theta}, a, b) \) where \((f_\xi)_{\xi<\theta}\) is a well-ordering of \( D \) and \( a < b \) are reals. The elements of the tree are of the form \( (u, T) \) with \( u \) a finite increasing subsequence of \((f_\xi)_{\xi<\theta}\) and \( T \) a finite dyadic tree in \( X \),

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where the length of \( u \) and the height of \( T \) are the same and which share certain properties. The partial order of this tree is naturally defined. The basic property of the tree is described by the following.

**Proposition A.** For every relatively compact subset \( D \) of \( \mathbb{R}^X \) the following are equivalent.

1. For every well-ordering \( (f_{\xi})_{\xi<\theta} \) of \( D \) and \( a < b \), the tree \( T((f_{\xi})_{\xi<\theta}, a, b) \) is well-founded.
2. The accumulation points of \( D \) in \( \mathbb{R}^X \) are Baire-1 functions.

Motivated by this we introduce the class of quasi-Rosenthal compacta as the compact subsets \( K \) of \( \mathbb{R}^X \) for which the set \( \text{Acc}(K) \) of accumulation points of \( K \) is a subset of \( B_1(X) \). Naturally defined examples show that this class is wider than the corresponding class of Rosenthal compacta. We also present some characterizations and results on quasi-Rosenthal compacta. Next for a sequence \( (f_n)_n \) in \( \mathbb{R}^X \) and a function \( f \in \text{Acc}(\{f_n\}_n) \cap B_1(X) \) we compare the quantity
\[
o((f_n)_n) + 1 \geq \alpha(f).
\]

Notice that (1) is expected and it holds for all ranks defined on a sequence and related to a rank of the limit function \( f \). For example similar results hold for the ranks \( \gamma((f_n)_n) \) and \( \alpha(f) \) (see [7]) or the ranks \( \nu((f_n)_n) \) and \( r_{ND}(f) \) (see [2]). However (2) is rather unexpected since usually the ranks defined on sequences do not recognize possible noise involved in the elements of the sequence. Thus \( \gamma((f_n)_n) \) or \( \nu((f_n)_n) \) could be arbitrarily larger than \( \alpha(f) \) and \( r_{ND}(f) \) respectively. As consequence of part (1) of the previous theorem we obtain the following.

**Theorem B.** For every sequence \( (f_n)_n \) in \( \mathbb{R}^X \) and every \( f \in B_1(X) \) the following hold.

1. If \( f \in \text{Acc}(\{f_n\}_n) \), then
\[
o((f_n)_n) + 1 \geq \alpha(f).
\]
2. If \( f = \lim_n f_n \), then
\[
o((f_n)_n) \leq \omega \cdot 2 \cdot \alpha(f).
\]

**Corollary C.** Let \( K = \{f_n\}_n \) be a quasi-Rosenthal compactum. Then
\[
\sup \{ \alpha(f) : f \in \text{Acc}(K) \} \leq o((f_n)_n) + 1.
\]

This result would yield an affirmative answer to Bourgain’s question provided that \( o((f_n)_n) < \omega_1 \). This is not true in general as the examples show. Kunen-Martin principle permits us to prove it under some additional regularity properties of the sequence \( (f_n)_n \). Namely we show the following (Theorem 23 in the main text), which answers Bourgain’s problem.
Theorem D. Let $\mathcal{K}$ be a Borel separable quasi-Rosenthal compactum. Then
\[ \sup \{ \alpha(f) : f \in \mathcal{K} \cap B_1(X) \} < \omega_1. \]

Here Borel separable means that there exists a countable dense subset consisting of Borel functions. We notice that the above result is sharp. More precisely we provide an example of a separable quasi-Rosenthal compactum $\mathcal{K}$ containing a countable dense subset consisting of characteristic functions of analytic sets and such that $\sup \{ \alpha(f) : f \in \text{Acc}(\mathcal{K}) \} = \omega_1$.

1. Preliminaries

In what follows $X$ is a Polish space and $d$ a compatible complete metric for $X$. By $B_1(X)$ (respectively $B(X)$) we denote the space of Baire-1 (respectively Borel) real-valued functions on $X$. By $\mathbb{N}$ we denote the set of all positive integers, while by $\omega$ the set of all non-negative integers. If $L$ is an infinite subset $\omega$, by $[L]$ we denote the set of all infinite subsets of $L$. For a well-ordered set $\theta$, by $[\theta]^{<\omega}$ we denote the set of all finite strictly increasing sequences of $\theta$.

The separation rank $\alpha$. The separation rank $\alpha(f)$ of a Baire-1 function has its roots in the work of Hausdorff, Kuratowski and Lavrentiev. We recall its definition taken from [7]. For $A, B \subseteq X$ one associates with them a derivative on closed sets, by
\[ K_{A,B}^\prime = (K \cap A) \cap (K \cap B). \]

Now, by recursion, define the iterated derivatives of $K$ by $K_{A,B}^{(0)} = K$, $K_{A,B}^{(\xi+1)} = (K_{A,B}^{(\xi)})^\prime$ and $K_{A,B}^{(\xi)} = \bigcap_{\zeta < \xi} K_{A,B}^{(\zeta)}$ if $\xi$ is a limit ordinal. Let
\[ \alpha(K, A, B) = \begin{cases} \text{least } \xi : & K_{A,B}^{(\xi)} = \varnothing, \text{ if such } \xi \text{ exists}, \\ \omega_1 : & \text{otherwise}, \end{cases} \]

The sets $[f \leq a]$, $[f > a]$ and $[f \geq a]$ have the obvious meaning.
and \( \alpha(A, B, \omega) = \alpha(A, B) < \omega_1 \) if and only if one can separate \( A \) from \( B \) by a set which is transfinite difference of closed sets (see [6], page 177).

Now let \( f : X \to \mathbb{R} \) be a function. For every \( a < b \) let

\[
\alpha(f, a, b) = \alpha([f < a], [f > b])
\]

and finally define the separation rank of \( f \) by

\[
\alpha(f) = \sup \{ \alpha(f, a, b) : a < b \}.
\]

The basic fact is the following (see [7]).

**Proposition 1.** A function \( f \) is Baire-1 if and only if \( \alpha(f) < \omega_1 \).

The above defined rank is slightly different from the one J. Bourgain originally defined in [3]. As it is shown in [5] the two variants are equivalent. Also observe that the rank is the same if we have defined \( \alpha(f, a, b) \) by \( \alpha([f \leq a], [f \geq b]) \). For every \( Y \subseteq X \) closed, \( \xi < \omega_1 \) and \( a < b \) by \( Y^{(\xi)}_{(f,a,b)} \) we denote the \( \xi \)-iterated derivative of \( Y \) with respect to \([f < a]\) and \([f > b]\). For the properties of \( \alpha \) and its relations with other ordinal ranks on \( B_1(X) \) we refer to [7].

**Trees and well-founded relations.**

By the term tree we mean a partial order set \((T, \prec)\) in the strict sense, such that for every \( t \in T \) the set \( \{ s \in T : s < t \} \) is well-ordered. Now let \( T \) be a well-founded tree. As usual set

\[
T' = \{ s \in T : \exists t \in T \text{ such that } s < t \}.
\]

By recursion we define \( T^{(0)} = T \), \( T^{(\xi+1)} = (T^{(\xi)})' \) and \( T^{(\xi)} = \bigcap_{\xi < \xi} T^{(\xi)} \) if \( \xi \) is a limit ordinal. The order \( o(T) \) of \( T \) is defined to be the least ordinal \( \xi \) for which \( T^{(\xi)} = \emptyset \). If \((S, \prec_S)\) and \((T, \prec_T)\) are well-founded trees, then a map \( \varphi : S \to T \) is called monotone if \( s_1 <_S s_2 \) implies that \( \varphi(s_1) <_T \varphi(s_2) \). Clearly in this case we have that \( o(S) \leq o(T) \).

Let \( X \) be a set and \( \prec \) a strict, well-founded relation on \( X \). By recursion we define \( \rho_\prec : X \to \text{Ord} \) as follows. We set \( \rho_\prec(x) = 0 \) if \( x \) is minimal, otherwise we set \( \rho_\prec(x) = \sup \{ \rho_\prec(y) + 1 : y < x \} \). Finally we define the rank of \( \prec \) to be \( \rho(\prec) = \sup \{ \rho_\prec(x) + 1 : x \in X \} \). We will need the following boundedness principle of analytic well-founded relations, due to Kunen and Martin (see [6] or [9]).

**Theorem 2.** Let \( X \) be a Polish space and \( \prec \) be a strict and well-founded relation. If \( \prec \) is analytic (as a subset of \( X \times X \)), then \( \rho(\prec) < \omega_1 \).

Note that if \((T, \prec)\) is a well-founded tree, then the relation \( \prec \) on \( T \) defined by \( t \prec s \) if \( s < t \), is strict and well-founded and \( o(T) = \rho(\prec) \).
2. **The tree** $T((f_ξ)_{ξ<θ}, a, b)$.

In this section we introduce the tree $T((f_ξ)_{ξ<θ}, a, b)$ and we define the class of quasi-Rosenthal compacta. We present some results related to the above two notions and examples of quasi-Rosenthal compacta which are not Rosenthal compacta. We start with the following definition.

**Definition 3.** Let $a < b$ reals, $θ$ an infinite ordinal and $D = (f_ξ)_{ξ<θ}$ be a long sequence of not necessarily distinct elements of $\mathbb{R}^X$. We define

$$T = T((f_ξ)_{ξ<θ}, a, b) \subseteq \bigcup_{k \in \omega} [θ]^k \times X^{D_k}$$

to be the set of all pairs $(u, T)$ for which the following hold. First $(∅, (t_∅)) \in T$ for every $t_∅ \in X$. Moreover, if $u = (ξ_1, ..., ξ_k)$, $T = (t_s)_{s \in D_k}$ and $k \geq 1$, then $(u, T) \in T$ if it satisfies the following conditions.

(C1) Either $t_0 = t_∅$ or $t_1 = t_∅$.
(C2) For every $s \in D_k$ with $|s| < k$, the following hold.
   (i) $d(t_{s-0}, t_{s-1}) \leq \frac{1}{2^{|s|-1}}$.
   (ii) $f_{ξ_{|s|+1}}(t_{s-0}) < a$ and $f_{ξ_{|s|+1}}(t_{s-1}) > b$ (hence $t_{s-0} ≠ t_{s-1}$).
   (iii) For every $s ≠ ∅$, if $f_{ξ_{|s|}}(t_s) < a$ then $t_s = t_s^1$, while if $f_{ξ_{|s|}}(t_s) > b$ then $t_{s-1} = t_s$.

The set $T$ is a tree under the following partial ordering. If $(u_1, T_1), (u_2, T_2) \in T$ with $T_1 = (t_1^s)_{s \in D_{k_1}}$ and $T_2 = (t_2^s)_{s \in D_{k_2}}$, then

$$(u_1, T_1) < (u_2, T_2)$$

if $u_1 ⊑ u_2$ and $T_1 \prec T_2$

where by $T_1 \prec T_2$ we mean that $t_1^s = t_2^s$ for every $s \in D_{k_1}$.

For every $Y \subseteq X$ non-empty let

$$T(Y, (f_ξ)_{ξ<θ}, a, b) \subseteq T$$

to be the set of all $(u, T) \in T$ for which $T \subseteq \bigcup_{k \in \omega} Y^{D_k}$. By convention, if $Y = ∅$, then we set $T(Y, (f_ξ)_{ξ<θ}, a, b) = ∅$.

**Remark 1.** Clearly $T(Y, (f_ξ)_{ξ<θ}, a, b)$ equipped with the induced partial order is a subtree of $T$. Also notice that if $Y' \subseteq Y$ and $a' ≤ a < b ≤ b'$, then $T(Y', (f_ξ)_{ξ<θ}, a', b')$ is a subtree of $T(Y, (f_ξ)_{ξ<θ}, a, b)$.

**Definition 4.** A compact subset $K$ of $\mathbb{R}^X$ is said to be a quasi-Rosenthal compactum if the set $\text{Acc}(K)$ of accumulation points of $K$ is a non-empty subset of $B_1(X)$.

**Remark 2.** Observe that for every sequence $(f_ξ)_{ξ<θ}$ in $K$ and every function $f \in \text{Acc}(\{f_ξ\}_{ξ<θ})$, there exists $ξ_f ≤ θ$ and a subnet of $(f_ξ)_{ξ<ξ_f}$ converging in $\mathbb{R}^X$ to $f$. Indeed, we define $ξ_f = \min \{ξ ≤ θ : f \in \{f_ξ\}_{ξ<ξ}\}$. It is easy to verify that $ξ_f$ is the desired ordinal.
Lemma 5. Let \((f_ξ)_{ξ<θ}\) be a pointwise bounded sequence of distinct elements of \(\mathbb{R}^X\) and suppose that the tree \(T((f_ξ)_{ξ<θ}, a, b)\) contains an infinite chain \(((u_κ, T_κ))_κ\).

We set \(N = \bigcup_κ u_κ = \{ξ_1 < ξ_2 < ...\}\) and \(T = \bigcup_κ T_κ\). Then there exist \(L ∈ [N]\) and \(f : T → \mathbb{R}\) such that

1. \(\lim_{n ∈ L} f_ξ|_T = f\).
2. \(T = A ∪ B\) with \(\overline{A} = \overline{B}\), \(A = [f ≤ a]\) and \(B = [f ≥ b]\).

In particular every \(g ∈ Acc\{\{f_ξ\}_{ξ ∈ L}\}\) is not a Baire-1 function.

Proof. We set \(T = \bigcup_{κ ∈ \mathbb{N}} T_κ = (t_κ)_κ ∈ D\). Let

\[A = \{t_κ : s ∈ D\}\) and \(B = \{t_κ^{-1} : s ∈ D\}\).

Since \(T\) is countable, we choose \(L ∈ [N]\) such that (1) is satisfied. The definition of the tree \(T((f_ξ)_{ξ<θ}, a, b)\) yields that \(A = [f ≤ a]\), \(B = [f ≥ b]\) and \(A = \overline{B}\). Using Proposition 1, for every \(g ∈ Acc\{\{f_ξ\}_{ξ ∈ L}\}\), we have that \(g|_T = f\) which implies that \(g\) is not a Baire-1 function. □

In the following theorem we provide equivalent characterizations of quasi-Rosenthal compacta.

Theorem 6. Let \(\mathcal{D}\) be a relatively compact subset of \(\mathbb{R}^X\). Then the following are equivalent.

1. \(K = \overline{\mathcal{D}}\) is a quasi-Rosenthal compactum.
2. For every sequence \((f_ξ)_{ξ<θ}\) of distinct members of \(\mathcal{D}\) and every \(a < b\), the tree \(T((f_ξ)_{ξ<θ}, a, b)\) is well-founded.
3. For every sequence \((f_κ)_{κ ∈ \mathbb{N}}\) of distinct members of \(\mathcal{D}\), there exists a subsequence pointwise convergent to a Baire-1 function.
4. For every infinite subset \(\mathcal{D}'\) of \(\mathcal{D}\), there exists a Baire-1 function \(f\) belonging to \(Acc(\mathcal{D}')\).

Proof. (1)⇒(2) Assume on the contrary that there exists a sequence \((f_ξ)_{ξ<θ}\) in \(\mathcal{D}\) and \(a < b\) such that the tree \(T((f_ξ)_{ξ<θ}, a, b)\) is not well-founded. Then Lemma 5 yields a contradiction.

(2)⇒(1) Let \(\mathcal{D} = (f_ξ)_{ξ<θ}\) be a well-ordering of \(\mathcal{D}\). Note that \(Acc(K) = Acc(\mathcal{D})\).

Assume on the contrary that there exists \(f ∈ Acc(\mathcal{D})\) such that \(f\) is not a Baire-1 function. Then \(α(f) = ω_1\) and so there exist \(a < b\) such that \(α(f, a, b) = ω_1\).

It follows easily that there exists \(A, B ⊆ X\) countable with \(\overline{A} = \overline{B}\) such that \(A ⊆ [f < a]\) and \(B ⊆ [f > b]\). We construct \(T = (t_κ)_{κ ∈ D} ∈ X D\) such that for every \(s ∈ D\) the following hold.

(i) \(t_σ ∈ A\).  
(ii) For every \(s ∈ D\), \(t_κ^{-0} ∈ A\) and \(t_κ^{-1} ∈ B\).  
(iii) For every \(s ∈ D\), if \(t_κ ∈ A\) then \(t_κ^{-0} = t_κ\), while if \(t_κ ∈ B\) then \(t_κ^{-1} = t_κ\).  
(iv) \(d(t_κ^{-0}, t_κ^{-1}) < \frac{1}{2^{|D|+1}}\).
We proceed by induction on the length of $s$. For $|s| = 0$, we choose $x \in A$ and we set $t_0 = x$. Suppose that $t_s$ have been defined for every $s \in D$ with $|s| \leq k$. For every $s \in D$ with $|s| = k$ and $t_s \in A$ we set $s^{-0}t_s = t_s$ and we choose $y_s \in B$ such that the condition (iv) above is satisfied for $t_{s^{-1}} = y_s$. The case $t_s \in B$ is treated similarly. The construction is completed.

Next we choose an increasing sequence $(\xi_k)_{k \in \mathbb{N}}$ such that $\xi_k < \xi_f$ (see Remark 2) and for every $s \in D$ with $|s| = k$ we have

$$f_{\xi_k}(t_s) < a \text{ if } f(t_s) < a \text{ and } f_{\xi_k}(t_s) > b \text{ if } f(t_s) > b.$$ 

We set $u_k = (\xi_1, \ldots, \xi_k)$ and $T_k = (t_s)_{s \in D_k}$ for every $k \in \mathbb{N}$. Then it is easily checked that $((u_k, T_k))_k$ is an infinite chain of $T(\langle \xi_k \rangle_{k < a}, a, b)$.

(3) $\Rightarrow$ (2) It is an immediate consequence of Lemma 5.

(1) $\Rightarrow$ (3) Let $(f_n)_n$ be a sequence in $D$. If $(f_n)_n$ has a pointwise convergent subsequence, then the limit function belongs to the accumulation points of $K$ and so it is a Baire-1 function. So, assume on the contrary that there exists a sequence $(f_n)_n$ of $D$ with no pointwise convergent subsequence. Then, Theorem 2 in [10] yields that there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ with no accumulation point in $B_1(X)$ which leads to a contradiction.

(1) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (2) This is again a consequence of Lemma 5. \hfill $\square$

In the next proposition we establish that quasi-Rosenthal compacta have countable tightness. This property is known for Rosenthal compacta (see [10]).

**Proposition 7.** Let $D$ be a subset of $\mathbb{R}^X$ such that $K = \overline{D}^p$ is a quasi-Rosenthal compactum. Then for every $g \in K$ there exists $D' \subseteq D$ countable with $g \in \overline{D}'^p$.

**Proof.** We may assume that $g \in Acc(D)$. We set

$$Seq(D) = \{ f \in Acc(D) : f \text{ is the limit of a sequence of distinct members of } D \}.$$ 

We claim that $g \in Seq(\overline{D}^p)$. Indeed, let $x_1, x_2, \ldots, x_k \in X$ and $\varepsilon > 0$ arbitrary. Then there exist a sequence $(f_n)_n$ of distinct members of $D$ such that $|f_n(x_i) - g(x_i)| < \varepsilon$ for every $i = 1, \ldots, k$ and every $n \in \mathbb{N}$. Theorem 6(3) yields that there exists a pointwise convergent subsequence $(f_{n_k})_k$ of $(f_n)_n$. If $f$ is the pointwise limit of $(f_{n_k})_k$, then clearly $f \in Seq(D)$ and $|f(x_i) - g(x_i)| \leq \varepsilon$ for every $i = 1, \ldots, k$ which proves the claim.

Now observe that $Seq(D)$ is a subset of $Acc(D) \subseteq B_1(X)$. So $Seq(D)$ is a relatively compact subset of $B_1(X)$. By the Main Theorem in [10], the result follows. \hfill $\square$

**Remark 3.** It seems well-known and follows by results in [13] that every separable Rosenthal compactum satisfies the Continuum Hypothesis. Indeed, by Theorem 5 in [13], every separable Rosenthal compactum $K$ either contains a discrete subspace of size continuum or it is an at most two-to-one continuous pre-image of a compact
metrizable space. In any case it is straightforward that \( \mathcal{K} \) has either \( \aleph_0 \) or \( 2^{\aleph_0} \) members.

**Example 1.** We shall present examples of separable quasi-Rosenthal compacta which show the variety of this class. Let \( \mathcal{C} = [0,1]^\mathbb{N} \) be the Cantor set and let \( X = \mathcal{C} \times \mathcal{C} \). For every \( \sigma \in \mathcal{C} \) let \( \mathcal{C}_\sigma = \{ \sigma \} \times \mathcal{C} \). We choose \( \Delta_\sigma \subseteq \mathcal{C}_\sigma \) such that \( \chi_{\Delta_\sigma} \) (as a function on \( X \)) is Baire-1. For every \( s \in D, D \) the dyadic tree, let

\[
\Delta_s = \bigcup_{\sigma \supseteq s} \Delta_\sigma
\]

and \( f_s = \chi_{\Delta_s} \). Let \( \mathcal{K} = \{ f_s \}_{s \in D} \). Then \( \mathcal{K} \) is separable and for every \( (f_{s_n})_n \) sequence in \( \{ f_s \}_{s \in D} \), there exists \( L \in [\mathbb{N}] \) such that \( (s_n)_{n \in L} \) are either pairwise incomparable or there exists \( \sigma \in \mathcal{C} \) such that \( s_n \supseteq \sigma \) for every \( n \in L \). In the first case the sequence \( (f_{s_n})_{n \in L} \) converges pointwise to 0 while in the second case it converges pointwise to \( \chi_{\Delta_\omega} \). Theorem 6 yields that \( \mathcal{K} \) is a separable quasi-Rosenthal compactum. Depending on the choice of \( \{ \Delta_\sigma \}_{\sigma \in \mathcal{C}} \) we obtain different spaces.

1. We may choose \( \Delta_\sigma \) with \( \alpha(\chi_{\Delta_\omega}) \geq \xi_\sigma \) and \( \sup \{ \xi_\sigma : \sigma \in \mathcal{C} \} = \omega_1 \). This space answers in negative J. Bourgain’s question stated for separable quasi-Rosenthal compacta.

2. If \( \aleph_0 < |\{ \sigma \in \mathcal{C} : \Delta_\sigma \neq \emptyset \}| < 2^{\aleph_0} \), then the corresponding \( \mathcal{K} \) satisfies that \( \aleph_0 < |\mathcal{K}| < 2^{\aleph_0} \). This yields that, under the negation of CH, there exist separable quasi-Rosenthal compacta not homeomorphic to any Rosenthal compactum (see Remark 3).

A variant of this example, based on techniques of universal sets from descriptive set theory, is presented after Theorem 23.

For the following proposition we recall that a sequence \( (\{A_k, B_k\})_k \) of subsets of a set \( S \) such that \( A_k \cap B_k = \emptyset \) for every \( k \in \mathbb{N} \), is called an independent sequence if for every \( F, G \) finite disjoint subsets of \( \mathbb{N} \) we have

\[
\left( \bigcap_{k \in F} A_k \right) \cap \left( \bigcap_{k \in G} B_k \right) \neq \emptyset.
\]

This definition is crucial for the proof of Rosenthal’s \( \ell^1 \) theorem (see [6] or [12]).

**Proposition 8.** Let \( (f_n)_n \) be a pointwise bounded sequence of continuous real-valued functions on \( X \). Suppose that there exist \( a < b \) such that the tree \( T = T((f_n)_n, a, b) \) is not well-founded and let \( ((u_k, T_k))_k \) be an infinite chain of \( T \). We set \( N = \bigcup_k u_k = \{ u_k : k \in \mathbb{N} \} \) and \( T = \bigcup_k T_k = (t_s)_{s \in D} \).

Then there exist a Cantor set \( C \subseteq T \) and a subsequence \( (f_{n_k'})_k \) of \( (f_{n_k})_k \) such that the sequence \( ((f_{n_k'})_k < a] \cap C, [f_{n_k'} > b] \cap C))_k \) is an independent sequence of disjoint pairs.

**Proof.** By induction, we construct a subtree \( T' = (t'_s)_{s \in D} \) of \( T \), an increasing sequence \( (l_k)_{k \in \omega} \) and a set of open balls \( \{ B_s : s \in D \} \) of \( X \) with the following properties.
(1) For every \( s, t'_s \in B_s \).
(2) If \(|s| = k\), then there exists \( s' \) with \(|s'| = l_k\) such that \( t'_s = t_{s'} \).
(3) If \(|s| = k\), then \( B_{s^0} \subseteq [f_{n_k+1} < a] \) and \( B_{s^1} \subseteq [f_{n_k+1} > b] \).
(4) \( \overline{B}_{s^0} \cup \overline{B}_{s^1} \subseteq B_s \).
(5) \( \text{diam} B_s \leq \frac{1}{2^k} \).

We start by setting \( t'_{s^0} = t_{s^0} \), \( t_{s^0} = 0 \) and \( B_0 \) be any open ball containing \( t_{s^0} \) with \( \text{diam} B_0 \leq \frac{1}{2} \). Suppose that the construction has been carried out for some \( k \in \omega \).

Let \( 2\varepsilon_k = \min\{\text{diam} B_s : s \in D_k\} \) and choose \( m \in \omega \) such that \( \frac{1}{2^{k+1}} < \varepsilon_k \). Let \( s \in D_k \) with \(|s| = k\). Then \( t'_s = t_s \) for some \( s' \) with \(|s'| = l_k\). Hence either \( f_{n_k}(t'_s) < a \) or \( f_{n_k}(t'_s) > b \). If \( f_{n_k}(t'_s) < a \) (respectively \( f_{n_k}(t'_s) > b \)), then we set \( s'' = s'' = s''_{s''} \) (respectively \( s'' = s''_{s''} \)) and we define \( t'_{s''} = t_{s''} \). Hence either \( t'_{s''} > b \) or \( t'_{s''} < a \). By the continuity of \( f_{n_k+1} \), as \( f_{n_k+1}(t''_{s''}) < a \) and \( f_{n_k+1}(t''_{s''}) > b \), we can choose \( B_{s^0} \) and \( B_{s^1} \) containing \( t'_{s''} \) and \( t'_{s''} \), respectively, such that \( B_{s^0} \subseteq [f_{n_k+1} < a] \), \( B_{s^1} \subseteq [f_{n_k+1} > b] \), \( \overline{B}_{s^0} \cup \overline{B}_{s^1} \subseteq B_s \) and \( \text{diam} B_s < \frac{1}{2^k} \) for \( i = 0, 1 \). The construction is completed.

For \( k \in \mathbb{N} \) we set \( n_k = n'_k \), \( A_k^\downarrow = [f_{n'_k} < a] \) and \( A_k^\uparrow = [f_{n'_k} > b] \) and \( C = \overline{T} \).

It is easily seen that \( (A_k^\downarrow \cap C, A_k^\uparrow \cap C)_k \) is an independent sequence of disjoint pairs.

**Remark 4.** The sequence \((f_{n'_k})_k\) resulting from Proposition 8 can be used to derive the following two well-known results (see [12]).

(1) The closure of \( \{f_{n_k}^\downarrow\}_k \) in \( R^X \) is homeomorphic to \( \beta \mathbb{N} \) and every accumulation point of \( \{f_{n_k}^\downarrow\}_k \) in \( R^X \) is not a Borel function.

(2) The sequence \((f_{n'_k})_k\) contains no pointwise convergent subsequence.

The properties of the tree \( T((f_n)_n, a, b) \) can also be used to derive the following well-known dichotomy (see [12]).

**Theorem 9.** Let \((f_n)_n\) be a pointwise bounded sequence of continuous real-valued functions on \( X \) and let \( \overline{\{f_n\}}^p \) be the closure of \( \{f_n\} \) in \( R^X \). Then one of the following mutually exclusive alternatives holds.

(i) \( \overline{\{f_n\}}^p \subseteq B_1(X) \).

(ii) \( \beta \mathbb{N} \) is homeomorphic to a subset of \( \overline{\{f_n\}}^p \).

**Proof.** Consider the following mutually exclusive cases.

**Case 1.** For every \( a < b \) the tree \( T((f_n)_n, a, b) \) is well-founded. Then, by Theorem 6, we get that the accumulation points of \( \{f_n\} \) in \( R^X \) belong to \( B_1(X) \).

**Case 2.** There exists \( a < b \) such that the tree \( T((f_n)_n, a, b) \) is not well-founded. Then, by Proposition 8 and the above remark, we get that \( \beta \mathbb{N} \) is homeomorphic to a subset of \( \overline{\{f_n\}}^p \).
3. The tree rank $o((f_n)_n)$ of a pointwise bounded sequence.

This section concerns the relation between $o((f_n)_n)$, defined below, and the separation rank $\alpha(f)$ when $f$ is an accumulation point of $\{f_n\}_n$. The exact relation is given in Theorem 11 below.

**Definition 10.** Let $f_n : X \to \mathbb{R}, n \in \mathbb{N}$, be a pointwise bounded sequence of functions. We define the tree rank $o((f_n)_n)$ of the sequence $(f_n)_n$ to be

$$o((f_n)_n) = \sup \left\{ o(T((f_n)_n, a, b)) : a < b \right\}$$

if the tree $T((f_n)_n, a, b)$ is well-founded for every $a < b$. Otherwise we set

$$o((f_n)_n) = (2^{\omega_0})^+.$$

The basic property of the tree rank is the following.

**Theorem 11.** Let $X$ be a Polish space. Then for every sequence $(f_n)_n$ in $\mathbb{R}^X$ and every $f \in B_1(X)$ the following hold.

1. If $f \in \text{Acc}((f_n)_n)$, then

$$o((f_n)_n) + 1 \geq \alpha(f).$$

2. If $f = \lim_n f_n$, then

$$o((f_n)_n) \leq \omega \cdot 2 \cdot \alpha(f).$$

The proof of Theorem 11 follows from a series of lemmas.

**Notation.** Let $f : X \to \mathbb{R}$ be a function, $Y \subseteq X$ closed and $a < b$. By $T(Y, f, a, b)$ we denote the tree $T(Y, (f_n)_n, a, b)$ where $f_n = f$ for every $n$. If $Y = X$, then we write $T(f, a, b)$ instead of $T(X, f, a, b)$. Note that if $f$ is a Baire-1 function, then the tree $T(f, a, b)$ is well-founded.

**Lemma 12.** Let $f : X \to \mathbb{R}$ be Baire-1, $a < b$, $T = T(f, a, b)$ and $\xi$ a countable ordinal. Let

$$S_\xi = T(X(f, a, b) \cap [f < a] \cup [f > b]), f, a, b).$$

Then $S_\xi \subseteq T^{(\xi)}$.

**Proof.** We proceed by induction on $\xi$. The case $\xi = 0$ is straightforward. Let $\zeta < \omega_1$ and suppose that the lemma is true for every $\xi < \zeta$. Assume that $\xi = \xi + 1$ and let $(u, T) \in S_{\xi+1}$. Then $T \subseteq X_{(f, a, b)}^{(\xi+1)} \cap ([f < a] \cup [f > b])$. Let $T = (t_s)_{s \in D_k}$ for some $k \in \omega$. Then for every $s$ with $|s| = k$ we have that $t_s \in X_{(f, a, b)}^{(\xi+1)}$ and either $f(t_s) < a$ or $f(t_s) > b$. If $f(t_s) < a$, then we choose $y \in X_{(f, a, b)}^{(\xi)}$ with $f(y) > b$ and $d(t_s, y) \leq \frac{1}{2^{k+1}}$ and we set $t_s^{-0} = t_s$ and $t_s^{-1} = y$. If $f(t_s) > b$, then with similar arguments, we choose $y' \in X_{(f, a, b)}^{(\xi)} \cap [f < a]$ with $d(t_s, y') \leq \frac{1}{2^{k+1}}$ and we set $t_s^{-0} = y'$ and $t_s^{-1} = t_s$. Let $T' = (t_s)_{s \in D_{k+1}}$ and $u' = u \cap n$ where $n > \max u$. Then
Let $T = (t, a, b)$ be such that $N \subseteq T^{(\xi)}$, which, under the notation of Lemma 12, gives that $S_{\xi} \subseteq S_{\xi}$ for every $\xi < \zeta$. Hence

$$S_{\xi} \subseteq \bigcap_{\xi < \zeta} S_{\xi} \subseteq \bigcap_{\xi < \zeta} T^{(\xi)} = T^{(\xi)}$$

and the lemma is proved. \qed

**Lemma 13.** Let $f : X \to \mathbb{R}$ Baire-1 and $a < b$. Then

$$\alpha(f, a, b) \leq o(T(f, a, b)) + 1.$$  

**Proof.** Let $\xi < \omega_1$ be such that $\alpha(f, a, b) \geq \xi + 2$. Then notice that

$$X_{(f,a,b)}(\xi) \cap ([f < a] \cup [f > b]) \neq \emptyset$$

which, under the notation of Lemma 12, gives that $S_{\xi} \neq \emptyset$. Hence, by Lemma 12, we get that $T^{(\xi)} \neq \emptyset$ and so $o(T(f, a, b)) \geq \xi + 1$. Therefore $\alpha(f, a, b) \leq o(T(f, a, b)) + 1$. \qed

**Lemma 14.** Let $\mathcal{K} = \{f_n\}_n$ be a quasi-Rosenthal compactum, $a < b$ and $f \in \text{Acc}(\mathcal{K})$. Then there exists a monotone map

$$\varphi : T(f, a, b) \to T((f_n)_n, a, b).$$

Consequently $o(T(f, a, b)) \leq o(T((f_n)_n, a, b))$.

**Proof.** Let $\mathcal{F}_2$ be the set of all finite subsets of $X \cap ([f < a] \cup [f > b])$ with cardinality greater or equal to 2. For every $F \in \mathcal{F}_2$, we set

$$N_F = \{n \in \mathbb{N} : \text{ for every } x \in F \text{ if } f(x) < a \text{ then } f_n(x) < a,$$

while if $f(x) > b$ then $f_n(x) > b\}.$$  

Note that $N_F$ is infinite. For every $F \in \mathcal{F}_2$ if $N_F = \{n_1 < n_2 < \ldots\}$ is the increasing enumeration of $N_F$, then we set $n_F = n_{|F|}$. Observe that if $F_1 \subsetneq F_2$, then $n_{F_1} < n_{F_2}$.

Now define the map $\varphi : T(f, a, b) \to T((f_n)_n, a, b)$ as follows. Let $(u, T) \in T(f, a, b)$. If $u = \emptyset$, then we set $\varphi((\emptyset, t_\emptyset)) = (\emptyset, t_\emptyset)$. If $u = (u_1, \ldots, u_k)$ and $T = (t_s)_{s \in D_k}$, then we set $\varphi((u, T)) = (u', T)$, where $u' = (n_1', \ldots, n_k')$ and $n_i' = n_{\{s \in D_k : |s| = i\}}$ for every $i = 1, \ldots, k$.

It is easy to see that $\varphi$ is a well-defined monotone map. \qed

Lemma 13 and Lemma 14 yield the following.

**Corollary 15.** Let $\mathcal{K} = \{f_n\}_n$ be a separable quasi-Rosenthal compactum. Then

$$\sup\{\alpha(f) : f \in \text{Acc}(\mathcal{K})\} \leq o((f_n)_n) + 1.$$
Suppose that \((f_n)_n\) is pointwise convergent to an \(f : X \to \mathbb{R}\). Let \(Y \subseteq X\) non-empty, \(a < b\), \(S = T(Y, (f_n)_n, a, b)\) and \(m \in \mathbb{N}\). Let \((u, T) \in S^{(\omega+m)}\) and \(x \in T\). Then either

(i) \(f(x) \leq a\) and there exists \(y \in Y\) with \(d(x, y) \leq \frac{1}{2|u|+m}\) and \(f(y) \geq b\), or

(ii) \(f(x) \geq b\) and there exists \(y \in Y\) with \(d(x, y) \leq \frac{1}{2|u|+m}\) and \(f(y) \leq a\).

\textbf{Proof.} By induction on \(m\). For \(m = 1\), let \((u, T) \in S^{(\omega+1)}\). Then there exists \((u', T') \in S^{(\omega)}\) such that \((u, T) < (u', T')\) and \(|u'| = |u| + 1\). Let \(x \in T\) and \(T' = (t'_s)_{s \in D_{|u|+1}}\). Then there exists \(s \in D_{|u|}\) such that \(x = t'_s\). We may assume that \(t'_{s-0} = t'_s = x\) (the case \(t'_{s-1} = t'_s = x\) is similarly treated). We set \(t'_{s-1} = y\).

Then \(d(x, y) \leq \frac{1}{2|u|+m+1}\). Since \((u', T') \in S^{(\omega)}\) we have that \((u', T') \in S^{(k)}\) for every \(k \in \mathbb{N}\). Hence, for every \(k \in \mathbb{N}\) we can choose \((u_k, T_k) \in S\) with \((u', T') < (u_k, T_k)\) and \(|u_k| = |u'| + k\). Set \(L = \bigcup_{k \in \mathbb{N}} u_k \setminus u'\). Clearly \(L \in [\mathbb{N}]\). Moreover, observe that for every \(l \in L\) we have \(f_l(x) < a\) and \(f_l(y) > b\). As \(f = \lim_{n \in \mathbb{N}} f_n\), we get that \(f(x) \leq a\) and \(f(y) \geq b\). The proof for the case \(m = 1\) is completed.

Suppose that the lemma is true for some \(m \in \mathbb{N}\). Let \((u, T) \in S^{(\omega+m+1)}\) and \(x \in T\). Pick \((u', T') \in S^{(\omega+m)}\) with \((u, T) < (u', T')\) and \(|u'\| = |u| + 1\). Setting \(T'' = (t''_s)_{s \in D_{|u'|+1}}\), we have that \(x \in T''\) and \(x = t''_s\) for some \(s \in D_{|u'|}\). By our inductive assumption, there exists \(y \in Y\) such that

\[
\begin{align*}
d(x, y) &\leq \frac{1}{2|u'|+m+1} = \frac{1}{2|u|+(m+1)} \\
\end{align*}
\]

and either \(f(x) \leq a\) and \(f(y) \geq b\) or \(f(x) \geq b\) and \(f(y) \leq a\). The proof of the lemma is completed. \(\square\)

\textbf{Lemma 17.} Suppose that \((f_n)_n\) is pointwise convergent to an \(f : X \to \mathbb{R}\). Let \(Y \subseteq X\) closed, \(a < b\) and \(S = T(Y, (f_n)_n, a, b)\). Then for every \(1 \leq \xi < \omega_1\) and every \(0 < \varepsilon < \frac{b-a}{2}\) we have

\[
S^{(\omega-2,\xi)} \subseteq T(Y^{(\xi)}_{(f,a+\varepsilon,b-\varepsilon)}, (f_n)_n, a, b).
\]

\textbf{Proof.} First we deal with the case \(\xi = 1\). Let \((u, T) \in S^{(\omega-2)}\) and \(x \in T\). Let \(m \in \mathbb{N}\). As \((u, T) \in S^{(\omega-2)} \subseteq S^{(\omega+m)}\), by Lemma 16, either

(i) \(f(x) \leq a < a + \varepsilon\) and there exists \(y \in Y\) with \(d(x, y) \leq \frac{1}{2m+1}\) and \(f(y) \geq b - \varepsilon\), or

(ii) \(f(x) \geq b > b - \varepsilon\) and there exists \(y \in Y\) with \(d(x, y) \leq \frac{1}{2m+1}\) and \(f(y) \leq a < a + \varepsilon\).

Since \(m\) is arbitrary, we have

\[
x \in Y \cap [f < a + \varepsilon] \cap Y \cap [f > b - \varepsilon] = Y^{(1)}_{(f,a+\varepsilon,b-\varepsilon)}.
\]

This shows that \(T \subseteq Y^{(1)}_{(f,a+\varepsilon,b-\varepsilon)}\) and in particular that

\[
S^{(\omega-2)} \subseteq T(Y^{(1)}_{(f,a+\varepsilon,b-\varepsilon)}, (f_n)_n, a, b).
\]
We proceed by induction on $\xi$. Let $\zeta < \omega_1$ and suppose that the lemma is true for every $\xi < \zeta$. If $\zeta = \xi + 1$, then by our inductive assumption we get
\[
S^{(\omega^2(\xi+1))} = (S^{(\omega^2 \xi)})^{(\omega^2)} \subseteq \left(T(Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)}, (f_n)_n, a, b)\right)^{(\omega^2)}.
\]
We set $Z = Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)}$. By the first case of our induction
\[
\left(T(Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)}, (f_n)_n, a, b)\right)^{(\omega^2)} \subseteq T(Z^{(1)}_{(f,a+\varepsilon, b-\varepsilon)}, (f_n)_n, a, b).
\]
By the above inclusions we conclude that
\[
S^{(\omega^2 \zeta)} = \bigcap_{\xi < \zeta} S^{(\omega^2 \xi)} \subseteq \bigcap_{\xi < \zeta} T(Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)}, (f_n)_n, a, b).
\]
Now suppose $\zeta$ is a limit ordinal. Then, by our inductive assumption, we have
\[
S^{(\omega^2 \zeta)} = \bigcap_{\xi < \zeta} S^{(\omega^2 \xi)} \subseteq \bigcap_{\xi < \zeta} T(Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)}, (f_n)_n, a, b).
\]
Note that
\[
\bigcap_{\xi < \zeta} T(Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)}, (f_n)_n, a, b) \subseteq T(Y^{(\zeta)}_{(f,a+\varepsilon, b-\varepsilon)}, (f_n)_n, a, b).
\]
Indeed, if $T \subseteq Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)}$ for every $\xi < \zeta$, then
\[
T \subseteq \bigcap_{\xi < \zeta} Y^{(\xi)}_{(f,a+\varepsilon, b-\varepsilon)} = Y^{(\zeta)}_{(f,a+\varepsilon, b-\varepsilon)}.
\]
Hence the proof is completed. $\square$

Lemma 17 yields the following.

**Corollary 18.** Let $(f_n)_n$ be a sequence of functions pointwise convergent to a Baire-1 function $f$. Then
\[
o((f_n)_n) \leq \omega \cdot 2 \cdot \alpha(f).
\]
By the above we conclude the following.

**Proof of Theorem 11.** Follows immediately by Corollaries 18 and 15. $\square$

We will give some consequences of Theorem 11. First let us make the following definition.

**Definition 19.** For every $\xi < \omega_1$ let
\[
B^{(\xi)}(X) = \{ f \in B^1(X) : \alpha(f, a, b) < \omega^\xi \text{ for every } a < b \}.
\]

Note that in the case of compact metrizable spaces, the above defined class coincides with the class of small Baire class $\xi$, defined by A. S. Kechris and A. Louveau in [7].

**Corollary 20.** Let $(f_n)_n$ be a sequence of real-valued functions on $X$ pointwise convergent to a Baire-1 function $f$. Then the following hold.
If \( \alpha(f) \) is a limit ordinal, then \( \alpha(f) \leq o((f_n)_n) \leq \omega \cdot 2 \cdot \alpha(f) \).

2. If \( \alpha(f) < \omega^{n+1} \) (respectively \( \alpha(f) \leq \omega^{n+1} \)) with \( n \in \omega \), then \( o((f_n)_n) < \omega^{n+2} \) (respectively \( o((f_n)_n) \leq \omega^{n+2} \)).

3. If \( \alpha(f) < \omega^\xi \) (respectively \( \alpha(f) \leq \omega^\xi \)) with \( \omega \leq \xi < \omega_1 \), then \( o((f_n)_n) < \omega^\xi \) (respectively \( o((f_n)_n) \leq \omega^\xi \)).

4. If \( o((f_n)_n) < \omega^\xi \), then \( \alpha(f) < \omega^\xi \) for every countable ordinal \( \xi \).

5. For every \( \omega \leq \xi < \omega_1 \), we have \( f \in B'_1(X) \) if and only if \( o(T((f_n)_n, a, b)) < \omega^\xi \) for every \( a < b \) and every sequence \((f_n)_n\) pointwise convergent to \( f \).

4. Borel separable quasi-Rosenthal compacta

The last section is devoted to the proof of the main result of the paper and to an example which shows that our results are the best possible. We have also included some open questions. We start with the following definition.

Definition 21. A quasi-Rosenthal compactum \( K \) will be called Borel separable if it has a countable dense subset of Borel functions.

In the following proposition, the well-known equivalence of (ii) and (iii) (see [1]) is connected to well-founded tree structures.

Proposition 22. Let \((f_n)_n\) be a pointwise bounded sequence of Borel functions. Then the following are equivalent.

(i) There exists a finer Polish topology \( \tau' \) on \( X \) with \( B(X) = B(X') \) where \( X' = (X, \tau) \), such that for every \( a < b \) the tree \( T(X', (f_n)_n, a, b) \) is well-founded.

(ii) For every \( L \in [\mathbb{N}] \) there exists \( L' \in [L] \) such that \((f_n)_{n \in L'}\) is pointwise convergent.

(iii) The closure of \((f_n)_n\) in \( \mathbb{R}^X \) is a subset of \( B(X) \).

Proof. Let \( \tau' \) be a finer Polish topology on \( X \) with \( B(X) = B(X') \), where \( X' = (X, \tau') \), and such that \( f_n \) is \( \tau' \)-continuous for every \( n \) (see [6]). Now the equivalence of (i) for this \( \tau' \) and (ii) is precisely the equivalence of (2) and (3) in Theorem 6. Similarly the equivalence of (i) (again for this \( \tau' \)) and (iii) is the equivalence of (1) and (2) in Theorem 6.

\( \square \)

Remark 5. Related to the above proposition, the following question is open for us. If \((f_n)_n\) is a pointwise bounded sequence of functions such that \( \text{Acc}((f_n)_n) \subseteq B(X) \), does this follow that \((f_n)_n\) is homeomorphic to a quasi-Rosenthal compactum? Let us observe that the stronger question of the existence of a finer Polish topology \( \tau' \) on \( X \) such that \( \text{Acc}((f_n)_n) \) becomes a quasi-Rosenthal compactum has a negative answer. Indeed, consider the variant of example 1 where \( \{ \Delta_\sigma : \sigma \in \mathcal{C} \} = B(\mathcal{C}) \). Then as in example 1 every \( f \in \text{Acc}((f_n)_{n \in D}) \) is a Borel function. As it is known every finer Polish topology on \( X \) has the same Borel sets. Hence assuming that for a finer Polish topology \( \tau' \) each \( \chi_{\Delta_\sigma} \) is a Baire-1 function, we conclude that each \( \Delta_\sigma \)
We enlarge the original topology of every Proof.

Assume that the tree $\subseteq C$ that $K$ compactum of $\tau$ for

We claim that Lemma 24. Note that by taking complements we get an example of a separable quasi-Rosenthal example 2. Theorem 23. Clearly Theorem 23 answers in the affirmative J. Bourgain’s question stated in [3]. We postpone its proof in order to give an example which shows that our theorem is sharp.

Example 2. Let $X = C \times C$ where $C = \{0, 1\}^\mathbb{N}$ is the Cantor set. Let $A \subseteq C \times C$ be a $C$-universal set for the class of $F_\sigma$ subsets of $C$ (see [6]). That is $A$ is $F_\sigma$ in $C \times C$, for every $\sigma \in C$ the section $A_\sigma = \{x \in C : (\sigma, x) \in A\}$ of $A$ is $F_\sigma$ and for every $F_\sigma$ subset $F$ of $C$ there exists $\sigma \in C$ such that $F = A_\sigma$. Let $\Pi = \{\sigma \in C : A_\sigma \neq \emptyset\}$. Then $\Pi$ is co-analytic subset of $C$. To see this observe that $\Pi = \{\sigma \in C : (X \setminus A)_\sigma \in F_\sigma\}$. As $X \setminus A$ is Borel in $C \times C$, by a classical theorem of Hurewicz (see [6], page 297), we get that $\Pi$ is co-analytic. For every $s \in D$, let as usual $C_s = \{\sigma \in C : s \subseteq \sigma\}$ (clearly $C_s$ is open in $C$). Now let $A_s = A \cap (\Pi \cap C_s) \times C_s$. Then $A_s$ is co-analytic in $X$. Define $f_s = \chi_{A_s}$ and let $K = \{\text{acc}\}_s \subseteq D$. As in example 1, it is easily verified that $K$ is a separable quasi-Rosenthal compactum. Moreover note that for every $\Delta \subseteq C$ which is $F_\sigma$ and $G_\delta$, there exists $\sigma \in \Pi$ such that $A_\sigma = \Delta$. It follows that for every such $\Delta \subseteq C$ there exists $\sigma \in C$ such that setting $\Delta_\sigma = \{\sigma\} \times \Delta$ we have that $\chi_{\Delta_\sigma} \in K$. As for every countable ordinal $\xi$ there exists $\Delta \subseteq C$ such that $\chi_\Delta$ is Baire-1 (that is $\Delta$ is $F_\sigma$ and $G_\delta$) and $\alpha(\chi_\Delta) \geq \xi$, we immediately get that

$$\sup\{\alpha(f) : f \in \text{Acc}(K)\} = \omega_1.$$ 

Note that by taking complements we get an example of a separable quasi-Rosenthal compactum having a dense subset consisting of characteristic functions of analytic sets and for which Theorem 23 is not valid.

Lemma 24. Let $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of Borel functions and $a < b$. Assume that the tree $T = T((f_n)_n, a, b)$ is well-founded. Then $\alpha(T) < \omega_1$.

Proof. We enlarge the original topology of $X$ to a Polish topology $\tau'$ in order to make the sequences of sets $\{|f_n < a|\}_n$ and $\{|f_n > b|\}_n \tau'$-clopen (see [6]). Then the tree $T$ is a subset of the Polish space

$$Y = \bigoplus_{k \in \omega} [\mathbb{N}]^k \times (X, \tau')^D.$$ 

We claim that $T$ is closed in $Y$. Indeed, let $\{(u_i, T_i)\}_i$ be a sequence such that $u_i \to u$, $T_i \to T$ and $(u_i, T_i) \in T$ for every $i$. There exists an $i_0$ such that $u = u_i$ for every $i \geq i_0$. Let $k = |u|$. Then $T_i \in (X, \tau')^D_k$ for every $i \geq i_0$ and so $T \in (X, \tau')^D_k$. 

The main theorem of this section is the following.

Theorem 23. Let $X$ be a Polish space and $K$ be a Borel separable quasi-Rosenthal compactum of $\mathbb{R}^X$. Then

$$\sup\{\alpha(f) : f \in K \cap B_1(X)\} < \omega_1.$$ 

Clearly Theorem 23 answers in the affirmative J. Bourgain’s question stated in [3]. We postpone its proof in order to give an example which shows that our theorem is sharp.
For every $i \geq i_0$ let $T_i = (t_{i,j})_{j \in D_k}$ and $T = (t_s)_{s \in D_k}$ and note that $t_{s,j} \rightarrow t_s$ in $\tau'$ for every $s \in D_k$. It is clear that if $k = 0$, then $(u, T) \in \mathcal{T}$.

So suppose that $k \geq 1$ and let $u = (n_1, \ldots, n_k)$. We will verify that $(u, T)$ satisfies conditions (C1) and (C2) of the definition of $\mathcal{T}$. For (C1), as for every $i \geq i_0$ either $t_{i,j} = t_{i,j}'$ or $t_{i,j}' = t_{i,j}$, there exists $I \in [\mathbb{N}]$ such that either $t_{0,j} = t_{i,j}'$ for all $i \in I$ or $t_{1,j}' = t_{i,j}$ for all $i \in I$. Hence either $t_{0,j} = t_{i,j}'$ or $t_1 = t_{i,j}$. So (C1) is clear.

To verify (C2), let $s \in D_k$ with $|s| < k$. As $\tau'$ is finer than the original topology we have that $t_{s,j} \rightarrow t_s$ in the original topology too. Hence $d(t_{s,0}, t_{s,1}) \leq \frac{1}{2^{|s|+1}}$ and so (C2)(i) is clear. Moreover, as the sets $[f_{n_i} < a]$ and $[f_{n_i} > b]$ are $\tau'$-closed we immediately get that $f_{n_i+1}(t_{s,0}) < a$ and $f_{n_i+1}(t_{s,1}) > b$. So (C2)(ii) is also satisfied. Finally, to show (C2)(iii), suppose that $s \notin \emptyset$ and that $f_{n_i}(t_s) < a$. Then, as $t_{s,j} \rightarrow t_s$ and $[f_{n_i}] < a$ is $\tau'$-open, we get that there exists $i_s \geq i_0$ such that $f_{n_{i_s}}(t_{s,j}) < a$ for all $i \geq i_s$. Hence $t_{i,j}' = t_{i,j}$ for all $i \geq i_s$ which gives that $t_{s,0} = t_s$. The case $f_{n_i}(t_s) > b$ is similarly treated. This completes the proof that $\mathcal{T}$ is closed in $Y$.

Now define the relation $\prec$ on $Y$ by

$$(u', T') \prec (u, T) \text{ if } (u, T), (u', T') \in \mathcal{T} \text{ and } (u, T) < (u', T').$$

Clearly $\prec$ is a strict well-founded relation on $Y$ and $o(T) = \rho(\prec)$. We will show that $\prec$, as a subset of $Y \times Y$, is closed. Indeed, let $((u_i', T'_i)_i, (u_i, T_i), (u, T)) \in Y$. Then define the relation $\triangleleft$ on $\mathcal{T}$ by

$$((u_i, T_i)_i, (u_i', T'_i)_i) \triangleleft (u, T) \text{ if } (u_i, T_i) \triangleleft (u_i', T'_i) \text{ for all } i.$$ 

Clearly $\triangleleft$ is a well-founded relation on $\mathcal{T}$. Hence, Kunen-Martin theorem yields that $o(T) = \rho(\triangleleft) < \omega_1$. $\square$

**Corollary 25.** Let $K = \overline{\bigcap_{n=1}^{\omega} T_n}$ be a Borel separable quasi-Rosenthal compactum. Then $o((f_n)_n) < \omega_1$.

**Proof.** Let $a < b$ and consider the tree $\mathcal{T}((f_n)_n, a, b)$. By Theorem 6, the tree $\mathcal{T}((f_n)_n, a, b)$ is well-founded. By Lemma 24, we get that $o(\mathcal{T}((f_n)_n, a, b)) < \omega_1$. Now note that

$$\sup \left\{ o(\mathcal{T}((f_n)_n, a, b)) : a < b \right\} = \sup \left\{ o(\mathcal{T}((f_n)_n, a, b)) : a < b \text{ rationals} \right\}. $$

Hence $o((f_n)_n) < \omega_1$ as desired. $\square$

**Proof of Theorem 23.** It follows from Corollaries 15 and 25 that

$$\sup \{ o(f) : f \in \text{Acc}(\mathcal{K}) \} < \omega_1.$$

As the isolated points of $K$ in $\mathcal{B}_1(X)$ are at most countable, the result follows. $\square$

**Remark 6.** We conclude the paper with some open problems.

(1) Even for separable Rosenthal compacta $\mathcal{K}$, it is unclear to us whether there exists an equivalence between the quantities $o((f_n)_n)$ and $\sup \{ o(f) : f \in \mathcal{K} \}$. However, the proper setting of this problem seems to be for the class
of Borel separable quasi-Rosenthal compacta. We notice that in the case where the sequence \((f_n)_n\) has finitely many accumulation points, then easy modifications of the proof of Theorem 11(2) yield a positive answer.

(2) For an arbitrary sequence \((f_n)_n\) of functions with \(T((f_n)_n, a, b)\) well-founded for every \(a < b\), we do not know if \(2^{\omega_1}\) is the best upper bound for \(\sigma((f_n)_n)\).

(3) As we have shown in Proposition 7, every quasi-Rosenthal compactum \(K\) has countable tightness. It remains open whether every accumulation point of a subset \(L\) of \(K\) is the limit of a sequence in \(L\). For quasi-Rosenthal compacta homeomorphic to a Rosenthal compactum this is a consequence of the Bourgain-Fremlin-Talagrand theorem [4].

References


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