Abstract—In this paper, we use a Markov model to develop a product form solution to efficiently analyze the throughput of arbitrary topology multihop packet radio networks that employ a carrier sensing multiple access (CSMA) protocol with perfect capture. We consider both exponential and nonexponential packet length distributions. Our method preserves the dependence between nodes, characteristic of CSMA, and determines the joint probability that nodes are transmitting. The product form analysis provides the basis for an automated algorithm that determines the maximum throughput in networks of size up to 100 radio nodes. Numerical examples for several networks are presented.

This model has led to many theoretical and practical extensions. These include determination of conditions for product form analysis to hold, extension to other access protocols, and consideration of acknowledgments.

I. INTRODUCTION

Broadcast packet radio networks carry packets of data between nodes equipped with radio transceivers via omnidirectional transmission. If two end users are not within communication range, intermediate radios will rebroadcast (relay) the packet, thus creating a multihop network. Various technologies and routing protocols have been proposed to establish reliable end-to-end paths for maximum network throughput, e.g., [1], [2]. They motivated several analytic and simulation models aimed at the evaluation of the throughput/ delay characteristics of single and multihop environments under different channel access techniques.

In [3], the carrier sense multiple access (CSMA) protocol was analyzed for centralized single-hop models, and in [4], this was extended to two-hop networks. Under CSMA, nodes do not transmit if they sense an ongoing transmission in their neighborhood. In a multihop environment, CSMA is subject to "hidden terminal" interference when two transmitters outside hearing range to each other transmit simultaneously on the same channel, so their transmissions collide. Hidden terminal interference can be eliminated with the busy tone multiple access protocol (BTMA) introduced in [5], in which a node detecting a transmission generates a busy tone to inhibit its neighbors from transmitting. More elaborate access protocols use receiver-directed orthogonal codes or other spread spectrum techniques to reduce the effect of collisions [6].

In analyzing multihop networks, the main difficulty is to account for the dependencies between nodes, even those that do not communicate directly. The node coupling is weaker in slotted ALOHA protocols for which several throughput studies have been reported. They include analyses for randomly dispersed networks [7], symmetric topology networks [8], and arbitrary topology networks [9]. In the latter work, the analysis was based on a discrete Markov chain model and assumed a priori product form state probabilities.

In unslotted CSMA networks, however, a transmitting node has an impact not only on its neighbors (who are not allowed to transmit), but indirectly on nodes several hops away. We show in this paper that under certain assumptions, a Markov model of multihop CSMA can be constructed which leads to product form state probabilities. In addition to the usual assumptions of Poisson scheduling and exponential packet length distribution (later generalized to nonexponential packet length distributions), our model assumes zero propagation delay (any two nodes within range can hear each other instantaneously) and perfect capture (i.e., once a successful transmission is initiated, the reception is immune from any subsequent interference).

The CSMA multihop analysis presented in this paper was first introduced in [10] and [11]. Its main feature is the product form state probabilities which result from its Markov formulation. This property is related to the notion of spatial Markov processes [12]. The problem was related to statistical mechanics definitions in [13]; similar models are found in closed queueing networks, and their computational aspects have been extensively studied, e.g., [14].

The product form Markov formulation has led to several theoretical and practical extensions. In [11], we extended the analysis to include general packet length distributions, and in [15], we presented an algorithmic procedure that is able to evaluate the throughput of CSMA networks on the order of 100 nodes with arbitrary topologies. In [16], we incorporated acknowledgment schemes to the CSMA analysis. In [17], the CSMA model was applied directly to conservative BTMA (nodes that sense a transmission broadcast a busy tone) by redefining the network connectivity to include two-hop links. In [18] and [19], Tobagi and Buzacco studied the underlying reversible Markov process and derived conditions for access protocols to have product form state probabilities. By redefining the state of the process, they were able to analyze additional protocols and to consider one-directional connectivity and nonperfect capture.

Unfortunately, many access protocols used by the packet radio community do not yield product form solutions. For such protocols, realistic size networks can be analyzed by either modifying the model to enforce product forms [19] or by using approximate techniques as in [20] for receiver-directed spread spectrum code division multiple access.

II. PACKET RADIO NETWORKS AND CSMA

A packet radio network is comprised of nodes equipped with radios suitable for broadcasting packetized data over a
limited distance. In general, the source and destination nodes cannot hear each other directly, and the message has to be relayed by one or more intermediate nodes. We assume that the network topology, traffic requirements, and routing do not change, at least for a sufficient period of time to establish steady-state conditions. We will not consider here how the routes are established or updated.

Throughout this paper, we will refer to the quantity $s_0$ as the "desired" rate. It is the rate at which we want packets to be transmitted between nodes $i$ and $j$ where nodes $i$ and $j$ can hear each other directly. In an actual problem, there are traffic requirements between nonadjacent nodes. We assume that a routing policy uniquely determines the desired link flows $s_0$ for the given end-to-end requirements. In general, transmissions of one node can be heard by many other nodes. The routing specifies which node must relay a packet when necessary.

Nodes are neighbors if they can successfully receive each other's transmissions. We assume two-way error-free connectivity. If node 1 can hear node 2, then node 2 can hear node 1. Packets fail to be received only if they collide with other packets. This assumption could be generalized. We assume in our models that nodes cannot receive and transmit simultaneously. Furthermore, they cannot receive more than one packet at a time. If two (or more) transmissions are heard simultaneously by a node (this is called a "collision"), at least one and possibly both transmitted packets are "lost" and must be retransmitted. When collisions occur, the retransmissions are scheduled at each node for a random time sufficiently far into the future so as to avoid repetition of the collisions. For this study, we assume that a packet can be retransmitted as many times as is necessary.

The CSMA protocol inhibits a node from initiating a scheduled transmission if it senses an ongoing transmission in its neighborhood. In this case, the node reschedules the transmission just as it does for collided packets. If at the scheduled transmission time the node is already transmitting a packet, the scheduled packet is rescheduled as above. Thus, packets are continually rescheduled and sometimes transmitted, until they are successfully delivered to the next node along their route.

Collision of transmissions from neighboring nodes may still take place despite the CSMA strategy. There are two ways for this to happen. The first is due to nonzero propagation delay. A collision can occur if a node senses the channel before another node's transmission is received. The effect of this is small when nodes are reasonably close or are not transmitting at high speed, since it depends upon the ratio of propagation delay to packet length. We assume zero propagation delay in this paper, and this assumption is crucial to the model. The second cause of collisions is the hidden terminal effect. Two nodes that cannot hear each other may try to transmit a packet simultaneously to a third node that can hear both.

We will assume that a node's transmissions to its neighbors are scheduled according to a Poisson point process. This implies that packets which either were inhibited from being transmitting or were unsuccessfully transmitted are rescheduled after a sufficiently long randomized time out to preserve the Poisson property. Kleinrock and Lam [21] demonstrated that for the ALOHA protocol, moderate retransmission times result in throughput values close to those predicted by the Poisson scheduling assumption. Hence, we have reason to believe that the discussion below is realistic.

We depict the topology of the network by a graph in which nodes are connected by a link if they can hear each other's transmissions (i.e., if they are neighbors). In Fig. 1, node 1 can hear node 2, but not node 3, node 2 can hear both nodes 1 and 3, and node 3 can hear node 2, but not node 1. If node 1 is transmitting to node 2 and node 3 begins transmitting, then the transmission from 1 to 2 may be lost depending upon the "capture" assumptions made. The conservative assumption that the 1 to 2 transmission is lost is known as zero capture. Alternatively, perfect capture assumes that this transmission is successfully received because the receiver has had time to "lock on" to it. In both cases, however, the 3 to 2 packet is lost. Note that under CSMA, if node 2 is transmitting, neither 1 nor 3 is allowed to transmit. For protocols employing orthogonal codes, a natural locking phenomenon exists that supports the assumption of perfect capture. For CSMA, however, perfect capture is incompatible with most sensing mechanisms. Thus, our assumption of perfect capture is idealistic and results in optimistic results. Nonperfect capture has been analyzed in [19]. The assumption of perfect capture leads to a much simpler model and allows consideration of large networks.

Our model is not valid when a protocol introduces a bias in the length statistics of packets rescheduled because of collisions. Protocols exhibiting this phenomenon are ALOHA and CSMA without capture in which long packets are more vulnerable to collisions, and thus are retransmitted more often [22].

III. THE PACKET RADIO MARKOV MODEL

The multihop nature of the network is modeled via a graph with the radio units (and repeaters) as nodes and links connecting nodes which can directly convey data and status information. Our assumptions as discussed earlier are as follows:

1) Nodes schedule transmissions to neighbors according to independent Poisson point processes.
2) Packet lengths are exponentially distributed and are generated independently at each transmission.
3) The propagation delay between neighboring nodes is zero.
4) Under CSMA, a node will transmit a scheduled packet if a) it is not currently transmitting or receiving a packet, and b) none of its neighbors is transmitting.
5) Nodes receive with perfect capture. A packet will be successfully received if the potential receiver and all of its neighbors are not transmitting at the start of the packet.
6) Links are error free.
7) Acknowledgments are obtained instantaneously.

By assuming independent Poisson scheduling at each node, we ignore the dynamics associated with relaying packets and storing packets in buffers at each node. Essentially, as in early ALOHA analyses, we are scheduling packets and computing how many of them are successfully transmitted and received. The assumption of exponential packet lengths is relaxed later in this paper. The assumptions of zero propagation delay and perfect capture allow us to determine at the scheduling instant...
whether a packet will be transmitted and whether the transmission will be successful. The assumption of error-free links can easily be changed by incorporating a probability of packet loss for each link. The effect of errors in receiving acknowledgments has been included in this model in [16].

With these assumptions, the network can be represented as a continuous time Markov chain, with the state at each time instant being the set of nodes which are transmitting. Certain transitions between states are not allowed because, for example, initiation of a transmission by neighbors of an active transmitter is not permissible under CSMA. Legal transitions occur with fixed exponential rates representing either packet scheduling or packet completion events. Under assumptions (1)–(7) and for given scheduling rates, the steady-state probabilities obey the global balance equations. An iterative procedure is used to solve for the scheduling rates that correspond to given desired link traffic rates, provided that the latter result in stable network operation. In many access protocols, the complexity of the state transitions precludes a systematic and efficient computation along these lines; for CSMA with perfect capture, the network Markov model leads to simple product form state probabilities, and thus allows for efficient computational procedures.

IV. CSMA ANALYSIS—EXPONENTIAL PACKET LENGTHS

In this section, we introduce the basic product form formulation for CSMA networks with perfect capture and exponentially distributed packet lengths.

Let \( i \) be a node, let \( N_i \) be the set of all neighbors of \( i \) (excluding \( i \)), and let \( N_i^e \) be the set of all \( i \)'s neighbors including \( i \). Let \( s_{ij} \) and \( g_{ij} \) denote the desired and scheduled packet rates, respectively, from node \( i \) to \( j \). The scheduled traffic includes successful as well as rescheduled packets, and is assumed to be Poisson. In order to have \( s_{ij} \) successful packets per second, \( g_{ij} > s_{ij} \) must be scheduled. Let \( P(A) \) be the probability that at a random instant all nodes in a set \( A \) are not transmitting (we call this an “idle” condition). For steady-state network operation, \( P(A) \) represents a time average. A scheduled packet from \( i \) to \( j \) will be successful if it finds the system in a state whereby both neighborhoods of \( i \) and \( j \) are idle. Thus, since Poisson arrivals see time averages,

\[
\frac{s_{ij}}{g_{ij}} = P(N_i \cup N_j).
\]

We will now demonstrate how to evaluate the probabilities \( P(A) \) under the Markov assumptions.

Let \( g_i \) be the total scheduling rate out of node \( i \) (assumed Poisson) and let \( 1/\mu_i \) be the average length of packets transmitted from \( i \) (assumed exponentially distributed),

\[
g_i = \sum_{j \in N_i} g_{ij}.
\]

The system can be viewed as a Markov process. The network state for given \( g_i \)'s and \( s_{ij} \)'s is fully described by the set of nodes \( D \) that are actively transmitting (we call this a “busy” condition). All nodes not in \( D \) (in \( D^c \)) are idle. The set of busy nodes provides sufficient information about the legal state transitions which can be either completions of a transmission with exponential rates \( \mu_i \) for \( i \in D \) or new transmissions with rate \( g_i \) for \( j \in D \) and \( j \in N_i, i \in D \). The latter condition results from the CSMA mechanism with zero propagation delay.

Under the rules of CSMA, the nodes in \( D \) are unconnected or, in a graph-theoretic sense, they form an “independent set” of nodes. Let \( N_D \) denote the set of all neighbors of nodes in \( D \) (including \( D \)), and let \( D + j, D - j \) denote independent sets formed by adding a node \( j \) to \( D \) or removing a node \( i \) and \( D \). Assuming that the network operation is stable for a given set of rates \( g_i \) and \( \mu_i \), the steady-state probabilities \( Q(D) \) of states \( D \) obey the global balance equations

\[
\left( \sum_{i \in D} \mu_i + \sum_{j \in N_D} g_{ij} \right) \cdot Q(D) = \sum_{i \in D} g_i Q(D - i) + \sum_{j \in N_D} \mu_j Q(D + j) \quad (3)
\]

It is easy to verify that (3) are consistent with the detailed balance equations

\[
Q(D + j) = \frac{g_{ij}}{\mu_j} Q(D), \quad j \in N_D.
\]

thus,

\[
Q(D) = \left( \prod_{i \in D} \frac{g_i}{\mu_i} \right) Q(\phi) \quad (5)
\]

where \( D \) is an independent set of nodes and \( \phi \) is the null state (i.e., no nodes are transmitting). Normalizing (5), we obtain that

\[
Q(\phi) = \left[ \sum_{i \in D} \prod_{i \in D} \frac{g_i}{\mu_i} \right]^{-1}. \quad (6)
\]

For a steady state to exist, \( \phi \) should be a positive recurrent state (i.e., \( Q(\phi) > 0 \)).

The product form formulation (5), (6) allows us to simplify the throughput evaluations. Still, the identification of independent sets \( D \) and the subsequent normalization (6) constitute NP-complete combinatorial problems. We will show how to take advantage of the structure of the problem to devise an efficient algorithm that enabled us to handle networks of arbitrary topology and moderate size (50–100 nodes).

To find the throughput, we need quantities like \( P(A) \) in (1), the probability that all nodes in a set \( A \) are idle (nodes not in \( A \) may or may not be idle). This can be found by summing \( Q(D) \) over all independent sets \( D \) that do not contain nodes in \( A \). Thus,

\[
P(A) = \sum_{D \subseteq A^c} Q(D) = \sum_{D \subseteq A^c} \left( \prod_{i \in D} \frac{g_i}{\mu_i} \right) \sum_{D \subseteq D} \left( \prod_{i \in D} \frac{g_i}{\mu_i} \right) \quad (7)
\]

where \( D \subseteq A^c \) refers to all independent sets contained in the complement of \( A \). We adopt the shorthand notation

\[
SP(B) = \sum_{D \subseteq B} \left( \prod_{i \in D} \frac{g_i}{\mu_i} \right); \quad SP(\phi) \triangleq 1 \quad (8)
\]

where \( SP \) refers to the sum of products. Thus,

\[
P(A) = \frac{SP(A^c)}{SP(V)} \quad (9)
\]

where \( V \) is the set of all nodes.

Using the \( SP \) notation, (1) becomes

\[
\frac{s_{ij}}{g_{ij}} = \frac{SP([N_i \cup N_j]^c)}{SP(V)}, \quad j \in N_i \quad (10)
\]

Equation (10) can be solved iteratively for the \( g_{ij} \)'s. In practice, the computations can be cumbersome for large networks due to the complexity in evaluating the sums of products in (8). These evaluations are made easier by the following two rules. Consider two sets of nodes \( A \) and \( B \) such
that no node in A can hear any node in B. Then by considering all products, it follows that

$$SP(A \cup B) = SP(A) SP(B).$$

(11)

Also,

$$SP(A) = SP(A - i) + \frac{g_i}{\mu_i} SP(A - N_i), \quad i \in A. \quad (12)$$

Equation (12) is justified by the observation that each independent subset of A either contains node i and none of i’s neighbors or does not contain node i. The terms containing i form \(g_i/\mu_i SP(A - N_i)\). The terms not containing i form \(SP(A - i)\).

More relations involving sums of products are reported in [11]. This algebra of events allows us to evaluate symmetric topologies and asymmetric topologies of modest size by hand. In particular, (12) is the basis for the automated recursive algorithm we have developed for evaluating large asymmetric networks [15].

We illustrate our analysis with the four-node chain of Fig. 1. The independent sets (states under CSMA) are \{\phi\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, and \{2, 4\}. The state transitions are also shown in the figure. The detailed balance equations (4) balance flows between any communicating states. As an example, between states \{1\} and \{1, 4\}, we have

$$\mu_4 Q(1, 4) = g_4 Q(1).$$

The steady-state probabilities are given in product form. For state \{1, 4\},

$$Q(1, 4) = \frac{g_1 g_4}{\mu_1 \mu_4} Q(\phi)$$

or by defining normalized rates \(G_i = g_i/\mu_i\), we have

$$Q(1, 4) = G_1 \cdot G_4 \cdot Q(\phi).$$

Notice that the probability of a state \(D\) is independent of the path from \(\phi\) (or any other state) to \(D\), which is Kolmogorov’s criterion for time reversibility in Markov processes and product form state probabilities [12].

In the same example and for transmission from 1 to 2, (1) and (10) translate into

$$S_{12} = P(1, 2, 3) = P(\phi) = \frac{SP(1, 2, 3)}{SP(1, 2, 3)}$$

and normalizing \(S_{12}/\mu_1\), \(G_{12} = g_{12}/\mu_1\) gives

$$S_{12} = \frac{SP(4)}{SP(1, 2, 3, 4)} = \frac{1 + G_4}{1 + G_1 + G_2 + G_3 + G_4 + G_1 G_2 G_4 + G_1 G_2 G_3 + G_1 G_4 + G_1 G_2 G_3 G_4}.$$

Similar equations can be written for all the transmit/receive pairs in the network. The resulting system of nonlinear equations can be solved iteratively for the unknown rates \(g_j\) in terms of the known desired traffic rates \(s_{ij}\) to determine whether the system can support the desired traffic, and to provide an average measure for the number of retransmissions per node pair. The latter is also a rough measure of the delays that packets experience in the network. The maximum traffic rates which can be supported determine the maximum throughput or capacity of the system.

With equal desired rates, the above equations can be solved directly. Let \(S_{12} = S_{34} = S_{34} = S_{13} = S_{14} = S\), so \(G_{12} = G_{14} = G_{34} = G_{43} = G_4\) and \(G_{23} = G_{12}\) by symmetry. Then

$$\frac{S}{G_{12}} = \frac{S}{G_1} = \frac{P(1, 2, 3)}{G_{12}} = \frac{P(1, 2, 3, 4)}{G_1}.$$

Thus,

$$SP(V) = \sum_{i \in D} \left( \prod_{j \in D} G_{ij} \right) = 1 + G_1 + G_2 + G_3 + G_4$$

and

$$P(1, 2, 3, 4) = \frac{1}{\Delta}, \quad P(1, 2, 3) = \frac{1 + G_4}{\Delta} = \frac{1 + G_1}{\Delta}.$$

Solving, we get

$$G_2 = G_1 (1 + G_1) \quad \text{and} \quad S = G_1 (1 + G_1)/\Delta$$

or

$$S = G_1 (1 + G_1)/(1 + 6G_1 + 7G_1^2 + 2G_1^3).$$

We can now find the maximum rate \(S\) to be 0.128, obtained when \(G_1 = 0.71\).

V. GENERAL PACKET LENGTHS

The node attribute in the previous section was specified by either busy or idle status because both scheduling and packet completion times were assumed memoryless (exponentially distributed). We can easily relax the exponential packet length assumption by modeling the length via an arbitrary combination of exponential stages. Now the node attribute has to also specify the stage of service of a packet under transmission from a busy node. We will show that the product form results of the previous section are still valid.

The method of stages, developed by Cox [23], decomposes arbitrary service length distributions into a combination of exponential servers. Cox demonstrated that any distribution with a rational Laplace transform can be represented by the general Erlang-branched configuration of Fig. 2. Here a node, which initiates a packet transmission, activates the first exponential server with rate \(\mu_1\). With a given probability \(1 - P_1\), the packet terminates transmission at this stage; otherwise, it enters a second exponential server. Since only one packet is transmitted at a time, a busy node will be assigned one out of \(n\) stages or attributes. The average packet transmission time \(1/\mu_1\) equals the average residency within the sequence of stages shown in Fig. 2. Thus,

$$\frac{1}{\mu_1} = \frac{1}{\mu_1} + \frac{P_1}{\mu_2} + \frac{P_2 P_1}{\mu_3} + \cdots + \frac{P_1 P_2 \cdots P_{n-1}}{\mu_n}.$$  (13)

The rational Laplace transform class includes or approximates all distributions found in practice [23]. For example, a single exponential server models exponential packet lengths, and a series of servers (\(P_1 = 1\)) approaches the fixed length distributing as \(n\) increases.

The network with service times decomposed into stages is a Markov process whose state must denote the attributes of all nodes (i.e., which nodes are busy and in what stage of transmission). Let \(D_k\) be the set of busy nodes at stage \(k\). The network state \(E\) is the collection of sets \(D_k\). The set \(D\) of busy nodes regardless of transmission stage is given by \(D = D_1 \cup D_2 \cup \cdots \cup D_k \cup \cdots \). Let \(N_D\) denote the set of all neighbors of nodes in \(D\), including \(D\) just as in Section IV. We will make use of the following notation in order to write state transition (global balance) equations:

\((D_1, \cdots, D_k, \cdots, D_n)\) denotes state \(E\)
The flow out of or exiting from a state. For example, when an idle node begins transmitting, the system moves from state $i$, of state all possible transitions and denoting the stationary probability attribute changes at a single node. Similar reasoning is used in the context of networks of queues [24].

Therefore, they provide the unique solution to local balance equations because they balance flows due to steady-state probabilities $Q(E)$. We observe that the global balance equations (14) can be decomposed into three sets of consistent equations:

$$
\begin{align*}
\mu_j^i Q(E) &= g_i Q(D_{i-1}, D_2, \ldots, D_n) \quad \forall i \in D_1 \\
\mu_j^k Q(E) &= \mu_j^k g_{k-1} P_k^j Q(D_{i-1}, \ldots, D_n), \quad \forall j \in D_k, 2 \leq k \leq n \\
g_m^i Q(E) &= \sum_{k=1}^n (1 - P_k^m) \mu_k^m Q(D_{i-1}, \ldots, D_{k+m}, \ldots, D_n), \quad m \in \mathbb{N}_D.
\end{align*}
$$

Equation (15) balances the flow out of state $E$ due to completion of the first stage of service at node $i$, with the flow into $E$ due to $i$ being activated from an idle condition. Equation (16) isolates transitions from and into $E$ due to a change at a node $j$ at an intermediate stage of service. Equation (17) balances flow out of $E$ due to activation of an idle node $m$, with flow into $E$ due to departures from all service stages at node $m$. Equations (15) and (16) constitute an independent set of equations that can be used to solve for $Q(E)$. They also satisfy (17) and (14), as can be seen by simple algebra. Therefore, they provide the unique solution to (14) for the steady-state probabilities $Q(E)$.

Equations (15)-(17) are called local balance equations because they balance flows due to attribute changes at a single node. Similar reasoning is used in the context of networks of queues [24].

It follows from the local balance equations that the steady-state probabilities $Q(E)$ have a product form, namely,

$$
Q(D_1, D_2, \ldots, D_n) = \prod_{i \in D_1} g_i \prod_{k \in D_k} \frac{g_{k-1}}{\mu_{k-1}^k} \prod_{i \in D_n} \frac{g_{n-1}^i}{\mu_{n-1}^i}.
$$

The stationary probability (ergodic time average) $Q(D)$ that all nodes in a set $D$ are busy regardless of stage of transmission is the sum of probabilities for all states $(D_1, D_2, \ldots, D_n)$ which includes a node $i$ in $D$ at some stage $k$.

$$
Q(D) = \sum_{D = D_1, D_2, \ldots, D_n} Q(D_1, D_2, \ldots, D_n)
$$

or

$$
Q(D) = Q(\phi) \prod_{i \in D} g_i \left[ \frac{1}{\mu_1^i} + \frac{P_1^i}{\mu_2^i} + \frac{P_1^i P_2^i}{\mu_3^i} + \cdots + \frac{P_1^i \ldots P_{n-1}^i}{\mu_n^i} \right].
$$

and from (13)

$$
Q(D) = Q(\phi) \prod_{i \in D} \frac{g_i}{\mu_i^i}.
$$

Equation (18) is identical to (5) obtained for exponential packet lengths. Thus, the stationary probabilities of sets of busy nodes have simple product forms, and depend on the average packet length (possibly different for each node) for all length distribution with a rational Laplace transform. These distributions need not be the same for each node.

Note that for the nonexponential case, the set $D$ in (18) is not a network state. It represents, however, the union of all states in which node $i$ in $D$ are busy; hence, $Q(D)$ denotes the fraction of time during which the set of nodes $D$ is busy. As a result, the product form for evaluations of the exponential case (where $D$ is a Markovian state) and the SP method described in the previous section are directly applicable to general packet lengths. These evaluations depend on the average packet length and not on its distribution.

So far, we have assumed that each node $i$ schedules packets with average length $1/\mu_i$ to all its neighbors. We can relax this assumption and extend the analysis to scenarios whereby a node $i$ transmits packets of different average length to each of its neighbors $j$. This is often the case in environments where some node pairs exchange short packets of control information while others communicate larger packets of data.

Let $g_i$ and $1/\mu_i$ be the scheduling rate and average packet length for the $i$ to $j$ transmission. We keep the structure which led to our generalization by breaking the node $i$ into a set of micronodes, each of which transmits to only one neighbor. Micronodes are connected in the resulting topology if they can hear each other. Micronodes belonging to the same node are fully connected and so are micronodes belonging to communicating nodes in the original topology. As an example, a five-node chain will be decomposed as in Fig. 3. By applying the previous results to the decomposed network, we find that for a successful transmission from 2 to 1,

$$
\frac{S(2)}{g_2} = P(1, 2, 2', 3, 3') = \frac{SP(4, 4', 5)}{SP(V)}
$$
with an increasing number of nodes, approaching
gies and a larger network of randomly generated topology.

computational procedure in Markovian CSMA networks with
length distributions.

node. The analysis depends on the average normalized
transforms which need not be the same for all neighbors of a

rational Laplace transforms (having rational Laplace

advantage of symmetry. In each case, we evaluated the
maximum rates given in Table I.

For a chain, the maximum one-way throughput decreases
with an increasing number of nodes, approaching $S = 0.086$
with more than ten nodes. An upper bound to this limit is 1/5
for the following reason. In CSMA, the transmissions of
neighbors may not overlap in time. Each node must transmit
successfully 2S packets per average packet transmission time S
in either direction. Consider five consecutive time intervals,
equal to packet transmission times. Two adjacent nodes
require four distinct intervals to transmit successfully in both
directions. Their neighbors can transmit inward simultaneously
in the fifth interval. Their outward transmission can
overlap with some of the first four intervals. Thus, we may
have at the most one success (in each direction) for every five
packet transmission intervals. However, the maximum
throughput is slightly smaller than 1/10, half of this bound,
due to collisions from transmissions two hops away, the
"hidden terminals." The maximum throughput of $S = 0.086$
for a chain is not a useful operating point. As in ALOHA, this
is the point at which delays become infinite and the system
becomes unstable. The network actually would have to be
operated at some lower throughput level.

It is instructive to compare the performance of a long chain
under CSMA to that of slotted ALOHA. Let $p$ be the
probability of a node’s transmission in one direction under
slotted ALOHA. Then $S = p(1 - 2p)^2$ with maximum value
0.074, approximately 14 percent less than that for CSMA.
There are two factors working here. CSMA will produce
fewer collisions since neighbors will not interfere with each
other (hidden terminals will still produce some collisions); on
the other hand, some potentially successful transmissions may
be prohibited. For instance, node 3 in Fig. 4(a) could transmit

Fig. 3. Micronode decomposition of a five-node chain.

where

$$SP(4, 4', 5) = 1 + \frac{g_{43}}{\mu_{43}} + \frac{g_{45}}{\mu_{45}} + \frac{g_{54}}{\mu_{54}}.$$  

Now define $G_i = g_{ij}/\mu_{ij}$ and $G_i = g_{ji}/\mu_{ji}$; in other

words, let $G_i$ be the average normalized scheduling rate of

node $i$ given by

$$G_i = \sum_j \frac{g_{ij}}{\mu_{ij}}. \quad (19)$$

Performing similar calculations for $SP(V)$, we can see that the
results are identical as in the original topology before

decomposition, but with $g_i/\mu_i$ replaced by (19). In general,

consider $SP(B^*)$ where $B^*$ is the set of micronodes associated

with the set of nodes $B$. Let node $i$ be in $B$, and denote its

micronodes by $\{ij\}$ where $j$ is a neighbor of $i$. Using (12),

we have

$$SP(B^*) = SP(B^* - \{i,j\}) + \frac{g_{ij}}{\mu_{ij}} SP(B^* - N^*_j), \quad (20)$$

where $N^*_j$ is the set of micronodes neighbors of $\{ij\}$ including

$\{ij\}$. Note because of the connectivity of micronodes, $N^*_j$

for a given $i$ is the same for all $j$, namely, the set of micronodes of

$N_i, N^*_j$. Let $i^*$ be the set of micronodes of $i$. Proceeding

recursively, we have

$$SP(B^*) = SP(B^* - i^*) + \sum_j \frac{g_{ij}}{\mu_{ij}} SP(B^* - N^*).$$

Using (20) and repeating the above process, we find that (5)

still holds for the original nodes, but with $g_i/\mu_i$ replaced by $G_i$
given in (19).

We summarize our results in the following theorem.

*Theorem:* The product form throughput evaluation and the
computational procedure in Markovian CSMA networks with
perfect capture and exponential packet length hold for arbitrary
packet length distributions (having rational Laplace transforms)
which need not be the same for all neighbors of a

node. The analysis depends on the average normalized
scheduling rates of nodes, independent of particular packet
length distributions.

**VI. Numerical Examples**

We consider here three simple examples of specific topologies
and a larger network of randomly generated topology.

The three topologies shown in Fig. 4 are a chain, a ring, and
a star for which we assume all $S_{ij} = S$ for all $i, j$ and take full
advantage of symmetry. In each case, we evaluated the
maximum rate $S$ given in Table I.

For a chain, the maximum one-way throughput decreases
with an increasing number of nodes, approaching $S = 0.086$
with more than ten nodes. An upper bound to this limit is 1/5

for the following reason. In CSMA, the transmissions of
neighbors may not overlap in time. Each node must transmit
successfully 2S packets per average packet transmission time S
in either direction. Consider five consecutive time intervals,
equal to packet transmission times. Two adjacent nodes
require four distinct intervals to transmit successfully in both
directions. Their neighbors can transmit inward simultaneously
in the fifth interval. Their outward transmission can
overlap with some of the first four intervals. Thus, we may
have at the most one success (in each direction) for every five
packet transmission intervals. However, the maximum
throughput is slightly smaller than 1/10, half of this bound,
due to collisions from transmissions two hops away, the
"hidden terminals." The maximum throughput of $S = 0.086$
for a chain is not a useful operating point. As in ALOHA, this
is the point at which delays become infinite and the system
becomes unstable. The network actually would have to be
operated at some lower throughput level.

It is instructive to compare the performance of a long chain
under CSMA to that of slotted ALOHA. Let $p$ be the
probability of a node’s transmission in one direction under
slotted ALOHA. Then $S = p(1 - 2p)^2$ with maximum value
0.074, approximately 14 percent less than that for CSMA.
There are two factors working here. CSMA will produce
fewer collisions since neighbors will not interfere with each
other (hidden terminals will still produce some collisions); on
the other hand, some potentially successful transmissions may
be prohibited. For instance, node 3 in Fig. 4(a) could transmit

![Fig. 3. Micronode decomposition of a five-node chain.](image1)

![Fig. 4. Simple network topologies.](image2)

**TABLE I**

**MAXIMUM ONE-WAY THROUGHPUT PER LEG (S) FOR A CHAIN OR RING, S AND N REFER TO THE ENTIRE NETWORK**

<table>
<thead>
<tr>
<th>Number of Nodes N</th>
<th>Chain</th>
<th>Ring</th>
<th>Star</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Legs, L</td>
<td>L=2</td>
<td>L=3</td>
<td>L=4</td>
</tr>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>0.167</td>
</tr>
<tr>
<td>2</td>
<td>0.500</td>
<td>0.500</td>
<td>0.111</td>
</tr>
<tr>
<td>3</td>
<td>0.167</td>
<td>0.167</td>
<td>0.097</td>
</tr>
<tr>
<td>4</td>
<td>0.058</td>
<td>0.073</td>
<td>0.092</td>
</tr>
<tr>
<td>5</td>
<td>0.097</td>
<td>0.100</td>
<td>0.106</td>
</tr>
<tr>
<td>6</td>
<td>0.086</td>
<td>0.083</td>
<td>0.086</td>
</tr>
<tr>
<td>7</td>
<td>0.094</td>
<td>0.085</td>
<td>0.086</td>
</tr>
<tr>
<td>8</td>
<td>0.094</td>
<td>0.085</td>
<td>0.086</td>
</tr>
<tr>
<td>9</td>
<td>0.091</td>
<td>0.086</td>
<td>0.086</td>
</tr>
<tr>
<td>10</td>
<td>0.091</td>
<td>0.086</td>
<td>0.086</td>
</tr>
<tr>
<td>∞</td>
<td>0.086</td>
<td>0.083</td>
<td>0.086</td>
</tr>
</tbody>
</table>

![Fig. 3. Micronode decomposition of a five-node chain.](image1)

![Fig. 4. Simple network topologies.](image2)
successfully to node 2 while node 4 is transmitting successfully to node 5. This is possible in ALOHA, but prohibited in CSMA (the price paid to control collisions).

For a ring with more than seven nodes, the maximum throughput is the same as that for a long chain. This is expected since the congestion is now in the middle so it is unimportant whether or not the chain is closed.

Now consider the star configuration in Fig. 4(c) as representing the center node (0) trying to transmit to some node or nodes far away via many repeaters. The results are shown in Table II where the throughput of the center node is given by LS. The maximum throughput increases with the number of legs, but for $L \geq 5$, there is no further improvement. Congestion at the central node is limiting its ability to increase its throughput. To reduce this congestion, we first considered connecting the first level nodes in a ring and then fully connecting them. These results are also summarized in Table II. When the first level is unconnected, the throughput saturates at 0.228. When the first level is fully connected, the maximum throughput with nine legs is 0.252, a 15 percent increase. For four legs, the ring-connected topology is best, providing some compromise between reducing collisions at the center and allowing simultaneous transmissions along the legs.

For larger networks, we developed an automated algorithm based on the properties presented in this paper. It generates and stores symbolically the terms in the SP expressions (8) using the recursion (12). The resulting SP table is used to iteratively solve (10) for the $a_i$ that satisfy given $s_i$ rates. For rates sufficiently below the maximum throughput (network capacity), we found that the iteration converges monotonically and rapidly. As the rates increase toward the network capacity, the convergence is still monotonic, but slows appreciably. For rates higher than capacity, the iteration does not converge and often diverges dramatically. The algorithm can uniformly scale the desired rates up to the maximum for which the iteration converges in order to determine the capacity of the network (under fixed proportions of desired rates or end-to-end requirements). We have uncovered no serious numerical problems with the procedure in the many examples that we evaluated. Details on the properties of the algorithm can be found in [15].

We illustrate the automated algorithm in terms of the 30-node network of Fig. 5. The nodes were placed at random in a 100 $\times$ 100 square. Connectivity was established by considering a transmitting range of 35 units, the same for all nodes. This resulted in 256 links. We assumed uniform end-to-end requirements and packet lengths normalized to 1. The link-desired rates were determined by routing requirements via minimum hop paths. The network generation and routing are imbedded within the automated algorithm. The SP table for this example included 362 terms symbolically stored and generated within 15 s on an IBM PC/XT with 8087 coprocessor. The maximum throughput was determined by performing a binary search for the factor that uniformly scales the end-to-end requirements to yield the maximum desired link rates. The search considered nine scale factors. For each factor, convergence of iteration (10) was checked. The number of iterations performed for each factor varied from 8 to 75, depending on whether rates were feasible or not, and if feasible, how close to the maximum rates. Overall, the algorithm performed 268 iterations and determined in 2:30 min that the maximum rate for each end-to-end requirement is 0.00065 packets/s. With 30 $\times$ 29 identical requirements, the network capacity is 0.00065 $\times$ 30 $\times$ 29 = 0.5655 packets/s.

VII. CONCLUSIONS

In this paper, we presented a methodology for the steady-state throughput analysis of CSMA multihop packet radio networks with perfect capture. Under the assumptions of zero propagation delay and independent Poisson scheduling rates, we derived product form state probabilities for the underlying Markov processes. The formulation holds for packet lengths distributed with distributions that have rational Laplace transforms. The analysis includes the dependencies that hidden terminals exercise on transmissions in networks with arbitrary connectivity and provides throughput evaluations under various topologies and static routing plans.

The basic algorithm was applied to simple chain, star, and ring networks to demonstrate how the analysis can be used to evaluate maximum throughput, to compare various topologies, and to understand the impact of increasing connectivity to alleviate hidden terminal interference. For larger networks, we have developed an automated algorithm that we illustrated by evaluating the maximum throughput of a 30-node randomly generated network.

REFERENCES


[25] R. Boorstyn has been Secretary of the IEEE Information Theory Group, Editor of Computer Communication of the IEEE TRANSACTIONS ON COMMUNICATIONS, Chairman of the Computer Communications Committee of the IEEE Communications Society, and a member of the Steering Committee of the IEEE INFOCOM Conferences. He is Associate Editor of the Networks journal. From 1972 to 1973 he was a participant in the IEEE Outstanding Lecture Tours Program. He was a member of the delegation to the first joint USSR-IEEE Workshop on Information Theory in Moscow in 1975.

Veli Sahin (S'77-M'82) was born in Elazığ, Turkey, in 1953. He received the B.S. degree in electronics and communications engineering from the Technical University of Istanbul, Turkey, in 1975, the M.S. degree in computer science, the M.S. degree in electrical engineering, and the Ph.D. degree in electrical engineering and computer science from the Polytechnic Institute of New York, Brooklyn, in 1980, and 1981, respectively.

While at Polytechnic, he was a Research Assistant in the Department of Electrical Engineering and Computer Science, where his research activities focused on modeling and analysis of multihop packet radio networks. From 1981 to 1985 he was a member of the Technical Staff at AT&T Bell Laboratories, Holmdel, NJ, and engaged in digital network synchronization, modeling and analysis of integrated network access architectures, ISDN architectures, and specifications and analysis of digital switching and transmission terminals in telephone networks. In March 1985 he joined Bell Communications Research, Red Bank, NJ, where he is a member of the Technical Staff in the Transmission Network Operations Division. He is responsible for the development of generic network elements functions and OSI protocol architectures for operations data networking, and ISDN basic access U interface specifications and data networking.

Dr. Sahin is a member of IEEE Computer and Communications Societies.