The general approach for the pricing of rainbow (or colored) options with fixed transaction costs is developed from the game theoretic point of view. The evolution of the underlying common stocks is considered in discrete time. The main result consists in the explicit calculation of the hedge price for a variety of the rainbow options including option delivering the best of $J$ risky assets and cash, calls on the maximum of $J$ risky assets and the multiple-strike options. The results obtained can be also used in the framework of real options.

**Keywords** Coloured (or rainbow) options; explicit formulas for hedge; transaction costs; interval model; submodular functions.

1. Model

One can distinguish two main types of options: European and American.

In case of European call options buyer is not engaged in financial activity and waits for the maturity date $n$ of the options. The contradicting interests of the investor and the buyer give optimization problem of minmax nature. In this paper we will discuss mostly European options (See subsection 3.4 bellow).

In our model, financial market is dealing with several securities: the risk-free bonds (or bank account) and $J$ common stocks, $J = 1, 2, \ldots$. In this paper we are concerned with the case $J > 1$, which means that we are discussing the so-called rainbow options. For a given $J$ these options are called $J$-color rainbow options. The prices of the units of these securities, $B_k$ and $S_{ik}^k$, $i \in \{1, 2, \ldots, J\}$ respectively, change in discrete moments of time $k = 1, 2, \ldots$ according to the recurrent equations $B_{k+1} = \rho B_k$, where the $\rho \geq 1$ is an interest rate which remains unchanged over time, and $S_{k+1}^{ik} = \xi_{k+1}^{ik} S_k^i$, where $\xi_{k+1}^{ik}, i \in \{1, 2, \ldots, J\}$ are unknown sequences taking values in some fixed intervals $M_i = [d_i, u_i] \subset \mathbb{R}$. It is worth noting difference
between the model considered and a colored version of the classical CRR model. In the latter a sequence \( \xi_k \) take values in the two point set \( \{d_i, u_i\} \), and it is supposed to be random with some given distribution. In our model any value in the interval \([d_i, u_i]\) is allowed and no probabilistic assumption is made.

The type of an option is specified by a given premium function \( f \) of \( J \) variables. The following are the standard examples Rubinstein (1995):

- option delivering the best of \( J \) risky assets and cash
  \[
  f(S^1, S^2, \ldots, S^J) = \max(S^1, S^2, \ldots, S^J, K),
  \]
- calls on the maximum of \( J \) risky assets
  \[
  f(S^1, S^2, \ldots, S^J) = \max(0, \max(S^1, S^2, \ldots, S^J) - K),
  \]
- multiple-strike options
  \[
  f(S^1, S^2, \ldots, S^J) = \max(0, S^1 - K_1, S^2 - K_2, \ldots, S^J - K_J),
  \]
- portfolio options
  \[
  f(S^1, S^2, \ldots, S^J) = \max(0, n_1S^1 + n_2S^2 + \cdots + n_JS^J - K),
  \]
- and spread options
  \[
  f(S^1, S^2) = \max(0, (S^2 - S^1) - K).
  \]

Here, the \( S^1, S^2, \ldots, S^J \) represent the expiration date values of the underlying assets, and \( K, K_1, \ldots, K_J \) represent the strike prices. In Rubinstein (1995), \( J = 2 \), but we shall be especially interested in the new features of the model arising under the assumption \( J > 2 \). The corresponding results for \( J = 2 \) were obtained in Kolokoltsov (1998). Let us stress that the assumptions on \( f \) that we shall make later on will include the first three options listed above. For the discussion of the last option we refer to Margrabe (1978) and Kulatilaka and Trigeorgis (1994).

Let \( X_k \) denote the capital of the investor at the time \( k = 1, 2, \ldots \). The investor is supposed to control the growth of his capital in the following way. At time \( n - 1 \) the investor determines his portfolio by choosing the numbers \( \gamma_{ni} \) of common stocks to be held. Then one can write

\[
X_{n-1} = \sum_{i=1}^{J} \gamma_{ni}^i S_{n-1}^i + \left( X_{n-1} - \sum_{i=1}^{J} \gamma_{ni}^i S_{n-1}^i \right),
\]

where the sum in bracket corresponds to the part of his capital laid on the bank account. The control parameter \( \gamma_n \) can take all real values, i.e. short selling and borrowing are allowed. The value \( \xi_n \) becomes known in the moment \( n \) and thus the capital at the moment \( n \) becomes

\[
X_n = \sum_{i=1}^{J} \gamma_{ni}^i \xi_n^i S_{n-1}^i + \rho \left( X_{n-1} - \sum_{i=1}^{J} \gamma_{ni}^i S_{n-1}^i \right),
\]

if no transaction costs are taken into consideration.
If \( n \) is the maturity date and hence the investor is obliged to pay the premium \( f \) to the buyer, the income of the investor equals

\[
X_n - f(S^1_n, S^2_n, \ldots, S^J_n).
\]

There are several types of transaction costs. The proportional transaction costs are dealt with in Grannan and Swidle (1996). They have the form

\[
TC = \sum_{i=1}^{J} \kappa |\gamma^i_n - \gamma^i_{n-1}| S^i_n,
\]

where the constant \( \kappa \) is between 0 and 1. In other models (see Olsder (2000)) the transaction costs are considered to be the function of the numbers of stock \((\gamma^i_n, \gamma^i_{n-1})\) only.

We shall consider the so-called fixed transaction costs, considered, for example, in Morton and Pliska (1995). It means that whenever a trade is made, the investor pays a transaction cost equal to a fixed fraction \( 1 - \beta \) of his entire portfolio value, where \( \beta \in (0, 1) \) is some constant.

So, the value of the portfolio \( \sum_{i=1}^{J} \gamma^i_n \xi^i_n S^i_{n-1} \) becomes \( \beta \sum_{i=1}^{J} \gamma^i_n \xi^i_n S^i_{n-1} \) when the transaction costs are met.

The capital of the investor in the time \( n \) becomes:

\[
X_n = \beta \sum_{i=1}^{J} \gamma^i_n \xi^i_n S^i_{n-1} + \rho \left( X_{n-1} - \sum_{i=1}^{J} \gamma^i_n S^i_{n-1} \right)
\]

To obtain the optimal strategies for an investor we will use the game theoretic approach. This game is played on the space of \((J + 1)\) nonnegative numbers \( X, S^1, \ldots, S^J \) with the final income specified by the function

\[
G(X, S^1, \ldots, S^J) = X - f(S^1, \ldots, S^J)
\]

The strategy of the investor is by definition any sequences of vectors \((\gamma_1, \ldots, \gamma_n)\) (with \( \gamma_j = (\gamma^1_j, \ldots, \gamma^J_j) \)) such that each \( \gamma^i_j \) could be chosen using the whole previous information: the sequences \( X_0, \ldots, X_{j-1} \) and \( S^i_0, \ldots, S^i_{j-1} \) (for every stock \( i = 1, 2, \ldots, J \)). It is supposed that the investor, selling an option by the price \( C = X_0 \) should organize the evolution of his capital in a way that would allow him to pay to the buyer some premium \( f(S^1_n, S^2_n, \ldots, S^J_n) \) depending on the prices \( S^1_n, S^2_n, \ldots, S^J_n \), in the prescribed moment \( n \) (maturity day).

The strategy \( \gamma^1_n, \ldots, \gamma^i_n, i = 1, \ldots, J \) of the investor is called a hedge, if for any sequence \((\xi_1, \ldots, \xi_n)\) (with \( \xi_j = (\xi^1_j, \ldots, \xi^J_j) \)) the investor is able to meet his obligations, i.e.

\[
G(X_n, S^1_n, \ldots, S^J_n) \geq 0.
\]

The minimal value of the capital \( X_0 \) for which the hedge exists is called the hedging price \( C_h \) of an option.

As we can not control parameters \( \xi^i_n \) (for \( i = 1, 2, \ldots, J \)), we will assume the worst possibility for the investor, i.e. that the nature minimises his income by means of \( \xi^i_n \) and the investor is looking for the maximum over \( \gamma^i_n \).
If the income is specified by a function \( G \) then the guaranteed income of the investor in one step of the game with the initial conditions \( X, S^1, \ldots, S^J \) is

\[
BG(X, S^1, \ldots, S^J) = \max_{\gamma} \min_{\xi} G \left( \rho X + \beta \sum_{i=1}^{J} \gamma^i \xi^i S^i - \rho \sum_{i=1}^{J} \gamma^i S^i, \xi^1 S^1, \ldots, \xi^J S^J \right)
\]  

(8)

From the theory of dynamic multistep games (see Bellman (1957)) it follows that the guaranteed income of the investor in the \( n \) step game with the initial conditions \( X_0, S^1_0, \ldots, S^J_0 \) is given by the formula

\[
B^n G(X_0, S^1_0, \ldots, S^J_0).
\]  

(9)

In our model \( G \) is given by (7). As the class of function \( G \) of the form

\[
\rho^k X - g(S^1, \ldots, S^J)
\]

is clearly invariant under the action of \( B \) it follows that in our model the guaranteed income in the \( n \) step game equals

\[
\rho^n X_0 - (B^n f)(S^1_0, \ldots, S^J_0),
\]

where the reduced Bellman operator is defined as:

\[
(B f)(S^1, \ldots, S^J) = \min_{\gamma} \max_{\xi} \left[ f(\xi^1 S^1, \xi^2 S^2, \ldots, \xi^J S^J) - \sum_{i=1}^{J} \gamma^i S^i(\beta \xi^i - \rho) \right].
\]  

(10)

The minimal value of \( X_0 \) for which this income is not negative (and which by definition is the hedge price \( C_h \)) is therefore given by

\[
C^n_h = \frac{1}{\rho^n} (B^n f)(S^1_0, \ldots, S^J_0).
\]  

(11)

This formula is the basic conclusion of the application of the game theory to the option pricing and can be used as a starting point for the construction of various schemes of numerical calculations.

Our main result in this paper is analytical. It consists in a rather mysterious possibility to calculate the operator (10) explicitly on the class of submodular convex functions. For other achievements in the interval model for options we refer to the recent review, Bernard (2005).

2. Results

We will denote vectors by bold letters, i.e. \( z = (z_1, z_2, \ldots, z_J) \), and to simplify writing we shall denote \( \xi z = (\xi^1 z_1, \ldots, \xi^J z_J) \) (which is not a usual scalar product).

The Bellman operator (10) could be written like

\[
(B f)(z) = \min_{\gamma} (B f)(z, \gamma),
\]
where

\[(Bf)(z, \gamma) = \max_{i} f(\xi'z_1, \ldots, \xi'z_J) - \sum_{i=1}^{J} \gamma_i z_i(\beta \xi_i - \rho). \tag{12}\]

For a set \(I \subset \{1, 2, \ldots, J\}\) let us denote by \(f_I\) the value of \(f(\xi'z_1, \xi'z_2, \ldots, \xi'z_J)\) with \(\xi_i = d_i\) for \(i \in I\) and \(\xi_i = u_i\) for \(i \notin I\).

Also, we shall write

\[B_I(z, \gamma) = f_I(z) - \sum_{i \in I} \gamma_i z_i(\beta d_i - \rho) - \sum_{i \notin I} \gamma_i z_i(\beta u_i - \rho).\]

For example, \(f_{\{1,3\}}(z) = f(d_1z_1, u_2z_2, d_3z_3)\) and \(B_{\{1,3\}}(z, \gamma) = f_{\{1,3\}}(z) - \gamma_1 z_1(\beta d_1 - \rho) - \gamma_2 z_2(\beta u_2 - \rho) - \gamma_3 z_3(\beta d_3 - \rho)\).

Using this notation, operator (12) for a convex function \(f\) can be written simply as

\[Bf(z, \gamma) = \max_{I \subset \{1, 2, \ldots, J\}} B_I(z, \gamma). \tag{13}\]

**Definition 1.** Let \(f : R_+^2 \to R^+\). The function \(f\) is called submodular if it satisfies the inequality:

\[f(z_1, \omega_2) + f(\omega_1, z_2) - f(z_1, z_2) - f(\omega_1, \omega_2) \geq 0 \tag{14}\]

for every \(z_1 < \omega_1\) and \(z_2 < \omega_2\).

Function \(f : R_+^2 \to R^+\) is submodular if it is submodular with respect to every two variables.

The set of all submodular convex functions we will denote by \(NS\).

Inequality (14) clearly implies

\[f_{\{1\}}(z) + f_{\{2\}}(z) - f_{\{1,2\}}(z) - f_{\emptyset}(z) \geq 0, \tag{15}\]

which will be frequently used in the paper.


**Remark 1.** If function \(f \in C^2(R_+^2)\), then the function \(f\) is submodular iff \(\frac{\partial^2 f}{\partial z_i \partial z_j} \leq 0\) for all \(i \neq j\).

To simplify our formulas let us introduce special notation for the increments of functions. Namely, we shall denote by \(\Delta_j f_{I \cup \{j\}}(z)\) the difference

\[f_I(z) - f_{\{j\} \cup I}(z)\]

for any \(I \subset \{1, 2, \ldots, J\}\) and \(j \notin I\).

Let us denote by \(r\) the adjusted interest rate:

\[r = \frac{p}{\beta}.\]

We will suppose that \(0 < d_i < r < u_i\) for all \(i \in \{1, 2, \ldots, J\}\). (If this condition is not satisfied then the solution of our problem is trivial, because if \(r \geq u_i\), say, then the corresponding bond \(i\) should not be taken into account.)
Now, let us introduce the following coefficients:

\[ \alpha_I = 1 - \sum_{j \in I} \frac{u_j - r}{u_j - d_j}, \text{ where } I \subset \{1, 2, \ldots, J\}. \]

For the case \( I = 3 \) the following coefficients will play the major role in our analysis:

\[
\begin{align*}
\alpha_{123} &= \left(1 - \frac{u_1 - r}{u_1 - d_1} - \frac{u_2 - r}{u_2 - d_2} - \frac{u_3 - r}{u_3 - d_3}\right), \\
\alpha_{12} &= \left(1 - \frac{u_1 - r}{u_1 - d_1} - \frac{u_2 - r}{u_2 - d_2}\right), \\
\alpha_{13} &= \left(1 - \frac{u_1 - r}{u_1 - d_1} - \frac{u_3 - r}{u_3 - d_3}\right), \\
\alpha_{23} &= \left(1 - \frac{u_2 - r}{u_2 - d_2} - \frac{u_3 - r}{u_3 - d_3}\right). 
\end{align*}
\]

In this section we will formulate our main result that concerns the case when \( J = 3 \). It is divided in two theorems. The form of \((Bf)(z)\) turns out to depend on the signs of coefficients \(I(16)\).

**Theorem 1.** Let \( f \in NS \).

(i) If \( \alpha_{123} \geq 0 \), then

\[
(Bf)(z) = \frac{1}{r} \left(\alpha_{123} f_0(z) + \frac{u_1 - r}{u_1 - d_1} f_{(1)}(z)\right) \\
+ \frac{u_2 - r}{u_2 - d_2} f_{(2)}(z) + \frac{u_3 - r}{u_3 - d_3} f_{(3)}(z) \right),
\]

(ii) If \( \alpha_{123} \leq -1 \), then

\[
(Bf)(z) = \frac{1}{r} \left(- (\alpha_{123} + 1) f_{(1,2,3)}(z) - \frac{d_1 - r}{u_1 - d_1} f_{(1,2,3)}(z) \right) \\
- \frac{d_2 - r}{u_2 - d_2} f_{(1,3)}(z) - \frac{d_3 - r}{u_3 - d_3} f_{(1,3)}(z) \right). 
\]

**Theorem 2.** Let \( f \in NS \) and suppose that \( 0 \geq \alpha_{123} \geq -1 \).

(i) If \( \alpha_{12} \geq 0 \), \( \alpha_{13} \geq 0 \) and \( \alpha_{23} \geq 0 \), then

\[
(Bf)(z) = \frac{1}{r} \max \left\{ \left(- \alpha_{123}\right) f_{(1,2)}(z) + \alpha_{13} f_{(2)}(z) + \alpha_{23} f_{(1)}(z) + \frac{u_3 - r}{u_3 - d_3} f_{(3)}(z) \right) \right), \\
\left(- \alpha_{123}\right) f_{(1,3)}(z) + \alpha_{12} f_{(3)}(z) + \alpha_{23} f_{(1)}(z) + \frac{u_2 - r}{u_2 - d_2} f_{(2)}(z) \right), \\
\left(- \alpha_{123}\right) f_{(2,3)}(z) + \alpha_{12} f_{(3)}(z) + \alpha_{13} f_{(2)}(z) + \frac{u_1 - r}{u_1 - d_1} f_{(1)}(z) \right) \right). 
\]
(ii) If \(\alpha_{ij} \leq 0, \alpha_{jk} \geq 0\) and \(\alpha_{ik} \geq 0\), where \(\{i, j, k\}\) is an arbitrary permutation of the set \(\{1, 2, 3\}\), then

\[
\mathcal{B} f(z) = \frac{1}{r} \max \left\{ \begin{array}{l}
(-\alpha_{ijk}) f_{i,j}(z) + \alpha_{ik} f_{i,k}(z) + \alpha_{jk} f_{i,j}(z) \\
+ \frac{u_k - r}{u_k - d_k} f_{i,k}(z) \\
- \frac{d_i - r}{u_i - d_i} f_{j,k}(z) \\
\alpha_{ik} f_{j,k}(z) + (-\alpha_{ij}) f_{i,j}(z) + \frac{u_k - r}{u_k - d_k} f_{i,k}(z) \\
- \frac{d_j - r}{u_j - d_j} f_{i,j}(z) \\
\end{array} \right\},
\]

(iii) If \(\alpha_{ij} \geq 0, \alpha_{jk} \leq 0\) and \(\alpha_{ik} \leq 0\), where \(\{i, j, k\}\) is an arbitrary permutation of the set \(\{1, 2, 3\}\), then

\[
\mathcal{B} f(z) = \frac{1}{r} \max \left\{ \begin{array}{l}
\alpha_{ij} f_{i,j}(z) + (-\alpha_{jk}) f_{i,j,k}(z) + \frac{u_i - r}{u_i - d_i} f_{i,k}(z) \\
- \frac{d_k - r}{u_k - d_k} f_{j,k}(z) \\
\alpha_{ij} f_{k,j}(z) + (-\alpha_{ik}) f_{i,j,k}(z) + \frac{u_j - r}{u_j - d_j} f_{i,k}(z) \\
- \frac{d_k - r}{u_k - d_k} f_{i,j}(z) \\
(\alpha_{123} + 1) f_{i,k}(z) - \alpha_{jk} f_{i,j,k}(z) - \alpha_{ik} f_{i,k}(z) \\
- \frac{d_k - r}{u_k - d_k} f_{i,j}(z)
\end{array} \right\}.
\]

Addition and scalar multiplication preserves the set of submodular functions and hence Theorem 1 can be applied recursively which allows to get an explicit expression for \(\mathcal{B}^n f(z)\) generalizing the CRR formula.

Denote by \(C_{n}^{ijk}\) the coefficient in the polynomial expansion

\[
(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^n = \sum_{i+j+k \leq n} C_{n}^{ijk} \epsilon_1^{n-i-j-k} \epsilon_2^i \epsilon_3^j \epsilon_4^k.
\]

The following Corollaries give us the expressions of hedge price \(C_h\) when interest rate \(r\) is close to upper bounds \(u_i\) (\(\alpha_{123} \geq 0\)), or when it is close to lower bounds \(d_i\) for all \(i = 1, 2, 3\).
Corollary 3. If $\alpha_{123} \geq 0$, the hedge price is equal to:

$$C_h = \frac{1}{\rho^n} \sum_{i,j,k \in P_n} C_{ij}^{n-k}(\alpha_{123})^{n-i-j-k} \left( \frac{u_1-r}{u_1-d_1} \right)^i \left( \frac{u_2-r}{u_2-d_2} \right)^j \left( \frac{u_3-r}{u_3-d_3} \right)^k \times f \left( d_1^n u_1^{n-i} S_0^1, d_2^n u_2^{n-j} S_0^2, d_3^n u_3^{n-k} S_0^3 \right),$$

(19)

and if $\alpha_{123} \leq -1$, hedge price is equal to:

$$C_h = \frac{1}{\rho^n} \sum_{i,j,k \in P_n} C_{ij}^{n-k}(-\alpha_{123} - 1)^{n-i-j-k} \left( \frac{r-d_1}{u_1-d_1} \right)^i \left( \frac{r-d_2}{u_2-d_2} \right)^j \left( \frac{r-d_3}{u_3-d_3} \right)^k \times f \left( d_1^n u_1^{n-i} S_0^1, d_2^n u_2^{n-j} S_0^2, d_3^n u_3^{n-k} S_0^3 \right),$$

(20)

where $P_n = \{ i, j, k : 0 \leq i + j + k \leq n \}$.

Proof. Following from (17) and (11) by induction.

This formula can be easily specified for concrete $f$ of form (1), (2) and (3). For example:

Corollary 4. Let function $f$ has form (3) with $J = 3$. If $\alpha_{123} \geq 0$, the hedge price is

$$C_h = \frac{1}{\rho^n} \sum_{i,j,k \in P_n} C_{ij}^{n-k}(\alpha_{123})^{n-i-j-k} \left( \frac{u_1-r}{u_1-d_1} \right)^i \left( \frac{u_2-r}{u_2-d_2} \right)^j \left( \frac{u_3-r}{u_3-d_3} \right)^k \times \max \left( d_1^n u_1^{n-i} S_0^1 - K_1, d_2^n u_2^{n-j} S_0^2 - K_2, d_3^n u_3^{n-k} S_0^3 - K_3 \right),$$

where $P_n = \{ 0 \leq i \leq \mu, 0 \leq j \leq \nu, 0 \leq k \leq \lambda : i + j + k \leq n \}$ and $\mu$ is the maximal integer $i$ such that $d_1^n u_1^{n-i} S_0^1 \geq K_1$, $\nu$ is the maximal integer $j$ such that $d_2^n u_2^{n-j} S_0^2 \geq K_2$ and $\lambda$ is the maximal integer $k$ such that $d_3^n u_3^{n-k} S_0^3 \geq K_3$.

If $\alpha_{123} \leq -1$, the hedge price is

$$C_h = \frac{1}{\rho^n} \sum_{i,j,k \in P_n} C_{ij}^{n-k}(-\alpha_{123} - 1)^{n-i-j-k} \left( \frac{r-d_1}{u_1-d_1} \right)^i \left( \frac{r-d_2}{u_2-d_2} \right)^j \left( \frac{r-d_3}{u_3-d_3} \right)^k \times \max \left( d_1^n u_1^{n-i} S_0^1 - K_1, d_2^n u_2^{n-j} S_0^2 - K_2, d_3^n u_3^{n-k} S_0^3 - K_3 \right)$$

where $P_n$ is the same as above but $\mu$ is the minimal integer $i$ such that $d_1^n u_1^{n-i} S_0^1 \geq K_1$, $\nu$ is the minimal integer $j$ such that $d_2^n u_2^{n-j} S_0^2 \geq K_2$ and $\lambda$ is the minimal integer $k$ such that $d_3^n u_3^{n-k} S_0^3 \geq K_3$.

Proof. Following from (19) and (3) by induction. □

Theorem 2 give us a result of one step game. Its application to the $n$-step game is not obvious, because it is not clear under what conditions the set of submodular
function is preserved by the Bellman operator $B$ from the Theorem 2. That means that investor needs to recalculate and find the best strategy after every step.

The proof of the Theorems 1 and 2 is very long. The sketch of the proof of Theorem 1 will be given in the last section. The full proof of both Theorems can be found in our preprint (Hucki and Kolokoltsov (2003)).

3. Discussion

3.1. Probabilistic interpretation

Let us define a Markov process $Z^t$, $t = 0, 1, 2, \ldots$, on $R^d_t$ by the following rule: for the $t$ and $Z^t = z \in R^d_t$ there are only four possibilities for the position of the process at the next time $t + 1$, namely $(u_1 z_1, u_2 z_2, u_3 z_3)$, $(d_1 z_1, u_2 z_2, u_3 z_3)$, $(u_1 z_1, d_2 z_2, u_3 z_3)$ and $(u_1 z_1, u_2 z_2, d_3 z_3)$ and they can occur with probabilities

\[ P_{i}^{u_1 z_1, u_2 z_2, u_3 z_3} = \alpha_{123}, \quad P_{i}^{d_1 z_1, u_2 z_2, u_3 z_3} = \frac{u_1 - r}{u_1 - d_1}, \]
\[ P_{i}^{u_1 z_1, d_2 z_2, u_3 z_3} = \frac{u_2 - r}{u_2 - d_2}, \quad P_{i}^{u_1 z_1, u_2 z_2, d_3 z_3} = \frac{u_3 - r}{u_3 - d_3}, \]

respectively. Since there are only finite number of possible jumps this Markov process is in fact a Markov chain.

**Theorem 5.** If $\alpha_{123} \geq 0$ then

\[ (B^n f)(z) = E_z f(z^n), \] (21)

where $E_z$ denotes the expectation of the process starting at the point $z$.

**Proof.** For $n = 1$ this follows from Theorem 1 and for $n > 1$ it follows from the above made observation that addition and scalar multiplication preserves the set of submodular functions and hence Theorem 1 can be applied recursively. \[ \square \]

**Proof.** Probabilities \((\frac{u_1 - r}{u_1 - d_1}, \frac{u_2 - r}{u_2 - d_2}, \frac{u_3 - r}{u_3 - d_3}, \alpha_{123})\) are risk-neutral probabilities. \[ \square \]

In the case when $0 \geq \alpha_{123} \geq -1$, observe that the Bellman operator (10) can be written in the form of the Bellman operator of a controlled Markov process, namely

\[ (B f)(z) = \max_{i=1,2,3} \sum_{j=1}^{4} P_{i}^{I_j(z)} f(I_j(z)). \] (22)

For example, for $i = 1$, $I_j(z)$, $j = 1, 2, 3, 4$, could be the points

\[ I_1(z) = (d_1 z_1, d_2 z_2, u_3 z_3), \quad I_2(z) = (d_1 z_1, u_2 z_2, u_3 z_3), \]
\[ I_3(z) = (u_1 z_1, d_2 z_2, u_3 z_3), \quad I_4(z) = (u_1 z_1, u_2 z_2, d_3 z_3) \]

and the corresponding probabilities of transitions from $z$ to $I_j(z)$ are given by

\[ P_{i}^{I_1(z)} = -\alpha_{123}, \quad P_{i}^{I_2(z)} = \alpha_{123}, \quad P_{i}^{I_3(z)} = \alpha_{13}, \quad P_{i}^{I_4(z)} = \frac{u_3 - r}{u_3 - d_3}. \]
Representation (22) shows in particular that in this case the solution cannot be written in form (21) and hence the obtained formula differs from what one can expect from the usual stochastic analyses approach to option pricing.

3.2. Unpredictable surplus

The estimation of unpredictable surplus is an important part of the modern analysis of the financial market (see, e.g. Lions (1995)). Let us give the formula for this surplus in the present game theoretic framework. Copying the previous argument one can see that the maximal income of the investor is given by the formula

$$\rho^n(X_0 - (B^n_{\text{min}} f)(S^1_0, \ldots, S^J_0)),$$

where

$$(B^n_{\text{min}} f)(z) = \frac{1}{\rho} \min_{\xi \in M} \{f(\xi z) - \sum_{i=1}^J \gamma^i z^i (\xi^i - \rho)\},$$

and $\gamma$ is the chosen hedge strategy.

Consequently the upper bound for the unpredictable surplus to the income of the investor is

$$(B^n f)(S^1_0, \ldots, S^J_0) - (B^n_{\text{min}} f)(S^1_0, \ldots, S^J_0).$$

3.3. Generalities

A surprisingly simple linear form (17) and (18) of min-max Bellman operator (10) arises the question whether it can be generalized to other options, see Kulatilaka and Trigeorgis (1994). For example, at the moment we even do not have a proof of the analogous result for the general $J > 3$. Another point to notice is an unexpectedly long and technical proof of Theorem 1 and 2 resulting from a number of strange coincidence and cancellations. This leads to the following question for the theory of multistep dynamic games. What is the general justification for these cancellations and/or what is the class of game theoretic Bellman operators that can be reduced to a simpler Bellman operator of a controlled Markov chain.

3.4. American options

In case of American call option buyer is an acting character in the market. Contract allows him to choose moment of exercising the options on his own (any moment before maturity date $n < N$). In selecting the corresponding hedging portfolio the investor must keep in mind the buyer’s freedom to exercise the option at any time. It is not difficult to modify our basic formula (8) for this case, namely instead of (8)
one should write
\[ BG(X, S^1, \ldots, S^J) = \max_{\gamma} \min_{\xi} \min_{\xi'} \left[ G \left( \rho X + \beta \sum_{i=1}^{J} \gamma^i \xi^i S^i \right) - \rho \sum_{i=1}^{J} \gamma^i S^i, \xi^1 S^1, \ldots, \xi^J S^J \right], G(X, S^1, \ldots, S^J) \]

Formula (9) remains unchanged. However, the further factorization with the reduced Bellman operator is not possible and one should calculate directly using formula (9).

4. Property of Submodular Functions

From now on we shall put \( \beta = 1 \) for simplicity. Hence, \( r = \rho \).

In this section we discuss some basic properties of submodular functions and the Bellman operator (10) which are valid for any dimension.

**Lemma 6.** If \( f(z) \in NS \) on set \( R^J_+ \), the inequality
\[ f_{I_1 \cup I_2 \cup I_3}(z) + f_{I_1}(z) \leq f_{I_1 \cup I_3}(z) + f_{I_1 \cup I_2}(z) \]  
holds for every disjoint subsets \( I_1, I_2, I_3 \) of \( \{1, 2, \ldots, J\} \).

**Proof.** Proof is done by trivial induction.

For the basis of mathematical induction we will take \( |I_2| = 1, |I_3| = 1 \), then inequality (23) holds for an arbitrary \( I_1 \) because of the submodular property (14).

Suppose now that (23) is true for any \( |I_2| \leq n \) and \( |I_3| = 1 \) and arbitrary \( I_1 \) such that \( I_1, I_2, I_3 \) are disjoint.

Let us prove this inequality for \( I_2 = I_2 \cup \{i\} \), where the set \( I_2 \) contains \( n \) elements and \( i \) is an arbitrary index not contained in \( I_1 \cup I_2 \cup I_3 \).

We should prove that
\[ f_{I_1 \cup I_2 \cup I_3}(z) + f_{I_1}(z) \leq f_{I_1 \cup I_3}(z) + f_{I_1 \cup I_2}(z). \]  
(24)

From our assumption with \( I_1 \cup \{i\} \) playing the role of \( I_1 \) we get
\[ f_{I_1 \cup \{i\} \cup I_2 \cup I_3}(z) + f_{I_1 \cup \{i\}}(z) \leq f_{I_1 \cup \{i\} \cup I_3}(z) + f_{I_1 \cup \{i\} \cup I_2}(z). \]  
(25)

By the assumption with \( \{i\} \) playing the role of \( I_2 \) we get
\[ f_{I_1 \cup \{i}\cup I_2}(z) + f_{I_1}(z) \leq f_{I_1 \cup \{i\}}(z) + f_{I_1 \cup I_2}(z). \]  
(26)

Now if we sum (25) and (26) together we get inequality (24).

So we have proved (24) and consequently (23) for any \( I_2 \).

Now, let us suppose that inequality (23) is true for any disjoint sets \( I_1, I_2 \) and \( |I_3| \leq n \).

Let us prove this inequality for \( I_3 = I_3 \cup \{i\} \), where the set \( I_3 \) contains \( n \) elements and \( i \) is any index not contained in \( I_1 \cup I_2 \cup I_3 \).
By the assumption with \( \{i\} \) playing the role of \( I_3 \) we have that inequality
\[
f_{I_1 \cup \{i\} \cup I_3}(\mathbf{z}) + f_{I_1}(\mathbf{z}) \leq f_{I_1 \cup \{i\}}(\mathbf{z}) + f_{I_1 \cup I_3}(\mathbf{z})
\] (27)
is true. Now if we sum (25) and (27) together we get the inequality
\[
f_{I_1 \cup I_2 \cup I_3}(\mathbf{z}) + f_{I_1}(\mathbf{z}) \leq f_{I_1 \cup I_3}(\mathbf{z}) + f_{I_1 \cup I_2}(\mathbf{z}).
\]
This completes the proof of (23) for every disjoint subsets \( I_1, I_2, I_3 \) of \( \{1,2,\ldots,J\} \).

**Proposition 7.** Let \( f(\mathbf{z}) \in NS \). If \( B_{I_1 \cup I_2 \cup I_3}(\mathbf{z},\gamma) \geq B_{I_1 \cup I_2}(\mathbf{z},\gamma) \) then \( B_{I_1 \cup I_3}(\mathbf{z},\gamma) \geq B_{I_1}(\mathbf{z},\gamma) \) holds for every disjoint \( I_1, I_2, I_3 \) subset of \( \{1,2,\ldots,J\} \).

**Proof.** The inequality \( B_{I_1 \cup I_2 \cup I_3}(\mathbf{z},\gamma) \geq B_{I_1 \cup I_2}(\mathbf{z},\gamma) \) is equivalent to
\[
f_{I_1 \cup I_2 \cup I_3}(\mathbf{z}) - \sum_{i \in I_1 \cup I_2 \cup I_3} \gamma_i z_i(d_i - \rho) - \sum_{i \not\in I_1 \cup I_2 \cup I_3} \gamma_i z_i(u_i - \rho)
\]
\[
\geq f_{I_1 \cup I_2}(\mathbf{z}) - \sum_{i \in I_1 \cup I_2} \gamma_i z_i(d_i - \rho) - \sum_{i \not\in I_1 \cup I_2} \gamma_i z_i(u_i - \rho)
\]
i.e.
\[
f_{I_1 \cup I_2 \cup I_3}(\mathbf{z}) - \sum_{i \in I_3} \gamma_i z_i(d_i - \rho) \geq f_{I_1 \cup I_2}(\mathbf{z}) - \sum_{i \in I_3} \gamma_i z_i(u_i - \rho).
\]
And this is equivalent to
\[
f_{I_1 \cup I_2 \cup I_3}(\mathbf{z}) - f_{I_1}(\mathbf{z}) \geq \sum_{i \in I_3} \gamma_i z_i(d_i - u_i).
\] (28)
Similar, the inequality \( B_{I_1 \cup I_3}(\mathbf{z},\gamma) \geq B_{I_1}(\mathbf{z},\gamma) \) is equivalent to
\[
f_{I_1 \cup I_3}(\mathbf{z}) - f_{I_1}(\mathbf{z}) \geq \sum_{i \in I_3} \gamma_i z_i(d_i - u_i).
\] (29)
But clearly (28) implies (29), because
\[
f_{I_1 \cup I_3}(\mathbf{z}) - f_{I_1}(\mathbf{z}) \geq f_{I_1 \cup I_2 \cup I_3}(\mathbf{z}) - f_{I_1 \cup I_2}(\mathbf{z}).
\]
The latter inequality being a consequence of Lemma 6.

**Lemma 8.** (on inclusions) If \( D,E,F \) are subsets of \( \{1,2,\ldots,J\} \) such that \( F \subset E \subset D \) and
\[
B_F(\mathbf{z},\gamma) = B_D(\mathbf{z},\gamma) = (B_f)(\mathbf{z},\gamma)
\]then
\[
B_F(\mathbf{z},\gamma) = B_E(\mathbf{z},\gamma) = B_D(\mathbf{z},\gamma) = (B_f)(\mathbf{z},\gamma).
\]

**Proof.** If
\[
B_D(\mathbf{z},\gamma) = (B_f)(\mathbf{z},\gamma)
\]
then
\[ B_D(z, \gamma) \geq B_{D\setminus E\cup F}(z, \gamma). \]

Consequently, by Proposition 7
\[ B_E(z, \gamma) \geq B_F(z, \gamma). \]
But \( B_F(z, \gamma) \) is the maximum in (12) so
\[ B_F(z, \gamma) \geq B_E(z, \gamma). \]
Consequently, \( B_F(z, \gamma) = B_E(z, \gamma) \). This completes the proof.

**Proposition 9.** There exists \( \gamma_0 \) such that
\[ (B f)(z, \gamma_0) = (B f)(z) \] (30)
Moreover, \( (B f)(z, \gamma_0) \geq 0 \). Any \( \gamma_0 \) satisfying (30) will be called optimal.

**Proof.** Clearly, for any \( \rho \in (d_i, u_i), i \in \{1, 2, \ldots, J\} \) and any \( \gamma_i \) we can choose \( \xi_i \) such that \( \gamma_i z_i(\xi_i - \rho) < 0 \).
Hence, for any \( \gamma \) we can find \( \xi \) such that
\[ f(\xi^1 z_1, \ldots, \xi^J z_J) - \sum_{i=1}^J \gamma_i z_i(\xi_i - \rho) > 0. \]
This means that maximum (12) is positive for every fixed \( \gamma \), which implies
\( (B f)(z) \geq 0 \).
From the positivity of \( f \) it follows that
\[ (B f)(z, \gamma_0) \geq 0 \]
and hence \( (B f)(z, \gamma_0) \to +\infty \) as \( \gamma \to \infty \). Consequently, as \( (B f)(z, \gamma) \) is continuous, we conclude that there exist \( \gamma_0 \) such that
\[ (B f)(z, \gamma_0) = \min_{\gamma} (B f)(z, \gamma) \]
and (30) holds.

5. **Auxiliary Results**

Now we shall stick to the case \( J = 3 \).
For any collection of four different subsets \( I_1, I_2, I_3, I_4 \) of \( \{1, 2, 3\} \) let \( \gamma^{\{I_1, I_2, I_3, I_4\}} \) denote any \( \gamma \) for which
\[ B_{I_1}(z, \gamma_0) = B_{I_2}(z, \gamma_0) = B_{I_3}(z, \gamma_0) = B_{I_4}(z, \gamma_0) \] (31)

Since (31) is a system of three linear equations on three variables \( \gamma \), it is clear that \( \gamma^{\{I_1, I_2, I_3, I_4\}} \) is uniquely defined in generic situation (but not always, of course). Let us denote the right hand side of (31) by \( B^{\{I_1, I_2, I_3, I_4\}} \).
Proposition 10. If \( \gamma_0 \) is optimal then there exist four different subsets \( I_1, I_2, I_3, I_4 \) of \( \{1, 2, 3\} \) such that

\[
\gamma_0 = \gamma^{\{I_1, I_2, I_3, I_4\}}
\]

Proof. If we suppose that \( \gamma_0 \) is optimal and there exist only one \( I \subset \{1, 2, 3\} \) such that

\[
(Bf)(z, \gamma_0) = B_{I_1}(z, \gamma_0),
\]

then we could take \( \gamma \) such that \( \gamma^i \neq \gamma_0^i \) for some \( i \in \{1, 2, 3\} \), then we have that

\[
B_{I_1}(z, \gamma_0) > B_{I_3}(z, \gamma)
\]

and that is a contradiction with (32). If there exist subsets \( I_1 \) and \( I_2 \) such that

\[
(Bf)(z, \gamma_0) = B_{I_1}(z, \gamma_0) = B_{I_2}(z, \gamma_0),
\]

then, in the best case, we are getting that one of the components of vector \( \gamma \) is fixed (obtained from \( B_{I_1}(z, \gamma_0) = B_{I_2}(z, \gamma_0) \)), but still we could change others components to get greater value. Similar situation we have for the three subsets \( I_1, I_2, I_3 \).

So, the optimal \( \gamma_0 \) need to be intersection of four hyper-planes, which have intersection for every two of them, defined by \( B_{I_1}(z, \gamma_0), B_{I_2}(z, \gamma_0), B_{I_3}(z, \gamma_0) \) and \( B_{I_4}(z, \gamma_0) \).

Let us denote

\[
\gamma^J_j = \frac{f_{I_1}(z) - f_{I_2 \cup \{j\}}(z)}{z_j (u_j - d_j)}.
\]

As any optimal \( \gamma_0 \) has the form \( \gamma^{\{I_1, I_2, I_3, I_4\}} \) for some sets \( \{I_1, I_2, I_3, I_4\} \) it follows that the number of candidates for optimal \( \gamma \) is finite, not exceeding 70 (the numbers of families \( \{I_1, I_2, I_3, I_4\} \) of four different subsets of \( \{1, 2, 3\} \)). Next results are meant to reduce the number of possibilities.

Proposition 11. Suppose that \( \gamma_0 = \gamma^{\{I_1, I_2, I_3, I_4\}} \).

(i) If \( \emptyset \not\in \{I_1, I_2, I_3, I_4\} \) then the only optimal \( \gamma \) is \( \gamma^{\{0, \{1\}, \{2\}, \{3\}\}} \).

(ii) If \( \{1, 2, 3\} \in \{I_1, I_2, I_3, I_4\} \) then the only optimal \( \gamma \) is \( \gamma^{\{1, 2, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}} \).

(iii) Set \( \{I_1, I_2, I_3, I_4\} \) can not be equal to \( \{i\}, \{j\}, \{i, k\}, \{j, k\} \) for any permutation \( i, j, k \) of \( \{1, 2, 3\} \).

Proof. (i) If we have family of sets \( \emptyset, \{1\}, \{2\}, \{3\} \), then by definition (31)
we have that

\[
B_{I_1}(z, \gamma) = B_{I_2}(z, \gamma) = B_{I_3}(z, \gamma) = (Bf)(z, \gamma).
\]
From $\mathcal{B}_0(z, \gamma) = \mathcal{B}_{(1)}(z, \gamma)$ we get $\gamma^1 = \frac{f_0(z) - f_{(1)}(z)}{z_1(u_1 - d_1)}$, from $\mathcal{B}_0(z, \gamma) = \mathcal{B}_{(2)}(z, \gamma)$ we get $\gamma^2 = \frac{f_0(z) - f_{(2)}(z)}{z_2(u_2 - d_2)}$, and from $\mathcal{B}_0(z, \gamma) = \mathcal{B}_{(3)}(z, \gamma)$ we get $\gamma^3 = \frac{f_0(z) - f_{(3)}(z)}{z_3(u_3 - d_3)}$.

Therefore, we have that $\gamma_0^{\{1\}, \{2\}, \{3\}} = (\gamma_1^0, \gamma_2^0, \gamma_3^0)$ (using the notation (33)).

Now, suppose that we have $\emptyset, \{i, j\} \in \{I_1, I_2, I_3, I_4\}$ for some $i, j \in \{1, 2, 3\}$.

This means that

$$B_{(i,j)}(z, \gamma) = B_0(z, \gamma) = (Bf)(z, \gamma),$$

then by Lemma 8 we have that

$$B_0(z, \gamma) = B_{(i,j)}(z, \gamma) = B_{(i)}(z, \gamma) = B_{(j)}(z, \gamma) = (Bf)(z, \gamma)$$

So, we get combination $\{\emptyset, \{i\}, \{j\}, \{i, j\}\}$.

From $B_{(i)}(z, \gamma) = B_0(z, \gamma)$ we are getting $\gamma_i = \frac{f_0(z) - f_{(i)}(z)}{z_1(u_1 - d_1)}$ and $B_{(j)}(z, \gamma) = B_{(i,j)}(z, \gamma)$ yields $\gamma_i = \frac{f_{(i)}(z) - f_{(i,j)}(z)}{z_1(u_1 - d_1)}$.

Similarly, from $B_{(j)}(z, \gamma) = B_0(z, \gamma)$ we are getting $\gamma_j = \frac{f_0(z) - f_{(j)}(z)}{z_j(u_j - d_j)}$ and $B_{(i,j)}(z, \gamma)$ yields $\gamma_j = \frac{f_{(j)}(z) - f_{(i,j)}(z)}{z_j(u_j - d_j)}$.

Now, if

$$f_0(z) - f_{(i)}(z) \neq f_{(j)}(z) - f_{(i,j)}(z)$$

the situation is impossible.

If $f_0(z) - f_{(i)}(z) = f_{(j)}(z) - f_{(i,j)}(z)$ then

$$(Bf)(z, \gamma) = f_0(z) - \frac{f_0(z) - f_{(i)}(z)}{u_i - d_i}(u_i - \rho) - \frac{f_0(z) - f_{(j)}(z)}{u_j - d_j}(u_j - \rho) - \gamma_k z_k(u_k - \rho).$$

However, this is a contradiction with the assumption that $\gamma$ is optimal, because we still have a free choice of $\gamma_k$.

If $\emptyset, \{i, j, k\} \in \{I_1, I_2, I_3, I_4\}$ then by Lemma 8 we are getting that

$$B_0(z, \gamma) = B_{(i)}(z, \gamma) = B_{(i,j)}(z, \gamma) = B_{(1,2,3)}(z, \gamma)$$

for all $i, j \in \{1, 2, 3\}$.

This means that for all $I \subseteq \{1, 2, 3\}$ the values $B_I(z, \gamma)$ have to be equal. In this case we are still getting optimal $\gamma_0^{\{0\}, \{1\}, \{2\}, \{3\}}$.

(ii) The proof is similar to (i).

(iii) From $B_{(i)}(z, \gamma) = B_{(i,k)}(z, \gamma)$ we get $\gamma_k = \frac{f_{(i)}(z) - f_{(i,k)}(z)}{z_k(u_k - d_k)}$

and $B_{(j)}(z, \gamma) = B_{(j,k)}(z, \gamma)$ yields $\gamma_k = \frac{f_{(j)}(z) - f_{(j,k)}(z)}{z_k(u_k - d_k)}$.

If

$$f_{(i)}(z) - f_{(i,k)}(z) \neq f_{(j)}(z) - f_{(j,k)}(z)$$

situation is impossible.
If \( f_{ij}(z) - f_{i,k}(z) = f_{ij}(z) - f_{i,k}(z) \) then

\[
(Bf)(\gamma, z) = f_{ij}(z) - \gamma_i z_i (d_i - \rho) - \frac{f_{ij}(z) - f_{i,j}(z) - \gamma_i (d_i - u_i)}{(u_j - d_j)} (u_j - \rho)
\]

which is a contradiction with the assumption that \( \gamma \) is optimal, because we still have a free choice of \( \gamma_i \).

As a direct consequence we have:

**Proposition 12.** If \( \gamma_0 = \gamma^{\{1,2,3\}} \), then there are the following possibilities for the collection \( \{I_1, I_2, I_3, I_4\} \):

(i) \( \{\emptyset, \{1\}, \{2\}, \{3\}\} \) and \( \gamma_0 = (\gamma_1^0, \gamma_2^0, \gamma_3^0) \) where we used notation (33).

(ii) \( \{\{1\}, \{2\}, \{3\}, \{i, j\}\} \) and \( \gamma_0 = (\gamma_i^1, \gamma_j^1, \gamma_k^1) \) where

\[
\gamma_k^{\{1\},\{2\},\{3\},\{i,j\}} = \frac{f_{ik}(z) + f_{i,j}(z) - f_{i}(z) - f_{i,j}(z)}{z_k (u_k - d_k)},
\]

where \( i \) is arbitrary index from \( \{1,2,3\} \).

(iii) \( \{\{i\}, \{2\}, \{1,3\}, \{2,3\}\} \) and \( \gamma_0 = (\gamma_i^2, \gamma_j^2, \gamma_k^2) \) where

\[
\gamma_i^{\{1\},\{2\},\{1,3\},\{2,3\}} = \frac{f_{i,j}(z) + f_{i,k}(z) - f_{i}(z) - f_{i,j}(z)}{z_i (u_i - d_i)},
\]

where \( i, j, k \) is arbitrary permutation of \( \{1,2,3\} \).

(iv) \( \{\{i\}, \{j\}, \{i, j\}, \{i, k\}\} \) and \( \gamma_0 = (\gamma_i^3, \gamma_j^3, \gamma_k^3) \).

(v) \( \{\{2\}, \{1,3\}, \{2,3\}, \{1,2\} \} \) and \( \gamma_0 = (\gamma_1^{2,3}, \gamma_2^{1,3}, \gamma_3^{1,2}) \).

Let us denote by \( \Gamma \) the set of all \( \gamma^{\{1,2,3\}} \) from Proposition 12. That means that the set \( \Gamma \) contains \( \gamma^{\{0,\{1\},\{2\},\{3\}\}}, \gamma^{\{1\},\{2\},\{3\},\{i,j\}} \) for every \( i \neq j \), \( \gamma^{\{1\},\{2\},\{1,3\},\{2,3\}} \) for every \( i \in \{1,2,3\} \) and \( \gamma^{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}} \). In particular, \( |\Gamma| = 14 \).

Direct consequence of Proposition 12 is that (10) can be written as

\[
(Bf)(z) = \frac{1}{\rho} \min_{\gamma \in \Gamma} B(z, \gamma)
\]

Hence proposition reduces the number of candidates for the optimum from 70 to 14.

Let \( \Gamma(z) \) be a subset of \( \Gamma \) defined by the following rule:

\[
\gamma^{\{1,2,3\}} \in \Gamma(z) \text{ iff } (Bf)(z, \gamma^{\{1,2,3\}}) = B_{I_1}(z, \gamma^{\{1,2,3\}}) = B_{I_2}(z, \gamma^{\{1,2,3\}}) = B_{I_3}(z, \gamma^{\{1,2,3\}}) = B_{I_4}(z, \gamma^{\{1,2,3\}}).
\]

Now we have the following form for the Bellman operator.

\[
(Bf)(z) = \frac{1}{\rho} \min_{\gamma \in \Gamma(z)} (Bf)(z, \gamma)
\]
The following proposition gives criteria for $\gamma \in \Gamma(z)$.

**Proposition 13.** If $f \in NS$ then

(a) $\gamma^{\{0,\{1\},\{2\},\{3\}\}, \gamma^{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}}} \in \Gamma(z)$ for every $z \in R^3$. 

(b) For all $i, j \in \{1, 2, 3\}$ such that $i \neq j$ one has $\gamma^{\{1\},\{2\},\{3\},\{i,j\}} \in \Gamma(z)$ if and only if the conditions

$$\Delta_i f_{\{i,k\}}(z) \geq \Delta_i f_{\{i,j\}}(z) \text{ and } \Delta_j f_{\{j,k\}}(z) \geq \Delta_j f_{\{i,j\}}(z)$$

are satisfied, where $\{k\} = \{1, 2, 3\} \backslash \{i, j\}$.

(c) For every $i \in \{1, 2, 3\}$ one has $\gamma^{\{1\},\{2\},\{3\},\{i\}} \in \Gamma(z)$ if and only if the conditions

$$\Delta_k f_{\{i,k\}}(z) \geq \Delta_k f_{\{i,j\}}(z) \text{ and } \Delta_j f_{\{i,k\}}(z) \geq \Delta_j f_{\{i,j\}}(z)$$

are satisfied, where $\{j, k\} = \{1, 2, 3\} \backslash \{i\}$.

(d) For every different $i, j, k \in \{1, 2, 3\}$ coefficient $\gamma^{\{i\},\{j\},\{i,j\},\{i,k\}} \in \Gamma(z)$ if and only if conditions

$$\Delta_k f_{\{i,k\}}(z) \geq \Delta_k f_{\{i,j\}}(z) \text{ and } \Delta_i f_{\{i,j\}}(z) \geq \Delta_i f_{\{i,k\}}(z)$$

are satisfied.

**Proof.** In all the cases we need to show that

$$B_{I_1, I_2, I_3, I_4} = \langle Bf \rangle (z, \gamma^{\{I_1, I_2, I_3, I_4\}}).$$

(a) First, let us show that

$$B_{\{0,\{1\},\{2\},\{3\}\}} = \langle Bf \rangle (z, \gamma^{\{0,\{1\},\{2\},\{3\}\}}).$$

To show previous inequality it is enough to proof that inequalities

\[
\begin{align*}
(\text{a}_1) & \quad B_{\{i,j\}}(z, \gamma^{\{0,\{1\},\{2\},\{3\}\}}) \leq B_{\{0,\{1\},\{2\},\{3\}\}} \\
(\text{a}_2) & \quad B_{\{1,2,3\}}(z, \gamma^{\{0,\{1\},\{2\},\{3\}\}}) \leq B_{\{0,\{1\},\{2\},\{3\}\}}
\end{align*}
\]

are true.

The inequality (a1) written explicitly is

$$f_{\{i,j\}}(z) - \gamma^0 z_i (d_i - \rho) - \gamma^0 z_j (d_j - \rho) - \gamma^0 z_k (u_k - \rho) \leq f_{\{i\}}(z) - \gamma^0 z_i (d_i - \rho) - \gamma^0 z_j (u_j - \rho) - \gamma^0 z_k (u_k - \rho).$$

This is equivalent to

$$f_{\{i,j\}}(z) - f_{\{i\}}(z) \leq \gamma^0 z_j (d_j - u_j).$$

Since $\gamma^0_j = \frac{f_{\emptyset}(z) - f_{\{j\}}(z)}{u_j - d_j}$, this is equivalent to

$$f_{\{i,j\}}(z) - f_{\{i\}}(z) \leq f_{\{j\}}(z) - f_{\emptyset}(z)$$

or written in terms of increments

$$\Delta_j f_{\{i,j\}}(z) \geq \Delta_j f_{\{j\}}(z).$$
and this is always true by submodular property.

The inequality \((a_2)\) written explicitly is
\[
\begin{align*}
\gamma_1^0 z_1 (d_1 - \rho) - \gamma_2^0 z_2 (d_2 - \rho) - \gamma_3^0 z_3 (d_3 - \rho) \\
\leq f_0(z) - \gamma_1^0 z_1 (u_1 - \rho) - \gamma_2^0 z_2 (u_2 - \rho) - \gamma_3^0 z_3 (u_3 - \rho),
\end{align*}
\]
i.e.
\[
\gamma_1^0 z_1 (u_1 - d_1) + \gamma_2^0 z_2 (u_2 - d_2) + \gamma_3^0 z_3 (u_3 - d_3)
\]
Due to notation \((33)\) with \(I = \emptyset\)
\[
\gamma_1^0 = \frac{f_0(z) - f_{(1)}(z)}{z_1 (u_1 - d_1)}, \quad \gamma_2^0 = \frac{f_0(z) - f_{(2)}(z)}{z_2 (u_2 - d_2)} \quad \text{and} \quad \gamma_3^0 = \frac{f_0(z) - f_{(3)}(z)}{z_3 (u_3 - d_3)}.
\]
Therefore, we have
\[
\begin{align*}
f_0(z) - f_{(1)}(z) + f_0(z) - f_{(2)}(z) + f_0(z) - f_{(3)}(z) \\
\leq f_0(z) - f_{(1,2,3)}(z).
\end{align*}
\]
Clearly, this could be written as
\[
\begin{align*}
0 \leq (-f_0(z) + f_{(1)}(z) + f_{(2)}(z) - f_{(1,2)}(z)) \\
+ (-f_0(z) + f_{(3)}(z) + f_{(1)}(z) - f_{(1,3)}(z)) \\
+ (-f_{(1)}(z) + f_{(1,3)}(z) + f_{(1,2)}(z) - f_{(1,2,3)}(z))
\end{align*}
\]
or in the terms of the increments
\[
\begin{align*}
0 \leq (\Delta_1 f_{(1,2)}(z) - \Delta_1 f_{(1)}(z)) + (\Delta_3 f_{(1,3)}(z) - \Delta_3 f_{(3)}(z)) \\
+ (\Delta_3 f_{(1,2,3)}(z) - \Delta_3 f_{(1,3)}(z))
\end{align*}
\]
This inequality is true because all brackets are always positive by the submodular property.

The proof that
\[
B_{\{1,2\},\{1,3\},\{2,3\}} = (Bf)(z; \gamma_{\{1,2\},\{1,3\},\{2,3\}})
\]
is similar to previous one.

(b) We need to show that
\[
\begin{align*}
(b_1) & \quad B_{\emptyset}(z, \gamma_{\{1\},\{2\},\{3\},\{i,j\}}) \leq B_{\{1\},\{2\},\{3\},\{i,j\}} \\
(b_2) & \quad B_{\{1,2,3\}}(z, \gamma_{\{1\},\{2\},\{3\},\{i,j\}}) \leq B_{\{1\},\{2\},\{3\},\{i,j\}} \\
(b_3) & \quad B_{\{i,k\}}(z, \gamma_{\{1\},\{2\},\{3\},\{i,j\}}) \leq B_{\{1\},\{2\},\{3\},\{i,j\}} \\
(b_4) & \quad B_{\{i,k\}}(z, \gamma_{\{1\},\{2\},\{3\},\{i,j\}}) \leq B_{\{1\},\{2\},\{3\},\{i,j\}}
\end{align*}
\]
The inequality \((b_1)\) written explicitly is
\[
\begin{align*}
f_0(z) - \gamma_i^0 z_i (u_i - \rho) - \gamma_j^0 z_j (u_j - \rho) - \gamma_k^0 z_k (u_k - \rho) \\
\leq f_{(i,j)}(z) - \gamma_i^0 z_i (d_i - \rho) - \gamma_j^0 z_j (d_j - \rho) - \gamma_k^0 z_k (u_k - \rho)
\end{align*}
\]
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Since, due to notation (33) and (34)
\[
\gamma_i^j = \frac{f_{(i,j)}(z) - f_{(i,j)}(z)}{z_i(u_i - d_i)} \quad \gamma_j^i = \frac{f_{(i,j)}(z) - f_{(i,j)}(z)}{z_j(u_j - d_j)}
\]
and
\[
\gamma_k^{1,2,3,1,2} = \frac{f_{(k)}(z) + f_{(i,j)}(z) - f_{(i)}(z) - f_{(j)}(z)}{z_k(u_k - d_k)}
\]
we are getting
\[
0 \geq f_0(z) - f_{(i,j)}(z) - (f_{(j)}(z) - f_{(i,j)}(z)) - (f_{(i)}(z) - f_{(i,j)}(z)),
\]
i.e.
\[
f_{(i)}(z) - f_{(i,j)}(z) \geq f_0(z) - f_{(j)}(z).
\]
Written in terms of increments this is equivalent to
\[
\Delta_j f_{(i,j)}(z) \geq \Delta_j f_{(j)}(z).
\]
This is true by submodular property.

The inequality \(b_2\) written explicitly is
\[
f_{(1,2,3)}(z) - \gamma_i^j z_i(d_i - \rho) - \gamma_j^i z_j(d_j - \rho) - \gamma_k^{1,2,3,1,2} z_k(d_k - \rho)
\leq f_{(i,j)}(z) - \gamma_i^j z_i(d_i - \rho) - \gamma_j^i z_j(d_j - \rho) - \gamma_k^{1,2,3,1,2} z_k(u_k - \rho),
\]
i.e.
\[
0 \leq f_{(i,j)}(z) - f_{(1,2,3)}(z) - \gamma_k^{1,2,3,1,2} z_k(u_k - d_k).
\]
Since, due to (35) we have
\[
0 \leq f_{(i,j)}(z) - f_{(1,2,3)}(z) - (-f_{(k)}(z) - f_{(i,j)}(z) + f_{(i)}(z) + f_{(j)}(z)),
\]
If we add \(f_{(j,k)}(z) - f_{(j,k)}(z)\) this is equivalent to
\[
0 \leq f_{(i,j)}(z) - f_{(1,2,3)}(z) - f_{(j)}(z) + f_{(j,k)}(z)
+ f_{(k)}(z) - f_{(i,j)}(z) + f_{(i)}(z) + f_{(i,j)}(z)
\]
or written in terms of increments
\[
0 \leq (\Delta_k f_{(1,2,3)}(z) - \Delta_k f_{(j,k)}(z))
+ (\Delta_j f_{(j,k)}(z) - \Delta_j f_{(i,j)}(z)).
\]
The first bracket is always positive by submodular property and the second is positive by the condition (36).

The inequality \(b_3\) is equivalent to
\[
f_{(i,k)}(z) - \gamma_i^j z_i(d_i - \rho) - \gamma_j^i z_j(u_j - \rho) - \gamma_k^{1,2,3,1,2} z_k(d_k - \rho)
\leq f_{(i,j)}(z) - \gamma_i^j z_i(d_i - \rho) - \gamma_j^i z_j(d_j - \rho) - \gamma_k^{1,2,3,1,2} z_k(u_k - \rho),
\]
i.e.
\[
f_{(i,k)}(z) - f_{(i,j)}(z) \leq \gamma_j^i z_j(u_j - d_j) - \gamma_k^{1,2,3,1,2} z_k(u_k - d_k).
\]
Since, $\gamma_i^{(i)} = \frac{f_i(z) - f_i(z)}{z_i(u_i - d_i)}$ and (34) this inequality becomes
\[ f_{(i,k)}(z) - f_{(i,j)}(z) \leq f_{(i)}(z) - f_{(i,j)}(z) + f_{(k)}(z) + f_{(i,j)}(z) - f_{(i)}(z) - f_{(j)}(z) \]
i.e.
\[ f_{(j)}(z) - f_{(i,j)}(z) \leq f_{(k)}(z) - f_{(i,k)}(z) \]
This is equivalent to the condition
\[ \Delta_i f_{(i,j)}(z) \leq \Delta_i f_{(i,k)}(z), \]
from the (36).
Proof for the case $(b_4)$ is similar to the case $(b_3)$.
(c) To prove this we need to show that
\[ (c_1) \quad \mathcal{B}_0(z, \gamma^{(1,2), \ldots, (1,3), (2,3), (3,1)}) \leq \mathcal{B}_{(1,2), (1,3), (2,3), (3,1)} \]
\[ (c_2) \quad \mathcal{B}_{(i,j)}(z, \gamma^{(1,2), (1,3), (2,3), (3,1)}) \leq \mathcal{B}_{(1,2), (1,3), (2,3), (3,1)} \]
\[ (c_3) \quad \mathcal{B}_{(i,k)}(z, \gamma^{(1,2), (1,3), (2,3), (3,1)}) \leq \mathcal{B}_{(1,2), (1,3), (2,3), (3,1)} \]
\[ (c_4) \quad \mathcal{B}_{(1,2,3)}(z, \gamma^{(1,2), (1,3), (2,3), (3,1)}) \leq \mathcal{B}_{(1,2), (1,3), (2,3), (3,1)} \]
are true.
The inequality (c1) is equivalent to
\[ f_0(z) - \gamma^{(1,2), (1,3), (2,3), (3,1)} z_i(u_i - \rho) - \gamma^{(i)} z_j(u_j - \rho) - \gamma^{(i)} z_k(u_k - \rho) \]
\[ \leq f_{(i)}(z) - \gamma^{(1,2), (1,3), (2,3), (3,1)} z_i(d_i - \rho) - \gamma^{(i)} z_j(u_j - \rho) - \gamma^{(i)} z_k(u_k - \rho) \]
i.e.
\[ 0 \leq f_{(i)}(z) - f_0(z) + \gamma^{(1,2), (1,3), (2,3), (3,1)} z_i(u_i - d_i). \]

Due to (35) we get
\[ 0 \leq f_{(i)}(z) - f_0(z) + f_{(i)}(z) + f_{(j,k)}(z) - f_{(i,k)}(z) - f_{(i,j)}(z) \]

If we add $+ f_{(k)}(z) - f_{(k)}(z)$ to the right hand side and write in terms of increments
\[ 0 \leq (\Delta_i f_{(i,j)}(z) - \Delta_i f_{(i,k)}(z)) + (\Delta_j f_{(i,j)}(z) - \Delta_j f_{(i,k)}(z)). \]

First bracket is positive by submodular property and the second one is positive by condition
\[ \Delta_j f_{(i,j)}(z) \geq \Delta_j f_{(i,k)}(z) \]
from (37).
The inequality (c2) is equivalent to
\[ f_{(j)}(z) - \gamma^{(1,2), (1,3), (2,3), (3,1)} z_i(u_i - \rho) - \gamma^{(i)} z_j(d_j - \rho) - \gamma^{(i)} z_k(u_k - \rho) \]
\[ \leq f_{(i,j)}(z) - \gamma^{(1,2), (1,3), (2,3), (3,1)} z_i(d_i - \rho) - \gamma^{(i)} z_j(d_j - \rho) - \gamma^{(i)} z_k(u_k - \rho) \]
This is equivalent to
\[ 0 \leq f_{(i,j)}(z) - f_{(j)}(z) + \gamma^{(1,2), (1,3), (2,3), (3,1)} z_i(u_i - d_i) \]
Due to (35) we get
\[ 0 \leq f_{(i,j)}(z) - f_{(j)}(z) + f_{(i)}(z) + f_{(j,k)}(z) - f_{(i,k)}(z) - f_{(i,j)}(z) \]

Written in terms of increments we get the condition
\[ \Delta_k f_{(i,k)}(z) \geq \Delta_k f_{(j,k)}(z). \]

The proof of the case (c3) is similar to the previous one.

The inequality (c4) is equivalent to
\[ f_{(i,j,k)}(z) - \gamma_i^{\{1,2\},\{1,3\},\{2,3\}} z_i (d_i - \rho) - \gamma_j^{\{i\}} z_j (d_j - \rho) - \gamma_k^{\{i\}} z_k (d_k - \rho) \leq f_{(j,k)}(z) - \gamma_i^{\{1,2\},\{1,3\},\{2,3\}} z_i (u_i - \rho) - \gamma_j^{\{i\}} z_j (d_j - \rho) - \gamma_k^{\{i\}} z_k (d_k - \rho), \]

i.e.
\[ 0 \leq f_{(j,k)}(z) - f_{(i,j,k)}(z) - \gamma_i^{\{1,2\},\{1,3\},\{2,3\}} z_i (u_i - d_i) \]

Due to \( \gamma_i^{\{1\}} = \frac{f_{(j,k)}(z) + f_{(i,j)}(z) - f_{(i,k)}(z) - f_{(i,j)}(z)}{z_i(u_i - d_i)} \) we get
\[ 0 \leq f_{(j,k)}(z) - f_{(i,j,k)}(z) - f_{(i)}(z) - f_{(j,k)}(z) + f_{(i,k)}(z) + f_{(i,j)}(z) \]

Written in terms of increments this is
\[ (\Delta_k f_{(i,k)}(z) - \Delta_k f_{(1,2,3)}(z)) \geq 0. \]

This is true by submodular property.

(d) To prove this we need to show that

\[
\begin{align*}
(d_1) \quad & B_{(i)}(z, \gamma^{\{i\}}) \leq B_{(i,j)}(z, \gamma^{\{i,j\}}) \\
(d_2) \quad & B_{(j)}(z, \gamma^{\{i\}}) \leq B_{(i,j)}(z, \gamma^{\{i\}}) \\
(d_3) \quad & B_{(i,j,k)}(z, \gamma^{\{i\}}) \leq B_{(i,j)}(z, \gamma^{\{i\}}) \\
(d_4) \quad & B_{(1,2,3)}(z, \gamma^{\{i\}}) \leq B_{(i,j)}(z, \gamma^{\{i\}}) 
\end{align*}
\]

Inequality (d1) written explicitly is
\[ f_{(i)}(z) - \gamma_i^{\{i\}} z_i (u_i - \rho) - \gamma_j^{\{i\}} z_j (u_j - \rho) - \gamma_k^{\{i\}} z_k (u_k - \rho) \leq f_{(i)}(z) - \gamma_i^{\{i\}} z_i (d_i - \rho) - \gamma_j^{\{i\}} z_j (u_j - \rho) - \gamma_k^{\{i\}} z_k (u_k - \rho), \]

i.e.
\[ 0 \leq f_{(i)}(z) - f_{(i)}(z) + \gamma_i^{\{i\}} z_i (u_i - d_i) \]

Due to (33) we have
\[ 0 \leq f_{(i,j)}(z) - f_{(i)}(z) + f_{(j)}(z) - f_{(i,j)}(z). \]

Written in terms of the increments we get the condition
\[ 0 \leq \Delta_i f_{(i,j)}(z) - \Delta_i f_{(i)}(z) \]

and this is one of submodular property inequalities.
Inequality $(d_2)$ written explicitly is
\[
\begin{align*}
  f_{(k)}(z) - \gamma_i^{(j)} z_i (u_i - \rho) - \gamma_j^{(i)} z_j (u_j - \rho) - \gamma_k^{(i)} z_k (d_k - \rho) \\
  \leq f_{(i,k)}(z) - \gamma_i^{(j)} z_i (d_i - \rho) - \gamma_j^{(i)} z_j (u_j - \rho) - \gamma_k^{(i)} z_k (d_k - \rho),
\end{align*}
\]
i.e.
\[
0 \leq f_{(i,k)}(z) - f_{(k)}(z) + \gamma_i^{(j)} z_i (u_i - d_i).
\]
Due to (33) we have
\[
0 \leq f_{(i,k)}(z) - f_{(i)}(z) + f_{(j)}(z) - f_{(i,j)}(z)
\]
Written in terms of the increments we get the condition
\[
0 \leq \Delta_i f_{(i,j)}(z) - \Delta_i f_{(i,k)}(z).
\]
Inequality $(d_3)$ written explicitly is
\[
\begin{align*}
  f_{(j,k)}(z) - \gamma_i^{(j)} z_i (u_i - \rho) - \gamma_j^{(i)} z_j (d_j - \rho) - \gamma_k^{(i)} z_k (d_k - \rho) \\
  \leq f_{(j)}(z) - \gamma_i^{(j)} z_i (u_i - \rho) - \gamma_j^{(i)} z_j (d_j - \rho) - \gamma_k^{(i)} z_k (u_k - \rho),
\end{align*}
\]
i.e.
\[
0 \leq f_{(j)}(z) - f_{(j,k)}(z) - \gamma_k^{(i)} z_k (u_k - d_k)
\]
Due to (33) we have
\[
0 \leq f_{(j)}(z) - f_{(i,k)}(z) - f_{(i)}(z) + f_{(i,k)}(z)
\]
Written in terms of the increments we get the condition
\[
0 \leq \Delta_k f_{(j,k)}(z) - \Delta_k f_{(i,k)}(z).
\]
And inequality $(d_4)$ written explicitly is
\[
\begin{align*}
  f_{(i,j,k)}(z) - \gamma_i^{(j)} z_i (d_i - \rho) - \gamma_j^{(i)} z_j (d_j - \rho) - \gamma_k^{(i)} z_k (d_k - \rho) \\
  \leq f_{(i,k)}(z) - \gamma_i^{(j)} z_i (d_i - \rho) - \gamma_j^{(i)} z_j (u_j - \rho) - \gamma_k^{(i)} z_k (d_k - \rho),
\end{align*}
\]
i.e.
\[
0 \leq f_{(i,k)}(z) - f_{(i,j,k)}(z) - \gamma_j^{(i)} z_j (u_j - d_j)
\]
Due to (33) we have
\[
0 \leq f_{(i,k)}(z) - f_{(i,j,k)}(z) - f_{(i)}(z) + f_{(i,j)}(z)
\]
Written in terms of the increments we get
\[
0 \leq \Delta_j f_{(1,2,3)}(z) - \Delta_j f_{(i,j)}(z).
\]
This is true by submodular property.

\[\square\]

**Lemma 14.** If we have conditions
\[
\Delta_i f_{(i,k)}(z) \geq \Delta_i f_{(i,j)}(z) \quad \text{and} \quad \Delta_j f_{(i,j)}(z) \geq \Delta_j f_{(i,k)}(z)
\]
then
\[ \Delta_k f_{k}^{(i)}(z) \geq \Delta_k f_{k}^{(j)}(z) \]
is true for all different \( i, j, k \in \{1, 2, 3\} \).

**Proof.** Adding together the first and the second inequality we get third one.  \( \square \)

6. **Proof of Theorem 1**

We will prove only the (17) (formula (18) can be shown similarly. It is not difficult to see that one of the following three cases is always realized:

(i) \[ \Delta_1 f_{1,3}^{(i)}(z) \geq \Delta_1 f_{1,2}^{(i)}(z) \] (39)
and
\[ \Delta_2 f_{1,2}^{(i)}(z) \geq \Delta_2 f_{2,3}^{(i)}(z), \] (40)

(ii) \[ \Delta_1 f_{1,3}^{(i)}(z) \geq \Delta_1 f_{1,2}^{(i)}(z) \] (41)
and
\[ \Delta_2 f_{1,2}^{(i)}(z) \geq \Delta_2 f_{2,3}^{(i)}(z), \] (42)

(iii) \[ \Delta_1 f_{1,2}^{(i)}(z) \geq \Delta_1 f_{1,3}^{(i)}(z) \] (43)
and
\[ \Delta_2 f_{1,2}^{(i)}(z) \geq \Delta_2 f_{1,3}^{(i)}(z). \] (44)

We shall consider only the case (i) (other being similar) By Lemma 14 we have
\[ \Delta_3 f_{1,3}^{(i)}(z) \geq \Delta_3 f_{2,3}^{(i)}(z). \] (45)

So, we need to show that
\[ B\{0,\{1\},\{2\},\{3\}\} = \min_{\gamma \in \Gamma(z)} B(z, \gamma). \] (46)

From Proposition 13 we have that the set \( \Gamma(z) \) is equal to
\[ \Gamma(z) = \{ \gamma^{0,\{1\},\{2\},\{3\}} , \gamma^{\{1\},\{2\},\{3\},\{2,3\} , \gamma^{\{1,2\},\{1,3\},\{2,3\},\{1\}} , \gamma^{\{1\},\{2\},\{1,2\},\{2,3\} , \gamma^{\{1\},\{3\},\{1,3\},\{2,3\} , \gamma^{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}} \}. \] (47)

According to notation (31), we need to show that
\[ B\{0,\{1\},\{2\},\{3\}\} = \min\{B\{0,\{1\},\{2\},\{3\}\} , B\{\{1\},\{2\},\{3\},\{2,3\} \} , B\{\{1,2\},\{1,3\},\{2,3\},\{1\} \} , B\{\{1\},\{3\},\{1,3\},\{2,3\} \} , B\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\} \} \} . \]
We will show the following inequalities:

(a1) \( \mathcal{B}_{\emptyset, \{1\}, \{2, 3\}} \leq \mathcal{B}_{\emptyset, \{1, 2\}, \{3\}} \)

(a2) \( \mathcal{B}_{\emptyset, \{1\}, \{2, 3\}} \leq \mathcal{B}_{\{1, 2\}, \{1, 3\}, \{2, 3\}} \)

(a3) \( \mathcal{B}_{\emptyset, \{1\}, \{2, 3\}} \leq \mathcal{B}_{\emptyset, \{1, 2\}, \{3\}} \)

(a4) \( \mathcal{B}_{\emptyset, \{1\}, \{2, 3\}} \leq \mathcal{B}_{\{1\}, \{1, 3\}, \{2, 3\}} \)

(a5) \( \mathcal{B}_{\emptyset, \{1\}, \{2, 3\}} \leq \mathcal{B}_{\{1\}, \{1, 3\}, \{2, 3\}} \)

(a1) We need to show that

\[
\alpha_{123} f_0(z) + \frac{u_1 - \rho}{u_1 - d_1} f_{\{1\}}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{\{2\}}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{\{3\}}(z) \\
\leq (\alpha_{123}) f_{\{2, 3\}}(z) + \alpha_{12} f_{\{3\}}(z) + \alpha_{13} f_{\{2\}} + \frac{u_1 - \rho}{u_1 - d_1} f_{\{1\}}(z).
\]

This is equivalent to

\[
0 \leq -\alpha_{123} f_0(z) - \alpha_{123} f_{\{2, 3\}}(z) + \left(\alpha_{12} - \frac{u_2 - \rho}{u_2 - d_2}\right) f_{\{2\}}(z) \\
+ \left(\alpha_{13} - \frac{u_3 - \rho}{u_3 - d_3}\right) f_{\{3\}}(z),
\]

i.e.

\[
0 \leq \alpha_{123} (f_{\{3\}}(z) - f_{\{2, 3\}}(z)) + f_{\{2\}}(z) - f_0(z).
\]

Written in terms of the increments we get

\[
0 \leq \alpha_{123} (\Delta_2 f_{\{2, 3\}}(z) - \Delta_2 f_{\{2\}}(z)).
\]

The term in bracket is positive by the submodular property.

(a2) Secondly, we need to prove the inequality

\[
\alpha_{123} f_0(z) + \frac{u_1 - \rho}{u_1 - d_1} f_{\{1\}}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{\{2\}}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{\{3\}}(z) \\
\leq (\alpha_{123} + 1) f_{\{1\}}(z) + (-\alpha_{12}) f_{\{1, 2\}}(z) + (-\alpha_{13}) f_{\{1, 3\}}(z) - \frac{d_1 - \rho}{u_1 - d_1} f_{\{2, 3\}}(z).
\]

This is equivalent to

\[
0 \leq -\alpha_{123} f_0(z) - \frac{u_2 - \rho}{u_2 - d_2} f_{\{2\}}(z) - \frac{u_3 - \rho}{u_3 - d_3} f_{\{3\}}(z) \\
- \left(\alpha_{123} + \frac{u_3 - \rho}{u_3 - d_3}\right) f_{\{1, 2\}}(z) - \left(\alpha_{123} + \frac{u_2 - \rho}{u_2 - d_2}\right) f_{\{1, 3\}}(z) \\
- \left(\alpha_{123} + \frac{u_2 - \rho}{u_2 - d_2} + \frac{u_3 - \rho}{u_3 - d_3}\right) f_{\{2, 3\}}(z) \\
+ \left(2\alpha_{123} + \frac{u_2 - \rho}{u_2 - d_2} + \frac{u_3 - \rho}{u_3 - d_3}\right) f_{\{1\}}(z),
\]
which after tidying up is

\[ 0 \leq -\alpha_{123}(f_{0}(z) - 2f_{(1)}(z) + f_{(1,2)}(z) + f_{(1,3)}(z) - f_{(2,3)}(z)) \]

\[ - \frac{u_2 - \rho}{u_2 - d_2} (f_{(2)}(z) - f_{(2,3)}(z) - f_{(1)}(z) + f_{(1,3)}(z)) \]

\[ - \frac{u_3 - \rho}{u_3 - d_3} (f_{(3)}(z) - f_{(2,3)}(z) - f_{(1)}(z) + f_{(1,2)}(z)). \]

The first bracket is equal to

\[ (f_{0}(z) - f_{(1)}(z) - f_{(2)}(z) + f_{(1,2)}(z) + f_{(2)}(z) - f_{(2,3)}(z) - f_{(1)}(z) + f_{(1,3)}(z)) \]

\[ = (\Delta_1 f_{(1)}(z) - \Delta_1 f_{(1,2)}(z)) + (\Delta_3 f_{(2,3)}(z) - \Delta_3 f_{(1,3)}(z)). \]

So, written in terms of the increments we get the inequality

\[ 0 \leq -\alpha_{123}(\Delta_1 f_{(1)}(z) - \Delta_1 f_{(1,2)}(z)) + (\Delta_3 f_{(2,3)}(z) - \Delta_3 f_{(1,3)}(z)) \]

\[ - \frac{u_2 - \rho}{u_2 - d_2} (\Delta_3 f_{(2,3)}(z) - \Delta_3 f_{(1,3)}(z)) \]

\[ - \frac{u_3 - \rho}{u_3 - d_3} (\Delta_2 f_{(2,3)} - \Delta_2 f_{(1,2)}(z)). \]

First bracket is negative by the submodular property. The rest is negative by conditions (45) and (40).

(a3) Third, we need to show that

\[ \alpha_{123} f_{0} + \frac{u_1 - \rho}{u_1 - d_1} f_{(1)}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{(2)}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{(3)}(z) \]

\[ \leq (-\alpha_{12}) f_{(1,2)}(z) + (\alpha_{13}) f_{(2)}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{(2,3)}(z) - \frac{d_2 - \rho}{u_2 - d_2} f_{(1)}(z). \]

This is equivalent to

\[ 0 \leq -\alpha_{123} f_{0} - \frac{u_3 - \rho}{u_3 - d_3} f_{(3)}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{(2,3)}(z) + \alpha_{123} f_{(2)}(z) \]

\[ + \left( -\alpha_{123} - \frac{u_3 - \rho}{u_3 - d_3} \right) f_{(1,2)}(z) + \left( \alpha_{123} + \frac{u_3 - \rho}{u_3 - d_3} \right) f_{(1)}(z), \]

which after tidying up is

\[ 0 \leq -\alpha_{123} f_{0}(z) - f_{(1)}(z) - f_{(2)}(z) + f_{(1,2)}(z) \]

\[ - \frac{u_3 - \rho}{u_3 - d_3} (f_{(3)}(z) - f_{(2,3)}(z) - f_{(1)}(z) + f_{(1,2)}(z)). \]

Written in terms of the increments we get

\[ 0 \leq \alpha_{123}(\Delta_1 f_{(1,2)}(z) - \Delta_1 f_{(1)}(z)) + \frac{u_3 - \rho}{u_3 - d_3} (\Delta_2 f_{(1,2)}(z) - \Delta_2 f_{(2,3)}(z)). \]

First bracket is positive by the submodular property and the second one by condition (40).
(a.) We need to prove that
\[
\alpha_{123} f_{\emptyset}(z) + \frac{u_1 - \rho}{u_1 - d_1} f_{(1)}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{(2)}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{(3)}(z)
\]
\[
\leq -\alpha_{13} f_{(1,3)}(z) + (\alpha_{12}) f_{(3)}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{(2,3)}(z) - \frac{d_3 - \rho}{u_3 - d_3} f_{(1)}(z).
\]
This is equivalent to
\[
\alpha_{123} f_{\emptyset}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{(2)}(z) - \left( \alpha_{123} + \frac{u_2 - \rho}{u_2 - d_2} \right) f_{(1)}(z)
\]
\[
\leq - \left( \alpha_{123} + \frac{u_2 - \rho}{u_2 - d_2} \right) f_{(1,3)}(z) + (\alpha_{123}) f_{(3)}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{(2,3)}(z),
\]
which after tidying is
\[
0 \leq \alpha_{123} (f_{(1)}(z) - f_{(1,3)}(z) - f_{\emptyset}(z) + f_{(3)}(z))
\]
\[
+ \frac{u_2 - \rho}{u_2 - d_2} (f_{(1)}(z) + f_{(1,3)}(z) - f_{(2)}(z) + f_{(2,3)}(z)).
\]
Written in terms of the increments we get
\[
0 \leq \alpha_{123} (\Delta_3 f_{(1,3)}(z) - \Delta_3 f_{(3)}(z)) + \frac{u_2 - \rho}{u_2 - d_2} (\Delta_3 f_{(1,3)}(z) - \Delta_3 f_{(2,3)}(z)).
\]
The first bracket is positive by the submodular property and the second one by condition (45).

(a5) finally, we need to prove the inequality
\[
\alpha_{123} f_{\emptyset}(z) + \frac{u_1 - \rho}{u_1 - d_1} f_{(1)}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{(2)}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{(3)}(z)
\]
\[
\leq (-\alpha_{123} - 1) f_{(1,2,3)}(z) - \frac{d_1 - \rho}{u_1 - d_1} f_{(2,3)}(z)
\]
\[
- \frac{d_2 - \rho}{u_2 - d_2} f_{(1,3)}(z) + \frac{d_3 - \rho}{u_3 - d_3} f_{(1,2)}(z).
\]
This is equivalent to
\[
0 \geq \alpha_{123} f_{\emptyset}(z) + \frac{u_1 - \rho}{u_1 - d_1} f_{(1)}(z) + \frac{u_2 - \rho}{u_2 - d_2} f_{(2)}(z) + \frac{u_3 - \rho}{u_3 - d_3} f_{(3)}(z)
\]
\[
- \left( \alpha_{123} + \frac{u_2 - \rho}{u_2 - d_2} + \frac{u_3 - \rho}{u_3 - d_3} \right) f_{(2,3)}(z)
\]
\[
- \left( \alpha_{123} + \frac{u_1 - \rho}{u_1 - d_1} + \frac{u_3 - \rho}{u_3 - d_3} \right) f_{(1,3)}(z)
\]
\[
- \left( \alpha_{123} + \frac{u_1 - \rho}{u_1 - d_1} + \frac{u_2 - \rho}{u_2 - d_2} \right) f_{(1,2)}(z)
\]
\[
- \left( 2\alpha_{123} + \frac{u_2 - \rho}{u_2 - d_2} + \frac{u_3 - \rho}{u_3 - d_3} + \frac{u_1 - \rho}{u_1 - d_1} \right) f_{(1,2,3)}(z),
\]
which after tidying up is

\[
0 \leq \alpha_{123}(-f_0(z) - 2f_{1,2,3}(z) + f_{1,2}(z) + f_{1,3}(z) + f_{2,3}(z)) \\
+ \frac{u_1 - \rho}{u_1 - d_1}(f_{1,2}(z) - f_{1,2,3}(z) - f_{1}(z) + f_{1,3}(z)) \\
+ \frac{u_2 - \rho}{u_2 - d_2}(f_{1,2}(z) - f_{1,2,3}(z) - f_{2}(z) + f_{2,3}(z)) \\
+ \frac{u_3 - \rho}{u_3 - d_3}(f_{1,3}(z) - f_{1,2,3}(z) - f_{3}(z) + f_{2,3}(z)).
\]

The first term is equal to

\[
-f_0(z) - 2f_{1,2,3}(z) + f_{1,2}(z) + f_{1,3}(z) + f_{2,3}(z) \\
= (\Delta_2 f_{1,2,3}(z) - \Delta_2 f_{1,3}(z) + (\Delta_1 f_{1,2}(z) - \Delta_1 f_{1,3}(z)) \\
+ (\Delta_1 f_{1,3}(z) - \Delta_1 f_{1,3}(z)).
\]

Shortly written, in terms of the increments, we get

\[
0 \leq \alpha_{123}((\Delta_2 f_{1,2,3}(z) - \Delta_2 f_{1,2}(z)) + (\Delta_1 f_{1,3}(z) - \Delta_1 f_{1,3}(z)) \\
+ (\Delta_1 f_{1,2,3}(z) - \Delta_1 f_{1,3}(z)) \\
+ \frac{u_1 - \rho}{u_1 - d_1}(\Delta_3 f_{1,2,3}(z) - \Delta_3 f_{1,3}(z)) \\
+ \frac{u_2 - \rho}{u_2 - d_2}(\Delta_3 f_{1,2,3}(z) - \Delta_3 f_{1,3}(z)) \\
+ \frac{u_3 - \rho}{u_3 - d_3}(\Delta_2 f_{1,2,3}(z) - \Delta_2 f_{1,3}(z)).
\]

finally, we could see that all brackets are positive by the submodular property. This completes the proof.

References


