A Full Envelope Small Commercial Aircraft Flight Control Design using Multivariable Proportional-Integral Control

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Abstract— This paper deals with the application of linear parameter varying control concepts to the design of a full envelope flight control system for commercial aircrafts. The proposed controller is fixed to have a multivariable proportional-integral structure. A linear matrix inequalities based technique is proposed to account for variations both in the reference model and in the plant. Some numerical simulations show the effectiveness of the proposed technique all over the aircraft operating envelope.

Index Terms— Flight Control Systems, Linear Parameter Varying Systems, $H_2/H_\infty$, Fixed Order Control, Model Following, Linear Matrix Inequalities, PI Control

I. INTRODUCTION

The application of modern techniques to the aircraft flight control provides benefits that are today widely recognized [1], [2]. On the other hand proportional-integral-derivative (PID) controllers are still the most common controllers that can be found on board especially for the category of commercial aircrafts [3], [4]. The reason why this happens is basically the confidence that people has with PID controllers, the consolidated hardware and software tools available for their design and implementation, the implications that more complex controllers may have during the aircraft certification procedures.

In this paper it is shown how recent theoretical developments can be successfully applied to design a multivariable proportional-integral (PI) flight control law for a commercial aircraft adopting a combination of three main control concepts with some realistic approximations:

- robust control of linear parameter varying (LPV) systems,
- $H_2/H_\infty$ control with pole clustering,
- model following.

The aircraft is a nonlinear system subject to uncertainties and disturbances. It is common to model it as an LPV system [5], [6], [7], [8], [9] to account both for nonlinearities and parametric uncertainties. If stability and performance are assured for all the time-varying realizations of the parameters within their bounding sets, it is reasonable that the control requirements are satisfied also on the original nonlinear uncertain system. The satisfaction of stability and performance requirements can be achieved adopting time invariant or gain scheduled controllers.

In flight control it is now common practice to translate performance requirements in the $H_2/H_\infty$ setting. Atmospheric disturbance, neglected dynamics, requirements on the control effort can be properly taken into account adopting this framework.

In [10] a linear matrix inequality (LMI) based procedure to synthesize a controller satisfying $H_2/H_\infty$ requirements and pole clustering, which can be readily extended to the LPV case, is provided. Necessary and sufficient conditions are available to solve the problem if the full state is measurable, or a dynamic compensator with the same order as the plant is designed. Some effort has been spent in the literature to obtain conditions to solve the fixed structure compensator design problem (see [11] for an application to flight control). Among others, in [12] sufficient conditions are provided to synthesize multivariable PID controllers for LPV systems with $H_\infty$ performance requirements. Under certain technical hypotheses, the PID gain synthesis is converted into a non-convex optimization problem involving bilinear matrix inequalities (BMI) for which some solution algorithms are today available [13], [14]. Moreover, under additional hypotheses, the BMI problem is converted into a convex optimization problem involving LMIs [15].

In this paper the technique proposed in [12], is extended to the case of $H_2/H_\infty$ performance requirements with pole clustering, and the PI approach is integrated into a model following control scheme, which is quite common in flight control applications. A numerical example shows how, in spite of the sufficiency of the proposed design conditions, a PI controller in the full envelope of a small commercial aircraft can be obtained with a reasonable effort. The proposed controller gain turns out to be LTI (linear time invariant) and is a valuable alternative to the gain scheduled approaches proposed in the literature [5], [6], [7], [8], [9].

II. THE CONTROLLER STRUCTURE

The aircraft can be modeled as a nonlinear system subject to uncertainties and disturbances [3], [4]. In the literature it is common to model it as an LPV system to account both for nonlinearities and parametric uncertainties [5], [6], [7], [8], [9]. The presence of a parameter vector, which may be physical or artificial, accounts both for the dependence of the Jacobian matrices of the linearized systems on the state and input, and for the physical uncertainties acting on the system.
We assume the following general model for the aircraft:

\[
\dot{x}_p = A_p(\pi)x_p + B_{1p}(\pi)\hat{w}_p + B_{2p}(\pi)u_p \\
y_p = x_p + D_{1p}(\pi)\hat{w}_p
\]  
(1)

where \(x_p \in \mathbb{R}^{n_p}\) is the state vector, \(u_p \in \mathbb{R}^{m_p}\), and \(\hat{w}_p \in \mathbb{R}^{m_w}\) are the control and disturbance input vectors, \(\pi \in \mathcal{P} \subseteq \mathbb{R}^q\) is the vector of parameters. As for the measured output \(y_p \in \mathbb{R}^{m_y}\), we assume that the full state is measurable. In the presence of atmospheric disturbances, this is instantaneously affected by the atmospheric wind velocities [3]. Model (1) is rewritten as

\[
\dot{x}_p = A_p(\pi)x_p + B_{1p}(\pi)w_p + B_{2p}(\pi)u_p \quad \pi \in \mathcal{P} \subseteq \mathbb{R}^q 
\]  
(2)

assuming a new state vector \(x_p = \dot{x}_p + D_{1p}(\pi)\hat{w}_p\), disturbance vector \(w_p = [\hat{w}_p^T \hat{w}_p^T]^T \in \mathbb{R}^{m_w^2}\), and parameter vector \(\pi = [\hat{x}_p^T \hat{x}_p^T]^T \in \mathbb{R}^q\). As it will be clear later, transformation of model (1) to model (2) is a necessary condition for the application of Theorem 2 transforming a bilinear matrix inequality design problem to a linear matrix inequality problem.

We assume that also the reference model for model following may have a LPV structure: this allows to continuously vary the desired behaviour of the aircraft with the operating conditions. We have

\[
\dot{x}_M = A_M(\pi_s(\pi))x_M + B_M(\pi_s(\pi))r = \\
= A_M(\pi)x_M + B_M(\pi)r \\
y_M = C_M(\pi_s(\pi))x_M + D_M(\pi_s(\pi))r = \\
= C_M(\pi)x_M + D_M(\pi)r 
\]  
(3)

where \(x_M \in \mathbb{R}^{n_M}\) is the state vector, \(r \in \mathbb{R}^{n_M}\) is the reference input vector, \(\pi_s\) being a vector of measurable parameters depending on \(\pi\) parameters. All the matrix valued functions appearing in (2), (3) are assumed to depend continuously on their arguments. We finally fix the structure of the PI multivariable controller:

\[
\dot{x}_I = r - G_Iy_p \\
u_p = K_py_p + K_mx_M + K_ix_I 
\]  
(4)

where \(x_I \in \mathbb{R}^{n_c}\) is the state of the multi-integrator, \(G_I\) is a matrix selecting the controlled outputs, \(K_p, K_M, K_I\) are constant controller gains. Systems (2),(3),(4) are connected as shown in Fig.1.

III. THE PROBLEM STATEMENT

To design a full envelope flight control system an LPV control allows to account for nonlinearities, parametric variations and/or uncertainties. \(H_2/H_\infty\) requirements are given to account for the presence of atmospheric disturbances and neglected dynamics (\(H_\infty\)), and limitations on the control energy (\(H_2\)); pole clustering helps to avoid high frequency and low damped modes in the closed loop.

The success of \(H_2/H_\infty\) control depends on the introduction of frequency weighting filters on the disturbance input and the controlled output [17]. According to the closed loop scheme shown in Fig.1 and denoting the state space matrices of the dynamic filters \(W_w, W_r, W_e, (A_{Fw}, B_{Fw}, C_{Fw}, D_{Fw}), (A_{Fr}, B_{Fr}, C_{Fr}, D_{Fr}), (A_{Fe}, B_{Fe}, C_{Fe}, D_{Fe})\) respectively, the interconnected system shown in the figure has the following state-space representation:

\[
\dot{x}_p = A_p(\pi)x_p + B_{1p}(\pi)C_{Fw}x_{Fw} + B_{1p}(\pi)D_{Fw}u_p' + \\
+ B_{2p}(\pi)u_p \\
\dot{x}_M = A_M(\pi)x_M + B_M(\pi)C_{Fr}x_{Fr} + B_M(\pi)D_{Fr}r' \\
\dot{x}_{Fw} = A_{Fw}x_{Fw} + B_{Fw}u_p' \\
\dot{x}_{Fr} = A_{Fr}x_{Fr} + B_{Fr}r' \\
\dot{x}_{Fu} = A_{Fu}x_{Fu} + B_{Fu}u_p \\
\dot{x}_{Fe} = A_{Fe}x_{Fe} + B_{Fe}C_{M}(\pi)x_M + B_{Fe}D_{M}(\pi)C_{Fr}x_{Fr} + \\
+ B_{Fe}D_{M}(\pi)D_{Fr}r' - B_{Fe}G_Ix_p \\
y = [y_p^T x_M^T x_I^T]^T \\
z_\infty = c' = C_{Fe}x_{Fe} + D_{Fe}C_M(\pi)x_M + \\
+ D_{Fe}D_M(\pi)C_{Fr}x_{Fr} + D_{Fe}D_M(\pi)D_{Fr}r' - D_{Fe}G_Ix_p \\
z_2 = u_p' = C_{Fu}x_{Fu} + D_{Fu}u_p
\]  
(5)

that can be rewritten, with clear meaning of matrices, as

\[
\dot{x} = A(\pi)x + B_u(\pi)w + B_u(\pi)u \\
z_\infty = z_{2w}(\pi)x + D_{2w}u \\
z_2 = z_{2y}(\pi)x + D_{2y}u \\
y = C_yx 
\]  
(6)

with

\[
x = [x_p^T x_M^T x_I^T x_{Fw}^T x_{Fr}^T u_p^T]^T \\
u = [u_p^T \pi^T]^T
\]

We can now formulate the controller design problem. We firstly define as \(D(\alpha_{min}, \zeta_{min}, \omega_{max})\) the sub-region of the complex plane determined by a maximum natural frequency \(\omega_{max}\), a minimum damping coefficient \(\zeta_{min}\) and a minimum decay rate \(\alpha_{min}\).

Problem 1

Given system (5) find a static output feedback control action in the form:

\[
u = [K_p \, K_M \, K_I][y_p \, x_M \, x_I]^T = K_y y
\]  
(7)

guaranteeing uniform exponential stability of the closed loop against all the time-varying realizations of the parameters \(\pi \in \mathcal{P}\), minimizing the \(H_2\) system norm on the \(w - z_2\) input-output channel, guaranteeing an \(H_\infty\) performance level \(\gamma\) on the \(w - z_2\) input-output channel, and guaranteeing that the linearized closed loop poles belong to a specified sub-region of the complex plane \(D(\alpha_{min}, \zeta_{min}, \omega_{max})\) \(\forall \pi \in \mathcal{P}\). If we find a solution to Problem 1, the PI controller gains shown in Fig.1 can be obtained by properly partitioning matrix \(K_y\).
IV. SOME MATHEMATICAL RESULTS

In order to solve Problem 1, an extension of the results derived in [12] for the design of PID controllers for LPV systems with $H_{\infty}$ performance requirements is briefly described. The following result can be readily derived from [10] to cope with $H_2/H_{\infty}$ performance requirements and pole clustering.

**Theorem 1**

Problem 1 admits a solution if the following optimization problem admits a solution in terms of two positive definite symmetric matrices $P$ and $Y$, and a gain matrix $K_y$ (the dependence of matrices on the parameter vector is omitted for brevity):

$$\min_{P,Y,K_y} tr(Y)$$

s.t.

$$\begin{align*}
\mathcal{L}^+(P,A_{cl}) + 2\alpha_{min} P &+ B_w \gamma^{-1} P C_z^T < 0 \quad (8) \\
\begin{bmatrix}
\mathcal{L}^+(Q,A) + B_w \gamma^{-1} Q C_z^T \\
\mathcal{L}^+(\hat{W},\hat{B}_u) + 2\alpha_{min} Q \\
\mathcal{L}^+(\tilde{W},\tilde{B}_u) + 2\alpha_{min} Q
\end{bmatrix} &< 0 \quad (14)
\end{align*}$$

The following Theorem which, at the price of some conservatism, allows a solution if

$$\begin{align*}
\begin{bmatrix}
\mathcal{L}^+(Q,A) + B_w \gamma^{-1} Q C_z^T \\
\mathcal{L}^+(\hat{W},\hat{B}_u) + 2\alpha_{min} Q
\end{bmatrix} &< 0 \quad (15)
\end{align*}$$

$$\begin{align*}
\begin{bmatrix}
\mathcal{L}^+(Q,A) + B_w \gamma^{-1} Q C_z^T \\
\mathcal{L}^+(\hat{W},\hat{B}_u) + 2\alpha_{min} Q
\end{bmatrix} &< 0 \quad (16)
\end{align*}$$

$$\begin{align*}
\begin{bmatrix}
\mathcal{L}^+(Q,A) + B_w \gamma^{-1} Q C_z^T \\
\mathcal{L}^+(\hat{W},\hat{B}_u) + 2\alpha_{min} Q
\end{bmatrix} &< 0 \quad (17)
\end{align*}$$

\forall \pi \in \mathcal{P} where $\Psi^+ = (\mathcal{L}^+(Q,A) + \mathcal{L}^+(\hat{W},\hat{B}_u))$, $\Psi^- = (\mathcal{L}^-(Q,A) + \mathcal{L}^-(\hat{W},\hat{B}_u))$, $A = T^{-1} A_T$, $\hat{B}_u = T^{-1} B_u$, $\hat{W} = T^{-1} W$, and $T$ is a nonsingular transformation matrix such that $C_z T = [I \ 0]$. The following is a possible value for the output feedback matrix gain $K_y = W^T Q^{-1} \in \mathbb{R}^{m_1 \times p_1}$. Sketch of the proof. LMI (14) can be rewritten as

$$\begin{align*}
\begin{bmatrix}
\mathcal{L}^+(Q,T^{-1}A_T) + B_w \gamma^{-1} Q T^T C_z^T \\
\mathcal{L}^+(\hat{W},\hat{B}_u) + 2\alpha_{min} Q
\end{bmatrix} &< 0 \quad (18)
\end{align*}$$

After defining the positive definite matrix $P = T Q T^{-1}$ and pre and post multiplying the above inequality by the matrix $\begin{bmatrix} [T^{-1} & I]\end{bmatrix}$ and its transpose we have

$$\begin{align*}
\begin{bmatrix}
P(A + B_u K C_y)^T + \Psi^- & B_w \gamma^{-1} P C_z^T \\
(P + A + B_u K C_y) & P + 2\alpha_{min} P
\end{bmatrix} &< 0 \quad (19)
\end{align*}$$

\forall \pi \in \mathcal{P}$ where $\pi = \begin{bmatrix} P(A + B_u K C_y)^T & P(A + B_u K C_y) & 2\alpha_{min} P \end{bmatrix}$ and $\Psi^- = \begin{bmatrix} B_w \gamma^{-1} P C_z^T \end{bmatrix}$ and its transpose we have

$$\begin{align*}
\begin{bmatrix}
P(A + B_u K C_y)^T & B_w \gamma^{-1} P C_z^T \\
(P + A + B_u K C_y) & P + 2\alpha_{min} P
\end{bmatrix} &< 0 \quad (20)
\end{align*}$$

where $\hat{W} = K C_y T Q$. Similar argumentation hold for inequalities (15-17). The new form of these inequalities allows to readily apply Theorem 1. (q.e.d.) The solvability of the LMI problem (13)-(17) is only a sufficient condition for the solution of the controller gain synthesis problem.

The conservatism of the proposed approach strongly depends on the choice of the system parameterization. For example, if we linearize the aircraft nonlinear model in the neighborhood of the state $\widehat{x}$ and the input $\widehat{u}$, and we assume $\pi = (\widehat{x}^T \widehat{u}^T)^T$, we run the risk to obtain very conservative conditions. In fact, among the different combinations of states and inputs in a prescribed hyper-rectangle, many of them may be not physically compatible. Due to the parameter dependence of the system matrices, the proposed LMI problem turns out to be infinite dimensional. However, in the case that this dependence is multi-affine and the parameter set $\mathcal{P}$ is a hyper-rectangle, it is necessary and sufficient to solve the LMIs on the $\mathcal{P}$ vertices. In our application example, we decided to artificially define the parameter vector $\pi$ so as to solve the synthesis problem simultaneously for a set of $N$ linearized plants. A common approach in control engineering practice is in fact to assure stability and performance for a grid of linearized systems. In
the Lyapunov setting this can be done with different Lyapunov functions, one for each plant (robust stability approach), or with one single Lyapunov function (quadratic stability approach). In particular a correspondence between the parametric approach and the quadratic stability gridding approach can be readily obtained assuming the following parametrization for the system matrices

\[
 M(\pi) = \sum_{k=1,\ldots,N} \left( M_k \pi_k \prod_{i \neq k} (1 - \pi_i) \right), \quad \pi \in [0,1]^N
\]

(18)

\( M_k \) being the matrix computed in the k-th of N linearization point. Each vertex of the hyper-rectangle \([0,1]^N = [0,1] \times \cdots \times [0,1] \) produces a linearized plant chosen for design.

The proposed approach does not theoretically assure stability and performance of the original nonlinear system out of the design conditions. However the quadratic stabilization approach guarantees stability and performance for all the linear parameter varying plants obtainable with \( \pi \in [0,1]^N \). This is something more than stability and performance properties obtainable with a robust stabilization approach. An additional degree of robustness can be obtained inserting linearized plants with uncertainties on aerodynamic and mass properties among the \( N \) design plants.

The pole placement constraints are imposed for each frozen value of the parameter vector \( \pi \) belonging to the set \( \mathcal{P} \). It is well known that, in the presence of time-varying parameters, even the stability of all the systems obtained with frozen values of \( \pi \in \mathcal{P} \) is not a sufficient condition for the stability of the parameter varying system [16]. Nevertheless in practice, if parameters are slowly varying, pole placement is a mean to avoid low damped response and high frequency dynamics. As for the required speed of convergence, this is effectively guaranteed also for the parameter varying system thanks to the \( 2\alpha_{\text{min}}Q \) term in LMI (14).

V. APPLICATION EXAMPLE

A. Mathematical Model, Disturbances and Uncertainties

We consider the small commercial aircraft assumed as a reference aircraft during the European project ADFCSII (Affordable Digital Flight Control System, phase II [18]). This is a conceptual aircraft with the following main characteristics: weight \( W \) from 11800 to 17100 kg, center of gravity position varying between 19% and 38.5% of the mean aerodynamic chord, wing area 39.1 \( m^2 \), mean aerodynamic chord \( c = 2.79 \) \( m \).

Under the usual hypotheses [3], the nonlinear equation of motion in the so-called polar form are:

\[
 W \dot{V} = T \cos(\alpha + \mu_T) \cos \beta - \bar{q} S C_D + W g_1
\]

(19)

\[
 V W \dot{\beta} = -T \cos(\alpha + \mu_T) \sin \beta + \bar{q} S C_Y - W V r + W g_2
\]

(20)

\[
 W V \cos \beta \dot{\alpha} = -T \sin(\alpha + \mu_T) - \bar{q} S C_L + W V q + W g_3
\]

(21)

\[
 I_z \dot{p} - I_{zz} \dot{r} = \bar{q} S b C_l + q r (I_y - I_z) + p q I_{xz}
\]

(22)

\[
 I_y \dot{q} = \bar{q} S c C_m + r p (I_z - I_x) + (v^2 - p^2) I_{xz}
\]

(23)

\[
 -I_{zz} \dot{p} + I_{xx} \dot{r} = \bar{q} S b C_n + p q (I_x - I_y) - q r I_{xz}
\]

(24)

\[
 \dot{\phi} = p + q \tan \theta \sin \phi + r \tan \theta \cos \phi
\]

(25)

\[
 \dot{\theta} = q \cos \phi - r \sin \phi
\]

(26)

with

\[
 g_1 = g(- \cos \alpha \cos \beta \sin \theta + \sin \beta \sin \phi \cos \theta + \sin \alpha \cos \beta \cos \phi \cos \theta)
\]

(27)

\[
 g_2 = g(\cos \alpha \sin \beta \sin \theta + \cos \beta \sin \phi \cos \theta - \sin \alpha \sin \beta \cos \phi \cos \theta)
\]

(28)

\[
 g_3 = g(\sin \alpha \sin \theta + \cos \alpha \cos \phi \cos \theta),
\]

where \( b \) is the wing span, \( g \) the gravity acceleration, \( \eta = 0.5 \rho V^2 \) the dynamic pressure, \( S \) the wing reference area, \( T \) the thrust, \( V \) the true air speed in the absence of external wind, \( \alpha \) the angle of attack, \( \beta \) the sideslip angle, \( \delta_T \) the throttle command, \( \delta_e, \delta_r \) the elevator, rudder, and aileron deflections, \( \mu_T \) the angle between the thrust direction and the X-body axis, \( \rho \) the air density (function of the height), \( I_x, I_y, I_z, I_{xx}, I_{yy}, I_{zz} \) the moments and products of inertia in body axes, \( (\phi, \theta, \psi) \) the roll, pitch, and yaw angles, \( (p, q, r) \) the roll, pitch, yaw rates.

Under the hypothesis of stationary aerodynamic field, we have the following functional relation between the aerodynamic coefficients and the state and control:

\[
 C_D = C_D(\alpha, q, Mach, \delta_e)
\]

\[
 C_Y = C_Y(\alpha, \beta, p, r, Mach, \delta_a, \delta_r)
\]

\[
 C_L = C_L(\alpha, q, Mach, \delta_e)
\]

\[
 C_l = C_l(\alpha, \beta, p, r, Mach, \delta_a, \delta_r)
\]

\[
 C_m = C_m(\alpha, q, Mach, \delta_e)
\]

\[
 C_n = C_n(\alpha, \beta, p, r, Mach, \delta_a, \delta_r).
\]

For a given aircraft geometry, a tabular form of these coefficients can be derived with experimental tests in wind gallery or by means of CFD (Computational Fluid Dynamics) calculations. The thrust is assumed to be a known function of the throttle command, which is assumed constant for the proposed attitude flight controller.

A full nonlinear simulator in the Matlab/Simulink environment was implemented during the ADFCSII project with sensors and actuators dynamics, uncertainties and external disturbances.

The 6 DOF model of a rigid aircraft can be readily brought into the LPV (2) assuming \( u_p = [\delta_e, \delta_a, \delta_r]^T, y_p = x_p = [V, a, q, \beta, p, r] \), \( \hat{w}_p = [u_{\text{wind}}, v_{\text{wind}}, w_{\text{wind}}]^T \) being the atmospheric wind velocity vector in the body frame. In the presence of external wind, the true air speed of the aircraft, the angle of attack and the sideslip angle has to be rewritten as:

\[
 V_{\text{TAS}} = \|V - V_{\text{wind}}\|
\]

\[
 \alpha = \arctan \left[ \frac{(w - w_{\text{wind}})}{(u - u_{\text{wind}})} \right]
\]

\[
 \beta = \arcsin \left[ \frac{(e - v_{\text{wind}})}{V_{\text{TAS}}} \right]
\]

where
The external atmospheric disturbances are generated with a Dryden continuous time turbulence model and a wind shear model implemented on the basis of the mathematical representation in the Military Specification MIL-F-8785C. Parameter $\pi$ can be artificially chosen as described in Section 4 to create a correspondence between the vertices of the parameter space and a number of linearized aircraft models. Aerodynamic uncertainties are introduced as multiplicative perturbations on some of the addenda composing the aerodynamic coefficients which are perturbed up to 30% of their nominal values. As for the mass properties, there is a nominal expected correspondence between mass [kg] and CG (Center of Gravity) position expressed in percentage of the mean aerodynamic chord: $X_{CG} = 19 + 3.7 \cdot 10^{-3}(W - 11800)$.

Four typical sets of perturbations are assigned in the controller design and testing, namely:

- **Reduced stability** high (low) weight: $-30\%$ on the stability derivatives $\frac{\partial C_1}{\partial \alpha}, \frac{\partial C_1}{\partial \delta}$, $-10\%$ on the stability derivatives $\frac{\partial C_2}{\partial \delta}, \frac{\partial C_2}{\partial \theta}, \frac{\partial C_2}{\partial \phi}$; weight $W = 17100$ (11800) kg.
- **Increased (decreased) control**: $+15\%$ ($-15\%$) on the following stability derivatives $\frac{\partial C_1}{\partial \delta}, \frac{\partial C_1}{\partial \beta}, \frac{\partial C_1}{\partial \phi}, \frac{\partial C_2}{\partial \theta}, \frac{\partial C_2}{\partial \phi}$, weight $W = 17100$ kg.

### B. Controller Design

We assume as controlled variables $\phi$, $\theta$, and $\beta$ which are mainly driven by the control commands $\delta_a$, $\delta_s$, and $\delta_r$ respectively. The reference models chosen for the three control channels are parameter independent: $W_{\theta,-\theta}(s) = (s + 1)^{-1}$, $W_{\phi,-\phi}(s) = (s + 1)^{-1}$, $W_{\beta,-\beta}(s) = (0.1s + 1)^{-1}$ where $\theta_r, \phi_r$, and $\beta_r$ are the requested values of $\theta, \phi$, and $\beta$ respectively.

First order models are often chosen in the literature (see for example [20]), however it may be possible that reference models are assumed to be second order due to handling qualities requirements. As for the time responses, a slower model is assumed on $\beta$ control channel to avoid saturation of the rudder command and/or unfeasibility in the LMI design problem. In fact flying at high $\beta$ values requires big effort on the rudder due to the stabilizing effect of the vertical tail.

Finally we note that the reference models are assumed to be LTI. If needed the reference model can be scheduled continuously with measurable parameters like Mach and altitude. The use of one parameter independent model all over the operating envelope does not represent a simplification for the controller synthesis and there is no loss of generality for the parameter varying case. Obviously the choice of LTI models leads to a LTI compensator which is very simple to implement.

In our design, dynamic weighting filters are limited to the diagonal elements of $W_c$. These are chosen to specify the frequency range of interest for the $H_\infty$ performance on the controlled outputs. The three diagonal terms are first-order low pass filters with a cut-off frequency of 1 rad/sec. The insertion of dynamic filters adds computational load to the LMI problem, hence it is preferable to keep the number and order of filters at minimum.

To take into account the dynamic response of the actuators and possible limitations of the control system hardware time response, we choose as pole clustering region $\mathcal{D}(0, 0.5, 40)$. With such a choice no constraint on the maximum time constant are imposed ($\alpha_{min} = 0$), the desired speed of response on the input-output channels of interest being imposed by the reference models. As for the maximum natural frequency and the minimum damping coefficient, these are limited by 40 rad/s and 0.5 respectively to avoid problems in the numerical implementation of the controller and low damped closed loop modes.

The 46 design points in the operating envelope used for the controller synthesis are shown in Fig.2 (plus). In correspondence of each design point, five linearized models are computed in forward wing-leveled flight for the nominal and the reduced stability (both high and low weight), increased and decreased control aircraft configurations. The resulting 226 linearized matrices are assumed to build a polytopic system as described in equation (18).

Since the level of coupling between lateral and longitudinal dynamics turns out to be negligible, as usual in flight control, we design two decoupled controllers: a lateral controller involving state variables $\phi, \beta, r$ and $p$, control variables $\delta_a$ and $\delta_r$, and controlled variables $\phi$ and $\beta$, and a longitudinal controller involving state variables $\alpha, \theta, q, V$, control variable $\delta_a$, and controlled variable $\theta$.

The LMI design problem arising in Theorem 2 is solved using the Matlab LMI Toolbox, the optimal value of $\gamma$ being achieved by means of a trial and error procedure based on numerical simulations.

### C. Numerical results

Fig. 3 show the performance obtained in the absence of external disturbances on a family of 226 testing configurations. The 44 operating points chosen for testing are drawn in Fig.2 (circle), 24 of them coinciding with the design conditions. Five different configurations are considered for each point: nominal, reduced stability (both high and low weight), increased control, and decreased control. Numerical simulations with the full nonlinear model of the aircraft are performed starting from forward flight equilibrium conditions for the 226 testing conditions. The aircraft is driven in the following way (see Fig.3):

- $\theta$ variation: a pilot demand of 10 deg (doublet)
- $\phi$ variation: a pilot demand of 10 deg (doublet)
- $\beta$ variation: a pilot demand of 10 deg

Fig.4 shows the requested value on $\delta_a$ for a $\phi$ doublet variation demand, obtained in the presence of severe atmospheric turbulence [19]. In this figure a comparison between the performance obtained with and without $H_2$ requirements in the controller synthesis is shown. Both syntheses guarantee the same $H_\infty$ performance level, however the $H_2$ requirements allows a reduction of about 30 % on the control power. The $H_2$ norm has a physically meaningful interpretation: if we consider $W(s)$ to be the matrix transfer function of a system driven by independent, zero mean, unit intensity, white noise, then the sum of the variances of the outputs is the square of the $H_2$ norm of $W(s)$. To have an idea of severe turbulence, consider that the speed components on the three
Fig. 5 shows the results obtained for a coordinate maneuver in presence of light atmospheric turbulence (maximum wind speed of about 3 m/s and standard deviation of about 0.5 m/s). Both simulations shown in Fig. 4 and 5 are performed without uncertainties, but with the detailed nonlinear simulator of the aircraft.

Finally a linearized plant of the aircraft is reported together with the controller matrices obtained with the proposed approach. Linear matrices are obtained for the following state/input equilibrium conditions: $\delta_e = 1.5 \text{ deg}$, $\delta_T = 39\%$, $\delta_a = 0 \text{ deg}$, $\delta_r = 0 \text{ deg}$, $V = 185.8036 \text{ m/s}$, $\alpha = 0.44 \text{ deg}$, $q = 0 \text{ deg/s}$, $\theta = 0.44 \text{ deg}$, $\beta = 0 \text{ deg}$, $p = 0 \text{ deg/s}$, $r = 0 \text{ deg/s}$, $\phi = 0 \text{ deg}$.

$$A = \begin{bmatrix} A_p & 0 & 0 & 0 \\ 0 & A_M & 0 & 0 \\ -G_I & 0 & 0 & 0 \\ B_{Fe}G_I & -B_{Fe}C_M & 0 & A_{Fe} \end{bmatrix},$$

$$B_w = \begin{bmatrix} B_{lp} \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C_z = \begin{bmatrix} 0 \\ C_{Fe} \end{bmatrix}, C_{z_2} = 0,$$

$$D_{z_\infty w} = 0, D_{z_2w} = 0, D_{z_2u} = I,$$

$$A_p = \begin{bmatrix} A_{lon} & 0 \\ 0 & A_{lat} \end{bmatrix},$$

$$A_{lon} = \begin{bmatrix} -1.10 \cdot 10^{-2} & 7.10 & 1.37 \cdot 10^{-7} & -9.81 \\ -5.66 \cdot 10^{-4} & -1.07 & 0.986 & 0 \\ 5.00 \cdot 10^{-4} & -4.46 & -0.790 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_{lat} = \begin{bmatrix} -0.185 & 6.69 \cdot 10^{-3} & -0.996 & 5.28 \cdot 10^{-2} \\ -10.7 & -3.38 & 2.07 & 0 \\ 2.15 & -0.137 & -0.117 & 0 \\ 0 & 1 & 7.56 \cdot 10^{-3} & 0 \end{bmatrix},$$

$$A_M = \text{diag} \left[ \begin{bmatrix} -1 & -0.1 & -1 \end{bmatrix} \right], B_M = C_M = I,$$

$$G_I = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{Fe} = -I, B_{Fe} = C_{Fe} = I,$$

$$B_{2p} = \begin{bmatrix} -3.95 \cdot 10^{-3} & 0 & 0 \\ -9.73 \cdot 10^{-4} & 0 & 0 \\ -6.93 \cdot 10^{-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3.73 \cdot 10^{-5} & 6.41 \cdot 10^{-4} \\ 0 & 0.365 & 6.84 \cdot 10^{-2} \\ 0 & 2.09 \cdot 10^{-2} & -3.77 \cdot 10^{-2} \\ 0 & 0 & 0 \end{bmatrix}. $$

$$K_p = \begin{bmatrix} K_{p_1} \\ 0 \\ 0 \\ K_{p_2} \end{bmatrix},$$

$$K_{p_1} = \begin{bmatrix} 0.122 & -34.7 & 131 & 296 \end{bmatrix},$$

$$K_{p_2} = \begin{bmatrix} 41.1 & -61.4 & -52.6 & -159 \end{bmatrix},$$

$$K_M = \begin{bmatrix} -36.9 & 0 & 0 \\ 0 & 3.94 & 4.35 \\ 0 & -25.3 & -1.77 \end{bmatrix},$$

$$K_I = \begin{bmatrix} -187 & 0 & 0 \\ 0 & -47.9 & 137 \\ 0 & 533 & -54.3 \end{bmatrix}.$$
\[
B_{1p} = \begin{bmatrix}
-4.95 \cdot 10^{-5} & 0 & -2.34 \cdot 10^{-5} & -1 & 0 & 0 \\
-1.05 \cdot 10^{-7} & 0 & 2.36 \cdot 10^{-7} & 0 & 0 & -5.38 \cdot 10^{-3} \\
2.51 \cdot 10^{-5} & 0 & -3.30 \cdot 10^{-5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]


