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The applied geometry of a general Liénard polynomial system✩

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A R T I C L E   I N F O

Article history:
Received 14 May 2012
Accepted 27 June 2012

Keywords:
General Liénard polynomial system
Field rotation parameter
Bifurcation
Singular point
Limit cycle

A B S T R A C T

In this work, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve the limit cycle problem for a general Liénard polynomial system with an arbitrary (but finite) number of singular points.

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doi:10.1016/j.aml.2012.06.026

1. Introduction

We consider Liénard equations
\[ \ddot{x} + f(x) \dot{x} + g(x) = 0 \] (1.1)
and the corresponding dynamical systems in the form
\[ \dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \] (1.2)

There are many examples in the natural sciences and technology in which this and related systems are applied [1–10]. Such systems are often used to model mechanical, electrical, or biomedical systems, and in the literature, many systems are transformed into ones of Liénard type to aid in investigations. They can be used, e.g., for certain mechanical systems, where \( f(x) \) represents a coefficient of the damping force and \( g(x) \) represents the restoring force or stiffness, when modeling wind–rock phenomena and surge in jet engines [2,8]. Such systems can also be used to model resistor–inductor–capacitor circuits with nonlinear circuit elements. Recently, for example, the Liénard system (1.2) has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunneling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator [10]. There are also some examples of using Liénard type systems in ecology and epidemiology [7].

In this work, we suppose that system (1.2), where \( f(x) \) and \( g(x) \) are arbitrary polynomials of \( x \), has an anti-saddle (a node or a focus, or a center) at the origin and write it in the form
\[ \dot{x} = y, \quad \dot{y} = -x (1 + \beta_1 x + \cdots + \beta_{2k} x^{2k}) + y (\alpha_0 + \alpha_1 x + \cdots + \alpha_{2k} x^{2k}). \] (1.3)

✩ This work was supported by the Netherlands Organization for Scientific Research (NWO). The author is also very grateful to the Max Planck Institute for Mathematics (Bonn) and the Johann Bernoulli Institute for Mathematics and Computer Science (Groningen) for hospitality and support during his stay in 2011–2012.

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doi:10.1016/j.aml.2012.06.026
2. Limit cycles of the general Liénard polynomial system

By means of our bifurcationally geometric approach [11–13], we will study the Liénard polynomial system (1.3). Its finite singularities are determined by the algebraic system

\[ x(1 + \beta_1 x + \cdots + \beta_{2l} x^{2l}) = 0, \quad y = 0. \]  

(2.1)

It always has an anti-saddle at the origin and, in general, can have at most \(2l+1\) finite singularities which lie on the \(x\)-axis and are distributed such that a saddle (or saddle–node) is followed by a node or a focus, or a center, and vice versa [1].

At infinity, system (1.3) has two singular points: a node at the "ends" of the \(x\)-axis and a saddle at the "ends" of the \(y\)-axis. For studying the infinite singularities, the methods applied in [1] for Rayleigh’s and van der Pol’s equations and also Erugin’s two-isocline method developed in [11] can be used (see [12,13]).

Following [11], we will study limit cycle bifurcations of (1.3) by means of a canonical system containing field rotation parameters of (1.3) [1,11].

**Theorem 2.1.** The Liénard polynomial system (1.3) with limit cycles can be reduced to the canonical form

\[ \dot{x} = y \equiv P(x, y), \quad \dot{y} = -x(1 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_{2l} x^{2l}) + y(\alpha_0 + x + \alpha_2 x^2 + \cdots + \alpha_{2k} x^{2k}) \equiv Q(x, y), \]  

(2.2)

where \(\beta_1, \beta_3, \ldots, \beta_{2l-1}\) are fixed and \(\alpha_0, \alpha_2, \ldots, \alpha_{2k}\) are field rotation parameters of (2.2).

**Proof.** Let all the parameters \(\alpha_i, i = 0, 1, \ldots, 2k\), vanish in system (2.2),

\[ \dot{x} = y, \quad \dot{y} = -x(1 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_{2l} x^{2l}), \]  

(2.3)

and consider the corresponding equation

\[ \frac{dy}{dx} = -x(1 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_{2l} x^{2l}) \equiv F(x, y). \]  

(2.4)

Since \(F(x, -y) = -F(x, y)\), the direction field of (2.4) is symmetric (as also is the vector field of (2.3)) with respect to the \(x\)-axis. It follows that for arbitrary values of the parameters \(\beta_j, j = 1, 2, \ldots, 2l\), system (2.3) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, all the even parameters \(\beta_j\) of system (1.3) can be supposed to be equal, to, e.g., \(\pm 1\): \(\beta_2 = \beta_4 = \beta_6 = \cdots = \pm 1\).

Let now all the parameters \(\alpha_i\) with even indexes and \(\beta_j\) with odd indexes vanish in system (2.2),

\[ \dot{x} = y, \quad \dot{y} = -x(1 \pm x^2 \pm \cdots \pm x^{2l}) + y(\alpha_1 x + \alpha_3 x^3 + \cdots + \alpha_{2k-1} x^{2k-1}), \]  

(2.5)

and consider the corresponding equation

\[ \frac{dy}{dx} = -x(1 \pm x^2 \pm \cdots \pm x^{2l}) + y(\alpha_1 x + \alpha_3 x^3 + \cdots + \alpha_{2k-1} x^{2k-1}) \equiv G(x, y). \]  

(2.6)

Since \(G(-x, y) = -G(x, y)\), the direction field of (2.6) is symmetric (as also is the vector field of (2.5)) with respect to the \(y\)-axis. It follows that for arbitrary values of the parameters \(\alpha_1, \alpha_3, \ldots, \alpha_{2k-1}\), system (2.3) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, all the odd parameters \(\alpha_i\) of system (1.3) can be supposed to be equal, to, e.g., 1: \(\alpha_1 = \alpha_3 = \cdots = \alpha_{2k-1} = 1\).

Inputting the odd parameters \(\beta_1, \beta_3, \ldots, \beta_{2l-1}\) into system (2.5),

\[ \dot{x} = y \equiv R(x, y), \quad \dot{y} = -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \cdots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y(x + \cdots + x^{2k-1}) \equiv S(x, y), \]  

(2.7)

and calculating the determinants

\[ \Delta_{\beta_1} = R_S'_{\beta_1} - SR'_{\beta_1} = -x^2 y, \]  
\[ \Delta_{\beta_3} = R_S'_{\beta_3} - SR'_{\beta_3} = -x^2 y, \]  
\[ \ldots, \Delta_{\beta_{2l-1}} = R_S'_{\beta_{2l-1}} - SR'_{\beta_{2l-1}} = -x^2 y, \]

we can see that the vector field of (2.7) is rotated symmetrically (in opposite directions) with respect to the \(x\)-axis and that the finite singularities (centers and saddles) of (2.7) moving along the \(x\)-axis (except the center at the origin) do not change their type or join in saddle–nodes. Therefore, we can fix the odd parameters \(\beta_1, \beta_3, \ldots, \beta_{2l-1}\) in system (2.2), fixing the position of its finite singularities on the \(x\)-axis.
To prove that the even parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ rotate the vector field of (2.2), let us calculate the following determinants:

$$
\Delta_{\alpha_0} = P Q'_{\alpha_0} - Q P'_{\alpha_0} = y^2 \geq 0,
$$
$$
\Delta_{\alpha_2} = P Q'_{\alpha_2} - Q P'_{\alpha_2} = x^2 y^2 \geq 0, \ldots, \Delta_{\alpha_{2k}} = P Q'_{\alpha_{2k}} - Q P'_{\alpha_{2k}} = x^{2k} y^2 \geq 0.
$$

By the definition of a field rotation parameter [1,11], for increasing each of the parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$, with the others fixed, the vector field of system (2.2) is rotated in the positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.2) is rotated in the negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (1.3), it is sufficient to consider the canonical system (2.2) containing only its even parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ which rotate the vector field of (2.2), with the others fixed. The theorem is proved. □

By means of the canonical system (2.2), let us study global limit cycle bifurcations of (1.3) and prove the following theorem.

**Theorem 2.2.** The general Liénard polynomial system (1.3) can have at most $k + l$ limit cycles, $k$ surrounding the origin and $l$ surrounding one by one the other singularities of (1.3).

**Proof.** According to Theorem 2.1, for the study of limit cycle bifurcations of system (1.3), it is sufficient to consider the canonical system (2.2) containing the field rotation parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ of (1.3) with the parameters $\beta_1, \beta_3, \ldots, \beta_{2l-1}$ fixed.

Let all of these parameters vanish:

$$
\dot{x} = y, \quad \dot{y} = -x (1 \pm x^2 \pm \cdots \pm x^l) + y (x + x^3 + \cdots + x^{2k-1}).
$$

System (2.8) is symmetric with respect to the $y$-axis and has centers as anti-saddles. Its center domains are bounded by either separatrix loops or digons of the saddles of (2.8) lying on the $x$-axis. If we input the parameters $\beta_1, \beta_3, \ldots, \beta_{2l-1}$ into (2.8) successively, we will get again system (2.7), the vector field of which is rotated symmetrically (in opposite directions) with respect to the $x$-axis. The finite singularities (centers and saddles) of (2.7) moving along the $x$-axis (except the center at the origin) do not change their type or join in saddle–nodes and the center domains will be bounded by separatrix loops of the saddles (or saddle–nodes) of (2.7) [1,11].

Let us input successively the field rotation parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ into system (2.7) beginning with the parameters at the highest degrees of $x$ and alternating with their signs (see [12,13]). So, begin with the parameter $\alpha_{2k}$ and let, for definiteness, $\alpha_{2k} > 0$:

$$
\dot{x} = y, \quad \dot{y} = -x (1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \cdots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y (x + x^3 + \cdots + x^{2k-1} + \alpha_{2k} x^{2k}).
$$

In this case, the vector field of (2.9) is rotated in the positive direction (counterclockwise) turning the center at the origin into a nonrough (weak) unstable focus. All the other centers become rough unstable foci [1,11].

Fix $\alpha_{2k}$ and input the parameter $\alpha_{2k-2} < 0$ into (2.9):

$$
\dot{x} = y, \quad \dot{y} = -x (1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \cdots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y (x + x^3 + \cdots + x^{2k-2} + \alpha_{2k} x^{2k})
$$

Then the vector field of (2.10) is rotated in the opposite direction (clockwise) and the focus at the origin immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of $x$ changes), generating a stable limit cycle. All the other foci will also generate stable limit cycles for some values of $\alpha_{2k-2}$ after changing the character of their stability. On further decreasing $\alpha_{2k-2}$, all the limit cycles will expand, disappearing on separatrix cycles of (2.10) [1,11].

Denote the limit cycle surrounding the origin by $\Gamma_1$, the domain outside the cycle by $D_1$, and the domain inside the cycle by $D_2$ and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a “trajectory concentration” surrounding this singular point. It is clear that, for decreasing parameter $\alpha_{2k-2}$, a semi-stable limit cycle cannot appear in the domain $D_2$, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [12,13].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain $D_1$. Suppose it appears in this domain for some values of the parameters $\alpha_{2k} > 0$ and $\alpha_{2k-2} < 0$. Return to system (2.7) and change the inputting order for the field rotation parameters. Input first the parameter $\alpha_{2k-2} < 0$:

$$
\dot{x} = y, \quad \dot{y} = -x (1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \cdots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y (x + x^3 + \cdots + \alpha_{2k-2} x^{2k-2} + x^{2k-1}).
$$

Fix it, with the condition $\alpha_{2k-2} = \alpha_{2k-2}'$. The vector field of (2.11) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter $\alpha_{2k} > 0$ into (2.11), we get again system (2.10), the vector field of which is rotated...
Under this rotation, a stable limit cycle \( \Gamma_1 \) will appear from a separatrix cycle for some value of \( \alpha_{2k} \). This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase for increasing \( \alpha_{2k} \) to the value \( \alpha_{2k}^* \). It follows that there are no values of \( \alpha_{2k-2}^* < 0 \) and \( \alpha_{2k}^* > 0 \) for which a semi-stable limit cycle could appear in the domain \( D_1 \).

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.10) for any values of the parameters \( \alpha_{2k-2} \) and \( \alpha_{2k} \) of different signs. Obviously, if these parameters have the same sign, system (2.10) has no limit cycles surrounding the origin at all. For the same reason, this system cannot have more than \( l \) limit cycles surrounding the other singularities (foci or nodes) of (2.10) one by one.

Let system (2.10) have the unique limit cycle \( \Gamma_1 \) surrounding the origin and \( l \) limit cycles surrounding the other anti-saddles of (2.10). Fix the parameters \( \alpha_{2k} > 0 \), \( \alpha_{2k-2} < 0 \) and input the third parameter, \( \alpha_{2k-4} > 0 \), into this system:

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x(1 + \beta_1 x^2 \pm \beta_3 x^4 + \cdots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y (x + x^3 + \cdots + \alpha_{2k-4} x^{2k-4} + \alpha_{2k-2} x^{2k-2} + x^{2k-1} + \alpha_{2k} x^{2k}).
\end{align*}
\] (2.12)

The vector field of (2.12) is rotated counterclockwise. Under this rotation, a stable limit cycle \( \Gamma_1 \) will appear from a separatrix cycle for some value of \( \alpha_{2k} \). This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase for increasing \( \alpha_{2k} \) to the value \( \alpha_{2k}^* \). It follows that there are no values of \( \alpha_{2k-2} < 0 \) and \( \alpha_{2k} < 0 \) for which a semi-stable limit cycle could appear in the domain \( D_1 \).

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.10) for any values of the parameters \( \alpha_{2k-2} \) and \( \alpha_{2k} \) of different signs. Obviously, if these parameters have the same sign, system (2.10) has no limit cycles surrounding the origin at all. For the same reason, this system cannot have more than \( l \) limit cycles surrounding the other singularities (foci or nodes) of (2.10) one by one.

Let system (2.10) have the unique limit cycle \( \Gamma_1 \) surrounding the origin and \( l \) limit cycles surrounding the other anti-saddles of (2.10). Fix the parameters \( \alpha_{2k} > 0 \), \( \alpha_{2k-2} < 0 \) and input the third parameter, \( \alpha_{2k-4} > 0 \), into this system:

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x(1 + \beta_1 x^2 \pm \beta_3 x^4 + \cdots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y (x + x^3 + \cdots + \alpha_{2k-4} x^{2k-4} + \alpha_{2k-2} x^{2k-2} + x^{2k-1} + \alpha_{2k} x^{2k}).
\end{align*}
\] (2.13)

The two parameters act in a similar way: they rotate the vector field of (2.13) counterclockwise, turning the origin into a nonrough unstable focus.

Fix these parameters with \( \alpha_{2k-4} = \alpha_{2k-2}^* \), \( \alpha_{2k} = \alpha_{2k}^* \) and input the parameter \( \alpha_{2k-2} < 0 \) into (2.13), getting again system (2.12). Since, by our assumption, this system has two limit cycles surrounding the origin for \( \alpha_{2k-2} > \alpha_{2k-2}^* \), there exists some value of the parameter, \( \alpha_{2k-2}^* \), for which a semi-stable limit cycle, \( \Gamma_1 \), appears in system (2.12) and then splits into a stable cycle, \( \Gamma_1 \), and an unstable cycle, \( \Gamma_2 \), for further decreasing \( \alpha_{2k-2} \). The domain \( D_2 \) formed, bounded by the limit cycles \( \Gamma_1 \) and \( \Gamma_2 \) and filled by the spirals, will enlarge since, by the properties of a field rotation parameter, the interior unstable limit cycle \( \Gamma_2 \) will contract and the exterior stable limit cycle \( \Gamma_1 \) will expand for decreasing \( \alpha_{2k-2} \). The distance between the spirals of the domain \( D_2 \) will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for \( \alpha_{2k-2} < \alpha_{2k-2}^* \) [12,13].

Thus, there are no such values of the parameters, \( \alpha_{2k-2}^* > 0 \), \( \alpha_{2k-2}^* < 0 \), and \( \alpha_{2k-4}^* > 0 \), for which system (2.12) would have an additional semi-stable limit cycle surrounding the origin. Obviously, there are no other values of the parameters \( \alpha_{2k} \), \( \alpha_{2k-2} \), and \( \alpha_{2k-4} \) for which system (2.12) would have more than two limit cycles surrounding this singular point. For the same reason, additional semi-stable limit cycles cannot appear around the other singularities (foci or nodes) of (2.12). Therefore, \( 2 + l \) is the maximum number of limit cycles in system (2.12).

Suppose that system (2.12) has two limit cycles, \( \Gamma_1 \) and \( \Gamma_2 \), surrounding the origin and \( l \) limit cycles surrounding the other anti-saddles of (2.12) (this is always possible if \( \alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} > 0 \)). Fix the parameters \( \alpha_{2k} \), \( \alpha_{2k-2} \), \( \alpha_{2k-4} \) and consider a more general system inputting the fourth parameter, \( \alpha_{2k-6} < 0 \), into (2.12):

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x(1 + \beta_1 x^2 \pm \beta_3 x^4 + \cdots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y (x + x^3 + \cdots + \alpha_{2k-4} x^{2k-4} + \alpha_{2k-2} x^{2k-2} + x^{2k-1} + \alpha_{2k} x^{2k}).
\end{align*}
\] (2.14)

For decreasing \( \alpha_{2k-6} \), the vector field of (2.14) will be rotated clockwise and the focus at the origin will immediately change the character of its stability, generating a third (stable) limit cycle, \( \Gamma_3 \). With further decreasing \( \alpha_{2k-6} \), \( \Gamma_3 \) will join with \( \Gamma_2 \) forming a semi-stable limit cycle, \( \Gamma_2 \), which will disappear in a "trajectory concentration" surrounding the origin; the cycle \( \Gamma_1 \) will expand, disappearing on a separatrix cycle of (2.14).

Let system (2.14) have three limit cycles surrounding the origin: \( \Gamma_1 \), \( \Gamma_2 \), \( \Gamma_3 \). Could an additional semi-stable limit cycle appear with decreasing \( \alpha_{2k-6} \) after whose splitting system (2.14) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain \( D_2 \) bounded by the cycles \( \Gamma_1 \) and \( \Gamma_2 \) or in the domain \( D_3 \) bounded by the origin and \( \Gamma_2 \) because of the increasing distance between the spiral coils filling these domains after decreasing \( \alpha_{2k-6} \). Consider two other domains: \( D_1 \) bounded on the inside by the cycle \( \Gamma_2 \) and \( D_3 \) bounded by the cycles \( \Gamma_2 \) and \( \Gamma_3 \). As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.
Suppose that for some set of values of the parameters \( \alpha \) there exists a semi-stable cycle. Return to system (2.7) again; input first the parameters \( \alpha < 0 \), \( \alpha < 0 \), and then the parameter \( \alpha > 0 \):

\[
\dot{x} = y, \quad \dot{y} = -x + \beta_1 x \pm \alpha_1 y^2 + \ldots + \beta_{2l-1} x^{2l-1} \pm x^{2l}
\]

and

\[
\alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \ldots \alpha_{2k-2} \alpha_{2k-4} \alpha_{2k-6} \alpha_{2k-8} \ldots
\]

Fix the parameters \( \alpha_{2k-6} \), \( \alpha_{2k-2} \) as the values \( \alpha_{2k-6}^\ast \), \( \alpha_{2k-2}^\ast \), respectively. With increasing \( \alpha \), a separatrix cycle formed around the origin will generate a stable limit cycle, \( \Gamma \). Fix \( \alpha \) as the value \( \alpha_{2k}^\ast \) and input the parameter \( \alpha > 0 \) into (2.15), getting system (2.14).

Since, by our assumption, (2.14) has three limit cycles for \( \alpha < \alpha_4 \), there exists some value of the parameter \( \alpha_{2k-4} \) (\( \alpha < \alpha_{2k-4} \)) for which a semi-stable limit cycle, \( \Gamma_{2k} \), appears in this system and then splits into an unstable cycle, \( \Gamma_2 \), and a stable cycle, \( \Gamma_3 \), with further increasing \( \alpha \). The domain \( D_3 \) formed, bounded by the limit cycles \( \Gamma_2 \), \( \Gamma_3 \), and, also, the domain \( D_1 \) bounded on the inside by the limit cycle \( \Gamma_1 \) will enlarge and the spirals filling these domains will untwist, excluding the possibility of the appearance of a semi-stable limit cycle there [12, 13].

All other combinations of the parameters \( \alpha_{2k} \), \( \alpha_{2k-2} \), \( \alpha_{2k-4} \), and \( \alpha_{2k-6} \) are considered in a similar way. It follows that system (2.14) can have at most \( 3 + l \) limit cycles.

If we continue the procedure of successive inputting of the even parameters, \( \alpha_2 \), \( \ldots \), \( \alpha_4 \), \( \alpha_0 \), into system (2.7), it is possible first to obtain \( k \) limit cycles surrounding the origin (\( \alpha_2 \) \( \alpha_4 \) \( \alpha_6 \) \( \alpha_8 \) \( \ldots \) \( \alpha_{2k} \) \( \alpha_{2k-2} \) \( \alpha_{2k-4} \) \( \alpha_{2k-6} \) \( \alpha_{2k-8} \) \( \ldots \) ) and then to conclude that the canonical system (2.2) (and thus, the Liénard polynomial system (1.3) as well) can have at most \( k + l \) limit cycles, \( k \) surrounding the origin and \( l \) surrounding one by one the anti-saddles (foci or nodes) of (2.2) (and (1.3) as well). The theorem is proved. \( \Box \)

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