Some coincidence point results in ordered $b$-metric spaces and applications in a system of integral equations

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**ABSTRACT**

The aim of this paper is to present some coincidence point results for four mappings satisfying generalized $(\psi, \varphi)$-weakly contractive condition in the framework of ordered $b$-metric spaces. An example and an application are also provided to support our results.

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1. Introduction and mathematical preliminaries

Let $(X, d)$ be a metric space and $f$ be a self mapping on $X$. If $x = fx$ for some $x$ in $X$, then $x$ is called a fixed point of $f$.

The function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function, if $\varphi$ is continuous and nondecreasing and $\varphi(t) = 0$ if and only if $t = 0$ [23].

A self mapping $f$ on $X$ is a weak contraction if the following contractive condition is satisfied:

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)),$$

for all $x, y \in X$, where $\varphi$ is an altering distance function.

The concept of a weakly contractive mapping was introduced by Alber and Guerre-Delabrere [8] in the setup of Hilbert spaces. Rhoades [37] considered this class of mappings in the setup of metric spaces and proved that every weakly contractive mapping defined on a complete metric space has a unique fixed point.

Let $f$ and $g$ be two self mappings on a nonempty set $X$. If $x = fx = gx$ for some $x$ in $X$, then $x$ is called a common fixed point of $f$ and $g$.

Zhang and Song [40] introduced the concept of a generalized $\varphi$-weak contractive mapping as follows.

Self mappings $f$ and $g$ on a metric space $X$ are called generalized $\varphi$-weak contractions, if there exists a lower semicontinuous function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$,

$$d(fx, gy) \leq N(x, y) - \varphi(N(x, y)),$$

where,

$$N(x, y) = \max\left\{d(x, y), \frac{1}{2}[d(x, gy) + d(y, fx)]\right\}.$$

Based on the above definition, they proved the following common fixed point result.

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Theorem 1.1 [40]. Let \( (X, d) \) be a complete metric space. If \( f, g : X \to X \) are generalized \( \varphi \)-weak contractive mappings, then there exists a unique point \( u \in X \) such that \( u = fu = gu \).

For further works in this direction, we refer to [1,16,24,36].

Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in so called spaces. For more details on fixed point results, its applications, comparison of different contractive conditions and related results in ordered metric spaces we refer the reader to [2–4,6,17,18,28–31,34,35,38] and the references mentioned therein.

Definition 1.2. Let \( f \) and \( g \) be two selfmaps on partially ordered set \( X \). A pair \((f, g)\) is said to be,

(i) weakly increasing if \( fx \preceq gx \) and \( gx \preceq fgx \) for all \( x \in X \) [9],
(ii) partially weakly increasing if \( fx \preceq gx \) for all \( x \in X \) [2].

Let \( X \) be a non-empty set and \( f : X \to X \) be a given mapping. For every \( x \in X \), let \( f^{-1}(x) = \{ u \in X : fu = x \} \).

Definition 1.3. Let \( (X, \preceq) \) be a partially ordered set and \( f, g, h : X \to X \) are mappings such that \( fx \subseteq hx \) and \( gx \subseteq hx \). The ordered pair \((f, g)\) is said to be:

(a) weakly increasing with respect to \( h \) if and only if for all \( x \in X, fx \preceq gy \) for all \( y \in h^{-1}(fx) \), and \( gx \preceq fy \) for all \( y \in h^{-1}(gx) \) [28],
(b) partially weakly increasing with respect to \( h \) if \( fx \preceq gy \), for all \( y \in h^{-1}(fx) \) [17].

Remark 1.4. In the above definition: (i) if \( f = g \), we say that \( f \) is weakly increasing (partially weakly increasing) with respect to \( h \), (ii) if \( h = I \) (the identity mapping on \( X \)), then the above definition reduces to the weakly increasing (partially weakly increasing) mapping (see [28,38]).

Jungck in [25] introduced the following definition.

Definition 1.5 [25]. Let \( (X, d) \) be a metric space and \( f, g : X \to X \). The pair \((f, g)\) is said to be compatible if and only if \( \lim_{n \to \infty} d(fgx_n, gfx_n) = 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \).

Definition 1.6 [22]. Let \( f, g : X \to X \) be given self-mappings on \( X \). The pair \((f, g)\) is said to be weakly compatible if \( f \) and \( g \) commute at their coincidence points (i.e. \( fgy = gfx \), whenever \( fx = gx \)).

Definition 1.7. Let \( (X, \preceq) \) be a partially ordered set and \( d \) be a metric on \( X \). We say that \( (X, d, \preceq) \) is regular if the following conditions hold:

(i) if a non-decreasing sequence \( x_n \to x \), then \( x_n \preceq x \) for all \( n \),
(ii) if a non-increasing sequence \( y_n \to y \), then \( y_n \preceq y \) for all \( n \).

In [17], Esmaily et al. established two coincidence point theorems for four mappings satisfying a generalized weakly contractive condition in ordered metric space.

Theorem 1.8. Let \( (X, d, \preceq) \) be a complete ordered metric space. Let \( f, g, S, T : X \to X \) be given mappings satisfying

(i) \( fx \subseteq TX \) and \( gx \subseteq SX \),
(ii) \( f, g, S \) and \( T \) are continuous,
(iii) the pairs \((f, S)\) and \((g, T)\) are compatible,
(iv) \((f, g)\) is partially weakly increasing with respect to \( T \) and \((g, f)\) is partially weakly increasing with respect to \( S \).

Suppose that for every \( x, y \in X \) such that \( Sx \) and \( Ty \) are comparable, we have,

\[
\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(N(x, y)),
\]

where

\[
M(x, y) = \max \left\{ d(Sx, Ry), d(Sx, fx), d(Ry, gy), \frac{d(Sx, gy) + d(Ry, fx)}{2} \right\}
\]
and
\[ N(x, y) = \max \{ d(Sx, Ry), d(Sx, gy), d(Ry, fx) \} \]
and \( \psi : [0, \infty) \to [0, \infty) \) is an altering distance function and \( \varphi : [0, \infty) \to [0, \infty) \) is a continuous function with \( \varphi(t) = 0 \) if and only if \( t = 0 \). Then, the pairs \((f, S)\) and \((g, T)\) have a coincidence point \( u \in X\), that is, \( fu = Su \) and \( gu = Tu \). Moreover, if \( Su \) and \( Tu \) are comparable, then \( u \in X \) is a coincidence point of \( f, g, S \) and \( T \).

Also, they showed that by replacing the continuity hypotheses on \( f, g, S \) and \( T \) with the regularity of \( X \) and reducing the compatibility of the pairs \((f, S)\) and \((g, T)\) to weak compatibility, the above theorem is still valid (See, Theorem 2.7 of [17]).

The concept of \( b\)-metric space was introduced by Czerwik in [15]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in \( b\)-metric spaces (see also [7,10–14, 19–21,32,33,39]).

Consistent with [15,39], the following definitions and results will be needed in the sequel.

**Definition 1.9** [15]. Let \( X \) be a (nonempty) set and \( s \geq 1 \) be a given real number. A function \( d : X \times X \to \mathbb{R}^+ \) is a \( b\)-metric iff, for all \( x, y, z \in X \), the following conditions are satisfied:

- \( b_1. d(x, y) = 0 \) iff \( x = y \),
- \( b_2. d(x, y) = d(y, x) \),
- \( b_3. d(x, z) \leq s[d(x, y) + d(y, z)] \).

The pair \((X, d)\) is called a \( b\)-metric space.

It should be noted that, the class of \( b\)-metric spaces is effectively larger than that of metric spaces, since a \( b\)-metric is a metric, when \( s = 1 \). Here, we present an example to show that in general a \( b\)-metric need not necessarily be a metric. (see, also, [39, p. 264]).

**Example 1.10** [5]. Let \((X, d)\) be a metric space, and \( \rho(x, y) = (d(x, y))^p \), where \( p > 1 \) is a real number. Then \( \rho \) is a \( b\)-metric with \( s = 2^{p-1} \).

However, if \((X, d)\) is a metric space, then \((X, \rho)\) is not necessarily a metric space.

For example, if \( X = \mathbb{R} \) is the set of real numbers and \( d(x, y) = |x - y| \) is the usual Euclidean metric, then \( \rho(x, y) = (x - y)^2 \) is a \( b\)-metric on \( \mathbb{R} \) with \( s = 2 \), but is not a metric on \( \mathbb{R} \).

Also, the following example of a \( b\)-metric space is given in [26].

**Example 1.11** [26]. Let \( X \) be the set of Lebesgue measurable functions on \([0, 1]\) such that

\[ \int_0^1 |f(x)|^2 \, dx < \infty. \]

Define \( D : X \times X \to [0, \infty) \) by

\[ D(f, g) = \int_0^1 |f(x) - g(x)|^2 \, dx. \]

As

\[ \left( \int_0^1 |f(x) - g(x)|^2 \, dx \right)^{\frac{1}{2}} \]

is a metric on \( X \), then, from the previous example, \( D \) is a \( b\)-metric on \( X \), with \( s = 2 \).

Khamsi [27] also showed that each cone metric space over the Banach space \( E \) with the normal cone \( P \) has a \( b\)-metric structure.

The purpose of this paper is to obtain some coincidence point theorems for four mappings satisfying a \((\psi, \varphi)\)-contractive condition in ordered \( b\)-metric spaces.

We also need the following definitions:

**Definition 1.12.** Let \( X \) be a nonempty set. Then \((X, d, \preceq)\) is called a partially ordered \( b\)-metric space if and only if \( d \) is a \( b\)-metric on a partially ordered set \((X, \preceq)\).

**Definition 1.13** [13]. Let \((X, d)\) be a \( b\)-metric space. Then a sequence \( \{x_n\} \) in \( X \) is called:

(a) \( b\)-convergent if and only if there exists \( x \in X \) such that \( d(x_n, x) \to 0 \), as \( n \to \infty \). In this case, we write \( \lim_{n \to \infty} x_n = x \).

(b) \( b\)-Cauchy if and only if \( d(x_n, x_m) \to 0 \), as \( n, m \to \infty \).
Proposition 1.14 (See, Remark 2.1 in [13]). In a $b$-metric space $(X, d)$ the following assertions hold:

$p_1$. A $b$-convergent sequence has a unique limit.

$p_2$. Each $b$-convergent sequence is $b$-Cauchy.

$p_3$. In general, a $b$-metric is not continuous.

Also very recently N. Hussain et al. have presented an example of a $b$-metric which is not continuous (see Example 3 in [19]).

Definition 1.15 [13]. The $b$-metric space $(X, d)$ is $b$-complete if every $b$-Cauchy sequence in $X$ be $b$-converges.

Definition 1.16 [13]. Let $(X, d)$ be a $b$-metric space. If $Y$ is a nonempty subset of $X$, then the closure $\overline{Y}$ of $Y$ is the set of limits of all $b$-convergent sequences of points in $Y$, i.e.,

$$\overline{Y} = \{x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ so that } \lim_{n \to \infty} x_n = x\}.$$ 

Taking into account of the above definition, we have the following concepts.

Definition 1.17 [13]. Let $(X, d)$ be a $b$-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in $Y$ which $b$-converges to an element $x$, we have $x \in Y$ (i.e. $\overline{Y} = Y$).

Definition 1.18. Let $(X, d)$ and $(X', d')$ be two $b$-metric spaces. Then a function $f : X \to X'$ is $b$-continuous at a point $x \in X$ if and only if it is $b$-sequentially continuous at $x$, that is, whenever $\{x_n\}$ is $b$-convergent to $x$, $\{f(x_n)\}$ is $b$-convergent to $f(x)$.

Since in general a $b$-metric is not continuous, we need the following simple lemma about the $b$-convergent sequences in the proof of our main result.

Lemma 1.19 [5]. Let $(X, d)$ be a $b$-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are $b$-convergent to $x, y$, respectively. Then we have,

$$\frac{1}{S} d(x, y) \leq \liminf_{n \to \infty} d(x_n, y_n) \leq \limsup_{n \to \infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n \to \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{S} d(x, z) \leq \liminf_{n \to \infty} d(x_n, z) \leq \limsup_{n \to \infty} d(x_n, z) \leq sd(x, z).$$

Motivated by the work in [1,2,16,17,24,36,38], we prove some coincidence point results for nonlinear generalized $(\psi, \varphi)$ weakly contractive mappings in partially ordered $b$-metric spaces.

2. Main results

Let $(X, \leq, d)$ be an ordered $b$-metric space and $f, g, R, S : X \to X$ be four self mappings. Throughout this paper, unless otherwise stated, for all $x, y \in X$, let

$$M_i(x, y) = \max \left\{ d(Sx, Ry), d(Sx, fx), d(Ry, gy), \frac{d(Sx, gy) + d(Ry, fx)}{2S} \right\}.$$ 

Theorem 2.1. Let $(X, \leq, d)$ be a partially ordered and complete $b$-metric space. Let $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$. Suppose that for every two comparable elements $Sx, Ry \in X$, we have,

$$\psi(s^2 d(fx, gy)) \leq \psi(M_i(x, y)) - \varphi(M_i(x, y)), \quad (2.1)$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Let $f, g, R$ and $S$ are continuous, the pairs $(f, S)$ and $(g, R)$ are compatible and the pairs $(f, g)$ and $(S, R)$ are partially weakly increasing with respect to $R$ and $S$, respectively. Then, the pairs $(f, S)$ and $(g, R)$ have a coincidence point $z$ in $X$. Moreover, if $Rz$ and $Sz$ are comparable, then $z$ is a coincidence point of $f, g, R$ and $S$.

Proof. Let $x_0$ be an arbitrary point of $X$. Choose $x_1 \in X$ such that $fx_0 = Rx_1$ and $x_2 \in X$ such that $gx_1 = Sx_2$. This can be done as $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$.
Continuing this way, construct a sequence \(\{z_n\}\) defined by:
\[
z_{2n+1} = Rx_{2n-1} = fx_{2n}
\]
and
\[
z_{2n+2} = Sx_{2n+1} = gx_{2n+1},
\]
for all \(n \geq 0\).

As \(x_1 \in R^{-1}(fx_0)\) and \(x_2 \in S^{-1}(gx_1)\), and the pairs \((f, g)\) and \((g, f)\) are partially weakly increasing with respect to \(R\) and \(S\), respectively, so we have,
\[
Rx_1 = fx_0 \geq gx_1 = Sx_2 \geq fx_2 = Rx_3.
\]
Repeating this process, we obtain \(z_{2n+1} \preceq z_{2n+2}\), for all \(n \geq 0\).

We will complete the proof in three steps.

Step 1. We prove that \(\lim_{k \to \infty} d(z_k, z_{k+1}) = 0\).

Define \(d_k = d(z_k, z_{k+1})\). Suppose \(d_{k_0} = 0\), for some \(k_0\). Then, \(z_{k_0} = z_{k_0+1}\). In the case that \(k_0 = 2n\), then \(z_{2n} = z_{2n+1}\) gives \(z_{2n+1} = z_{2n+2}\). Indeed,
\[
\psi(s^2d(z_{2n+1}, z_{2n+2})) = \psi(s^2d(fx_{2n}, gx_{2n+1}))
\]
\[
\leq \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})),
\]
(2.2)

where,
\[
M_s(x_{2n}, x_{2n+1}) = \max \left\{ d(Sx_{2n}, Rx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Rx_{2n+1}, gx_{2n+1}), \frac{d(Sx_{2n}, gx_{2n+1}) + d(Rx_{2n+1}, fx_{2n})}{2s} \right\}
\]
\[
= \max \left\{ d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s} \right\}
\]
\[
= \max \left\{ d(z_{2n}, z_{2n+1}), \frac{d(z_{2n}, z_{2n+2})}{2s} \right\}
\]
\[
\leq \max \left\{ d(z_{2n}, z_{2n+1}), 0 + \frac{d(z_{2n+1}, z_{2n+2})}{2} \right\} = d(z_{2n+1}, z_{2n+2}).
\]
Thus, from 2.2,
\[
\psi(s^2d(z_{2n+1}, z_{2n+2})) \leq \psi(d(z_{2n+1}, z_{2n+2})) - \varphi \left( \max \left\{ d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s} \right\} \right)
\]
\[
\leq \psi(s^2d(z_{2n+1}, z_{2n+2})) - \varphi \left( \max \left\{ d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s} \right\} \right)
\]
implies that \(\varphi \left( \max \left\{ d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s} \right\} \right) = 0\), that is, \(z_{2n} = z_{2n+1} = z_{2n+2}\). Similarly, if \(k_0 = 2n + 1\), then \(z_{2n+1} = z_{2n+2}\) gives \(z_{2n+2} = z_{2n+3}\). Consequently, the sequence \(\{z_k\}\) becomes constant for \(k \geq k_0\) and hence \(\lim_{k \to \infty} d(z_k, z_{k+1}) = 0\).

Suppose that,
\[
d_k = d(z_k, z_{k+1}) > 0,
\]
for each \(k\). We now claim that the following inequality holds.
\[
d(k, z_{k+1}) \leq d(z_k, z_{k+1}) = M_s(x_k, x_{k+1}),
\]
for each \(k = 1, 2, 3, \ldots\)

Let \(k = 2n\) and for an \(n \geq 0\) \(d(z_{2n+1}, z_{2n+2}) \geq d(z_{2n}, z_{2n+1}) > 0\). Then, as \(Sx_{2n} \preceq Rx_{2n-1}\), using (2.1) we obtain that,
\[
\psi(s^2d(z_{2n+1}, z_{2n+2})) = \psi(s^2d(fx_{2n}, gx_{2n+1}))
\]
\[
\leq \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})),
\]
(2.5)

where,
\[
M_s(x_{2n}, x_{2n+1}) = \max \left\{ d(Sx_{2n}, Rx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Rx_{2n+1}, gx_{2n+1}), \frac{d(Sx_{2n}, gx_{2n+1}) + d(Rx_{2n+1}, fx_{2n})}{2s} \right\}
\]
\[
= \max \left\{ d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s} \right\}
\]
\[
\leq \max \left\{ d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2} \right\}
\]
\[
= d(z_{2n+1}, z_{2n+2}).
\]
Hence, since \( \psi \) is nondecreasing, (2.5) implies that
\[
\psi(s^2 d(z_{2n+1}, z_{2n+2})) \leq \psi(d(z_{2n+1}, z_{2n+2})) - \varphi(M_1(x_{2n}, x_{2n+1})),
\]
which is possible only if \( M_1(x_{2n}, x_{2n+1}) = 0 \), that is, \( d(z_{2n+1}, z_{2n+2}) = 0 \), a contradiction to (2.3). Hence,
\[
d(z_{2n+1}, z_{2n+2}) \leq d(z_{2n}, z_{2n+1})\]
and \( M_1(x_{2n}, x_{2n+1}) = d(z_{2n}, z_{2n+1}) \).

Therefore, (2.4) is proved for \( k = 2n \).

Similarly, it can be shown that
\[
d(z_{2n+1}, z_{2n+2}) \leq d(z_{2n}, z_{2n+1}) = M_1(x_{2n}, x_{2n+1}).
\]
(2.6)

Hence \( \{d(z_k, z_{k+1})\} \) is a nondecreasing sequence of nonnegative real numbers. Therefore, there is an \( r \geq 0 \) such that
\[
\lim_{k \to \infty} d(z_k, z_{k+1}) = r.
\]
(2.7)

Now, taking limit as \( k \to \infty \) in (2.4), we obtain,
\[
\lim_{k \to \infty} M_1(x_k, x_{k+1}) = r.
\]
(2.8)

Letting \( n \to \infty \) in (2.5), using (2.7), (2.8) and the continuity of \( \psi \) and \( \varphi \), we have \( \psi(s^2 r) \leq \psi(r) - \varphi(r) \). Therefore \( \varphi(r) = 0 \).

Hence,
\[
\lim_{k \to \infty} d(z_k, z_{k+1}) = 0,
\]
(2.9)

from our assumptions about \( \varphi \).

Step II. We now show that \( \{z_n\} \) is a \( b \)-Cauchy sequence in \( X \). That is, for every \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that for all \( m, n \geq k \), \( d(z_m, z_n) < \varepsilon \).

We assume on contrary that there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{z_{m(k)}\} \) and \( \{z_{n(k)}\} \) of \( \{z_n\} \) such that \( n(k) > m(k) \geq k \) and

(a) \( m(k) = 2t \) and \( n(k) = 2t' + 1 \), where \( t \) and \( t' \) are nonnegative integers,

(b)
\[
d(z_{m(k)}, z_{n(k)}) \geq \varepsilon
\]
and

(c) \( n(k) \) is the smallest number such that the condition (b) holds; i.e.,
\[
d(z_{m(k)}, z_{n(k)-1}) < \varepsilon.
\]
(2.10)

From triangle inequality and (2.10) and (2.11), we have,
\[
\varepsilon \leq d(z_{m(k)}, z_{n(k)})
\leq s[d(z_{m(k)}, z_{n(k)-1}) + d(z_{m(k)-1}, z_{n(k)})]
< s \varepsilon + s d(z_{m(k)-1}, z_{n(k)}).
\]
(2.12)

Taking limit as \( k \to \infty \) in (2.12), from (2.9) we obtain that,
\[
\varepsilon \leq \lim_{k \to \infty} \sup \{d(z_{m(k)}, z_{n(k)})\} \leq s \varepsilon.
\]
(2.13)

Using triangle inequality again we have,
\[
\varepsilon \leq d(z_{m(k)}, z_{n(k)}) \leq s[d(z_{m(k)}, z_{m(k)-1}) + d(z_{m(k)-1}, z_{n(k)})]
\leq s^2[d(z_{m(k)}, z_{m(k)}) + d(z_{m(k)}, z_{m(k)-1})] + s d(z_{m(k)}, z_{m(k)-1}).
\]
(2.14)

Taking limit as \( k \to \infty \) in (2.14) and using (2.9) and (2.13), we have,
\[
\varepsilon \leq s \lim_{k \to \infty} \sup \{d(z_{m(k)}, z_{n(k)+1})\} \leq s^3 \varepsilon
\]
or, equivalently,
\[
\frac{\varepsilon}{s} \leq \lim_{k \to \infty} \sup \{d(z_{m(k)}, z_{n(k)+1})\} \leq s^2 \varepsilon.
\]
(2.15)

Using triangle inequality again we have,
\[
\varepsilon \leq s \lim_{k \to \infty} \sup \{d(z_{m(k)}, z_{n(k)})\} \leq s^2 \varepsilon.
\]
(2.16)
Making $k \to \infty$ in the above inequality, we have,
\[ \varepsilon \leq s \limsup_{k \to \infty} d(z_{m(k)}, z_{m(k)+1}) \leq s^3 \varepsilon \]
or, equivalently,
\[ \frac{\varepsilon}{s} \leq \limsup_{k \to \infty} d(z_{m(k)}, z_{m(k)+1}) \leq s^2 \varepsilon. \tag{2.17} \]

Finally,
\[ d(z_{m(k)+1}, z_{m(k)+1}) \leq s[d(z_{m(k)+1}, z_{m(k)}) + d(z_{m(k)}, z_{m(k)+1})] \]
\[ \leq sd(z_{m(k)+1}, z_{m(k)}) + s^2[d(z_{m(k)+1}, z_{m(k)}) + d(z_{m(k)}, z_{m(k)+1})]. \tag{2.18} \]

Taking limit as $k \to \infty$, and using (2.13), we have,
\[ \limsup_{k \to \infty} d(z_{m(k)+1}, z_{m(k)+1}) \leq s^3 \varepsilon. \]

Consider,
\[ d(z_{m(k)}, z_{n(k)}) \leq s[d(z_{m(k)}, z_{m(k)+1}) + d(z_{m(k)+1}, z_{n(k)})] \]
\[ \leq sd(z_{m(k)}, z_{m(k)+1}) + s^2[d(z_{m(k)+1}, z_{n(k)}) + d(z_{m(k)}, z_{n(k)})]. \tag{2.19} \]

Making $k \to \infty$ and using (2.9) and (2.13), we have,
\[ \frac{\varepsilon}{s^2} \leq \limsup_{k \to \infty} d(z_{m(k)+1}, z_{n(k)+1}). \tag{2.20} \]

As $Sx_{m(k)} \preceq Rx_{n(k)}$, from (2.1), we have,
\[ \psi(s^3 d(z_{m(k)+1}, z_{n(k)+1})) = \psi\left(s^3 d(f x_{m(k)}, g x_{n(k)})\right) \leq \psi\left(M_s(x_{m(k)}, x_{n(k)})\right) - \varphi\left(M_s(x_{m(k)}, x_{n(k)})\right), \tag{2.21} \]

where,
\[ M_s(x_{m(k)}, x_{n(k)}) = \max\left\{ d(Sx_{m(k)}, Rx_{n(k)}), d(Sx_{m(k)}, gx_{n(k)}), d(Rx_{n(k)}, gx_{n(k)}), \frac{d(Sx_{m(k)}, gx_{n(k)}) + d(Rx_{n(k)}, gx_{n(k)})}{2s} \right\} \]
\[ = \max\left\{ d(z_{m(k)}, z_{n(k)}), d(z_{m(k)}, z_{n(k)+1}), d(z_{m(k)+1}, z_{n(k)+1}), \frac{d(z_{m(k)+1}, z_{n(k)+1}) + d(z_{m(k)}, z_{n(k)+1})}{2s} \right\}. \]

Taking limit as $k \to \infty$ and using (2.9), (2.13), (2.15), (2.17) and (2.20), we have,
\[ \varepsilon = \max\left\{ \varepsilon, \frac{s^2 \varepsilon + s^2 \varepsilon}{2s} \right\} \leq \limsup_{k \to \infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{ \frac{s^2 \varepsilon + s^2 \varepsilon}{2s} \right\} = s \varepsilon. \]

Also, we can show that,
\[ \varepsilon = \max\left\{ \varepsilon, \frac{s^2 \varepsilon + s^2 \varepsilon}{2s} \right\} \leq \liminf_{k \to \infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{ \frac{s^2 \varepsilon + s^2 \varepsilon}{2s} \right\} = s \varepsilon. \]

Taking limit as $k \to \infty$ in (2.21), we have,
\[ \psi(s\varepsilon) = \psi(s(s^3 \varepsilon/s)) \leq \psi\left(s^3 \limsup_{k \to \infty} d(z_{m(k)+1}, z_{n(k)+1})\right) \]
\[ \leq \psi\left(\limsup_{k \to \infty} M_s(x_{m(k)}, x_{n(k)})\right) - \varphi\left(\liminf_{k \to \infty} M_s(x_{m(k)}, x_{n(k)})\right) \leq \psi(s\varepsilon) - \varphi(\varepsilon), \tag{2.22} \]

which implies that $\varphi(\varepsilon) \leq 0$, hence, $\varepsilon = 0$, a contradiction. Hence $\{z_n\}$ is a $b$-Cauchy sequence.

Step III. We will show that $f, g, R$ and $T$ have a coincidence point.
Since $\{z_n\}$ is a $b$-Cauchy sequence in the complete $b$-metric space $X$, there exists $z \in X$ such that,
\[ \lim d(z_{2n+1}, z) = \lim d(Rx_{2n+1}, z) = \lim d(fx_{2n}, z) = 0 \tag{2.23} \]
and
\[
\lim_{n \to \infty} (z_{2n+2}, z) = \lim_{n \to \infty} (Sx_{2n+2}, z) = \lim_{n \to \infty} (gSx_{2n+1}, z) = 0. \tag{2.24}
\]

Hence,
\[
Sx_{2n} \to z \quad \text{and} \quad fx_{2n} \to z, \text{ as } n \to \infty. \tag{2.25}
\]

As \((f, S)\) is compatible, so,
\[
\lim_{n \to \infty} (Sfx_{2n}, Sfx_{2n}) = 0. \tag{2.26}
\]

Moreover, from \(\lim_{n \to \infty} d(fx_{2n}, z) = 0\), \(\lim_{n \to \infty} d(Sx_{2n}, z) = 0\), and the continuity of \(S\) and \(f\) and Lemma 1.19, we obtain,
\[
\lim_{n \to \infty} (Sfx_{2n}, Sz) = 0 = \lim_{n \to \infty} (fSx_{2n}, fz). \tag{2.27}
\]

By the triangle inequality, we have,
\[
d(Sz, fz) \leq s[d(Sz, Sfx_{2n}) + d(Sfx_{2n}, fz)] \\
\quad \leq sd(Sz, Sfx_{2n}) + s^2[d(Sfx_{2n}, fSx_{2n}) + d(fSx_{2n}, fz)]. \tag{2.28}
\]

Taking limit as \(n \to \infty\) in (2.28) and again using Lemma 1.19, we obtain:
\[
d(Sz, fz) \leq 0,
\]
which implies that \(fz = Sz\), that is \(z\) is a coincidence point of \(f\) and \(S\).

Similarly, it can be proved that \(gz = Rz\). Now, let \(Rz\) and \(Sz\) are comparable. By (2.1) we have,
\[
\psi(s^2 d(fz, gz)) \leq \psi(M_i(z, z)) - \varphi(M_i(z, z)), \tag{2.29}
\]

where,
\[
M_i(z, z) = \max \left\{ d(Sz, Rz), d(Sz, fz), d(Rz, gz), \frac{d(Sz, gz) + d(Rz, fz)}{2s} \right\} \\
= d(Sz, Rz) = d(fz, gz).
\]

Hence (2.29) gives,
\[
\psi(s^3 d(fz, gz)) \leq \psi(d(fz, gz)) - \varphi(d(fz, gz)) = 0.
\]

Therefore \(fz = gz = Tz = Rz\). \(\Box\)

In the following theorem, we omit the assumption of continuity of \(f, g, R\) and \(S\) and replace the compatibility of the pairs \((f, S)\) and \((g, R)\) by weak compatibility of the pairs.

**Theorem 2.2.** Let \((X, \preceq, d)\) be a regular partially ordered \(b\)-metric space, \(f, g, R, S : X \to X\) be four mappings such that \(f(X) \subseteq R(X)\) and \(g(X) \subseteq S(X)\) and \(RX\) and \(SX\) are \(b\)-closed subsets of \(X\). Suppose that for every two comparable elements \(x, y \in X\), we have,
\[
\psi(s^2 d(fx, gy)) \leq \psi(M_i(x, y)) - \varphi(M_i(x, y)), \tag{2.30}
\]

where \(\psi, \varphi : [0, \infty) \to [0, \infty)\) are altering distance functions. Then, the pairs \((f, S)\) and \((g, R)\) have a coincidence point \(z\) in \(X\) provided that the pairs \((f, S)\) and \((g, R)\) are weakly compatible and the pairs \((f, g)\) and \((g, f)\) are partially weakly increasing with respect to \(R\) and \(S\), respectively. Moreover, if \(Rz\) and \(Sz\) are comparable, then \(z \in X\) is a coincidence point of \(f, g, R\) and \(S\).

**Proof.** Following the proof of Theorem 2.1, there exists \(z \in X\) such that:
\[
\lim_{k \to \infty} (z_k, z) = 0. \tag{2.31}
\]

Since \(R(X)\) is \(b\)-closed and \\(\{z_{2n+1}\} \subseteq R(X)\), therefore \(z \in R(X)\). Hence, there exists \(u \in X\) such that \(z = Ru\) and
\[
\lim_{n \to \infty} (z_{2n+1}, Ru) = \lim_{n \to \infty} (Rx_{2n+1}, Ru) = 0. \tag{2.32}
\]

Similarly, there exists \(v \in X\) such that \(z = Ru = Sv\) and
\[
\lim_{n \to \infty} (z_{2n}, Sv) = \lim_{n \to \infty} (Sx_{2n}, Sv) = 0. \tag{2.33}
\]

Now we prove that \(v\) is a coincidence point of \(f\) and \(S\). Since \(Rx_{2n+1} \to z = Sv\), as \(n \to \infty\), from regularity of \(X\), \(Rx_{2n+1} \preceq Sv\). Therefore, from (2.1), we have
\[
\psi(s^2 d(fv, gx_{2n+1})) \leq \psi(M_i(v, x_{2n+1})) - \varphi(M_i(v, x_{2n+1})), \tag{2.34}
\]
where,
\[ M_s(x, x_{2n+1}) = \max \left\{ d(Sv, Rx_{2n+1}), d(Sv, f v), d(Rx_{2n+1}, gx_{2n+1}), \frac{d(Sv, gx_{2n+1}) + d(Rx_{2n+1}, f v)}{2s} \right\} \]

Taking limit as \( n \to \infty \) in (2.34) and using Lemma 1.19, we obtain that
\[ \psi(\delta f v, z) = \psi \left( s^2 \frac{1}{s} \delta f v, z \right) \leq \psi (\delta (f v, z)) + \varphi (\delta (f v, z)) \]

which implies that \( f v = z = Sv \).

As \( f \) and \( S \) are weakly compatible, we have \( fz = fs v = Sf v = Sz \). Thus, \( z \) is a coincidence point of \( f \) and \( S \).

Similarly, it can be shown that \( z \) is a coincidence point of the pair \((g, R)\).

The rest of the proof follows from similar arguments as in Theorem 2.1. \( \square \)

Taking \( f = g \) in Theorem 2.1, we obtain the following coincidence point result:

**Corollary 2.3.** Let \((X, \le, d)\) be a partially ordered and complete \( b \)-metric space and \( f, R, S : X \to X \) be three mappings such that \( f(X) \subseteq R(X) \cup S(X) \). Suppose that for every two comparable elements \( Sx, Ry \in X \), we have
\[ \psi (s^2 \delta fx, fy) \leq \psi (M_s(x, y)) - \varphi (M_s(x, y)) \quad (2.35) \]

where,
\[ M_s(x, y) = \max \left\{ d(Sx, Rx), d(Sx, fx), d(Ry, fy), \frac{d(Sx, fy) + d(Ry, fx)}{2s} \right\} \]

and \( \psi, \varphi : [0, \infty) \to [0, \infty) \) are altering distance functions. Then, \( f, S \) and \( R \) have a coincidence point \( z \) in \( X \) provided that \( f, R \) and \( S \) are continuous, the pairs \((f, S)\) and \((f, R)\) are compatible and \( f \) is weakly increasing with respect to \( R \) and \( S \).

Taking \( S = R \) in Theorem 2.1, we obtain the following result:

**Corollary 2.4.** Let \((X, \le, d)\) be a partially ordered and complete \( b \)-metric space and \( f, g, R : X \to X \) be three mappings such that \( f(X) \cup g(X) \subseteq R(X) \). Suppose that for every two comparable elements \( Rx, Ry \in X \), we have,
\[ \psi (s^2 \delta fx, gy) \leq \psi (M_s(x, y)) - \varphi (M_s(x, y)) \quad (2.36) \]

where,
\[ M_s(x, y) = \max \left\{ d(Rx, Ry), d(Rx, fx), d(Ry, gy), \frac{d(Rx, gy) + d(Ry, fx)}{2s} \right\} \]

and \( \psi, \varphi : [0, \infty) \to [0, \infty) \) are altering distance functions. Then, \( f, g \) and \( R \) have a coincidence point \( z \) in \( X \) provided that \( f, g \) and \( R \) are continuous, the pairs \((f, R)\) and \((g, R)\) are compatible and the pair \((f, g)\) is weakly increasing with respect to \( R \).

Taking \( R = S \) and \( f = g \) in Theorem 2.1, we obtain the following coincidence point result:

**Corollary 2.5.** Let \((X, \le, d)\) be a partially ordered and complete \( b \)-metric space and \( f, R : X \to X \) be two mappings such that \( f(X) \subseteq R(X) \). Suppose that for every comparable elements \( Rx, Ry \in X \), we have,
\[ \psi (s^2 \delta fx, fy) \leq \psi (M_s(x, y)) - \varphi (M_s(x, y)) \quad (2.37) \]

where,
\[ M_s(x, y) = \max \left\{ d(Rx, Ry), d(Rx, fx), d(Ry, fy), \frac{d(Rx, fy) + d(Ry, fx)}{2s} \right\} \]

and \( \psi, \varphi : [0, \infty) \to [0, \infty) \) are altering distance functions. Then, the pair \((f, R)\) has a coincidence point \( z \) in \( X \) provided that \( f \) and \( R \) are continuous, the pair \((f, R)\) is compatible and \( f \) is weakly increasing with respect to \( R \).

**Example 2.6.** Let \( X = [0, \infty) \) and \( d \) on \( X \) be given by \( d(x, y) = |x - y|^2 \), for all \( x, y \in X \). We define an ordering “\( \le \)” on \( X \) as follows:
\[ x \le y \iff y - x, \quad \forall x, y \in X. \]

Define self-maps \( f, g, S \) and \( R \) on \( X \) by
\[ fx = \ln(\sqrt{x^2 + 1} + x) = \sinh^{-1} x, \quad Rx = \sinh x, \quad gx = \sinh^{-1} \left( \frac{x}{2} \right), \quad Sx = \sinh 6x. \]
To prove that \((f, g)\) is partially weakly increasing with respect to \(R\), let \(x, y \in X\) be such that \(y \in R^{-1}fx\), that is \(Ry = fx\). By the definition of \(f\) and \(R\), we have \(\sinh^{-1}x = \sinh 3y\) and \(y = \frac{\sinh^{-1}x}{3}\). As \(\sinh x \geq (\sinh^{-1}x)\), for all \(x \geq 0\), therefore \(6x \geq \sinh^{-1}(\sinh^{-1}x)\), or,

\[
fx = \sinh^{-1}x \geq \sinh^{-1}\left(\frac{1}{6}\sinh^{-1}(\sinh^{-1}x)\right) = \sinh^{-1}\left(\frac{1}{2}y\right) = gy.
\]

Therefore, \(fx \leq gy\). Hence \((f, g)\) is partially weakly increasing with respect to \(R\).

To prove that \((g, f)\) is partially weakly increasing with respect to \(S\), let \(x, y \in X\) be such that \(y \in S^{-1}gx\), that is \(Sy = gx\). Hence, we have \(\sinh^{-1}\frac{x}{2} = \sinh 6y\) and so, \(y = \frac{\sinh^{-1}(\sinh^{-1}x)}{6}\). As \(\sinh x \geq (\sinh^{-1}x)\), for all \(x \geq 0\), therefore \(3x \geq \frac{x}{2} \geq \sinh^{-1}(\sinh^{-1}\frac{x}{2})\), or, \(\frac{x}{2} \geq \frac{\sinh^{-1}(\sinh^{-1}x)}{6}\), so,

\[
gx = \sinh^{-1}\left(\frac{x}{2}\right) \geq \sinh^{-1}\left(\frac{1}{6}\sinh^{-1}(\sinh^{-1}\frac{x}{2})\right) = \sinh^{-1}(y) = fy.
\]

Therefore, \(gx \leq fy\).

Furthermore, \(fx = gx = SX = RX = (0, \infty)\) and the pairs \((f, S)\) and \((g, R)\) are compatible. Indeed, let \(\{x_n\}\) is a sequence in \(X\) such that for some \(t \in X\), \(\lim_{n \to \infty} d(fx_n, tx) = \lim_{n \to \infty} d(gx_n, ty) = 0\). Therefore, we have,

\[
\lim_{n \to \infty} |\sinh^{-1}x_n - t| = \lim_{n \to \infty} |\sinh 6x_n - t| = 0.
\]

Continuity of \(\sinh^{-1}\) and \(\sinh\) implies that,

\[
\lim_{n \to \infty} |x_n - \sinh t| = \lim_{n \to \infty} \left|x_n - \frac{\sinh^{-1}t}{6}\right| = 0
\]

and the uniqueness of the limit gives that \(\sinh t = \frac{\sinh^{-1}t}{6}\). But,

\[
\sinh t = \frac{\sinh^{-1}t}{6} \iff t = 0.
\]

So, we have \(t = 0\). Since \(f\) and \(S\) are continuous, we have

\[
\lim_{n \to \infty} d(fTx_n, Tfx_n) = \lim_{n \to \infty} |fTx_n - Tfx_n| = 0.
\]

Define \(\psi, \varphi : [0, \infty) \to [0, \infty)\) as \(\psi(t) = bt\) and \(\varphi(t) = (b - 1)t\) for all \(t \in [0, \infty)\), where \(b\).

Using the mean value theorem simultaneously for the functions \(\sinh^{-1}\) and \(\sinh\), we have

\[
\psi(2^d(fx, gx)) = 4b|fx - gy|^2 = 4b\left|\sinh^{-1}x - \sinh^{-1}\left(\frac{y}{2}\right)\right|^2
\]

\[
\leq 4b|y - \frac{y}{2}|^2
\]

\[
\leq 4b|y - \frac{y}{2}|^2
\]

\[
\leq 6|\sinh 6x - \sinh 3y|^2
\]

\[
\leq |x - \varphi(y)|^2
\]

\[
= d(Sx, Ry) \leq M(x, y)
\]

\[
= \psi(M(x, y)) - \varphi(M(x, y)).
\]

Thus, \((2.1)\) is satisfied for all \(x, y \in X\). Therefore, all the conditions of Theorem 2.1 are satisfied. Moreover, \(0\) is a coincidence point of \(f, g, R\) and \(S\).

Let \(\Lambda\) be the set of all functions \(\mu : [0, \infty) \to [0, \infty)\) satisfying the following conditions:

(I) \(\mu\) is a positive Lebesgue integrable mapping on each compact subset of \([0, \infty)\).

(II) For all \(\varepsilon > 0\), \(\int_0^\varepsilon \mu(t)dt > 0\).

Remark 2.7. Suppose that there exists a \(\mu \in \Lambda\) such that mappings \(f, g, R\) and \(S\) satisfy the following condition.

\[
\int_0^{\psi(2^d(fx, gx))} \mu(t)dt \leq \int_0^{\psi(M(x, y))} \mu(t)dt - \int_0^{\psi(M(x, y))} \mu(t)dt.
\]

Then, \(f, g, R\) and \(S\) have a coincidence point, if the other conditions of Theorem 2.1 are satisfied.
For this, define the function $\Gamma(x) = \int_0^a \mu(t)dt$. Then, (2.38) becomes

$$
\Gamma(\psi(s^2d(fx,gy))) \leq \Gamma(\psi(M_s(x,y))) - \Gamma(\phi(M_s(x,y))).
$$

Taking $\psi_1 = \Gamma \phi$ and $\phi_1 = \Gamma \phi$, it is easy to verify that $\psi_1$ and $\phi_1$ are altering distance functions.

Taking $R = S = I_X$ (the identity mapping on $X$) in Theorems 2.1 and 2.2, we obtain the following common fixed point result.

**Corollary 2.8.** Let $(X, \preceq, d)$ be a partially ordered and complete $b$–metric space. Let $f, g : X \to X$ be two mappings. Suppose that for every comparable elements $x, y \in X$,

$$
\psi(s^2d(fx,gy)) \leq \psi(M_s(x,y)) - \phi(M_s(x,y)),
$$

where,

$$
M_s(x,y) = \max \left\{ d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy) + d(y,fx)}{2s} \right\}
$$

and $\psi, \phi : [0, \infty) \to [0, \infty)$ are altering distance functions. Then, the pair $(f, g)$ have a common fixed point $z$ in $X$ provided that the pair $(f, g)$ is weakly increasing and either,

a. $f$ and $g$ are continuous, or,

b. $X$ is regular.

**Remark 2.9.** Theorems 2.1 and 2.7 from [17] are special cases from our main results (Theorems 2.1 and 2.2), if we take $s = 1$.

3. Existence of a common solution for a system of integral equations

Consider the following system of integral equations:

$$
\begin{align*}
 x(t) &= \int_a^b K_1(t, r, x(r))dr, \\
 y(t) &= \int_a^b K_2(t, r, x(r))dr.
\end{align*}
$$

(3.1)

where $b > a \geq 0$. The purpose of this section is to present an existence theorem for a solution to 3.1 that belongs to $X = C[a, b]$ (the set of continuous real functions defined on $[a, b]$), by using the obtained result in Corollary 2.8.

Here, $K_1, K_2 : [a, b] \times [a, b] \times R \to R$. The considered problem can be reformulated in the following manner.

Let $f, g : X \to X$ be the mappings defined by:

$$
\begin{align*}
 fx(t) &= \int_a^b K_1(t, r, x(r))dr, \\
 gy(t) &= \int_a^b K_2(t, r, x(r))dr,
\end{align*}
$$

for all $x \in X$ and for all $t \in [a, b]$.

Then the existence of a solution to 3.1 is equivalent to the existence of a common fixed point of $f$ and $g$. Obviously, $X$ with the $b$-metric given by

$$
d(u, v) = \max_{t \in [a, b]} |u(t) - v(t)|^p,
$$

for all $u, v \in X$ is a complete $b$-metric space.

We endow $X$ with the partial ordered $\preceq$ given by:

$$
x \preceq y \iff x(t) \leq y(t),
$$

for all $t \in [0, T]$. Moreover, in [31], it is proved that $(X, \preceq)$ is regular.

Now, we will prove the following result.

**Theorem 3.1.** Suppose that the following hypotheses hold:

(i) $K_1, K_2 : [a, b] \times [a, b] \times R \to R$ are continuous;

(ii) for all $t, r \in [a, b]$ and $x \in X$, we have,
\begin{align*}
K_1(t, r, x(r)) & \leq K_2(t, r, \int_a^b K_1(r, t, x(t))dt) \\
\text{and} \\
K_2(t, r, x(r)) & \leq K_1(t, r, \int_a^b K_2(r, t, x(t))dt).
\end{align*}

(iii) for all $r, t \in [a, b]$ and $x, y \in X$ with $x \preceq y$ we have,
$$|K_1(t, r, x(r)) - K_2(t, r, y(r))| \leq \xi(t, r)(\ln(1 + |x(r) - y(r)|^p)),$$
where $\xi$ is a continuous function satisfying
$$\sup_{t \in [a, b]} \left( \int_a^b \xi(t, r)^p dr \right) = \frac{1}{2^{2p-2}(b-a)}.$$

Then, the integral Eqs. 3.1 have a solution $x \in X$.

**Proof.** From condition (ii), the ordered pairs $(f, g)$ and $(g, f)$ are partially weakly increasing, let $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Now, let $x, y \in X$ be such that $x \preceq y$. From condition (iii), for all $t \in [a, b]$ we have,
$$\begin{align*}
(2^{2p-2}|f(t) - g(t)|^p & \leq 2^{2p-2p} \left( \int_a^b |K_1(t, r, x(r)) - K_2(t, r, x(r))|dr \right)^p \\
& \leq 2^{2p-2p} \left( \left( \int_a^b \xi(t, r)^p dr \right)^{\frac{1}{p}} \left( \int_a^b |K_1(t, r, x(r)) - K_2(t, r, x(r))|^p dr \right)^{\frac{1}{p}} \right)^p \\
& \leq 2^{2p-2p}(b-a)^{\frac{p}{p}} \left( \int_a^b \xi(t, r)^p (\ln(1 + |x(r) - y(r)|^p))^p dr \right) \\
& \leq 2^{2p-2p}(b-a)^{\frac{p}{p}} \left( \int_a^b \xi(t, r)^p (\ln(1 + d(x, y))^p dr \right) \\
& \leq 2^{2p-2p}(b-a)^{\frac{p}{p}} \left( \int_a^b \xi(t, r)^p (\ln(1 + M_s(x, y))^p dr \right) \\
& = 2^{2p-2p}(b-a)^{p-1} \left( \int_a^b \xi(t, r)^p dr \right) (\ln(1 + M_s(x, y))^p \\
& < \ln(1 + M_s(x, y))^p \\
& = M_s(x, y)^p - (\ln(1 + M_s(x, y))^p.
\end{align*}$$

Hence,
$$\begin{align*}
(s^2d(fx, gy))^p & = (s^2 \sup_{t \in [a, b]} |f(t) - g(t)|^p]^p \leq M_s(x, y)^p - [M_s(x, y)^p - (\ln(1 + M_s(x, y))^p].
\end{align*}$$

Taking $\psi(t) = t^p$ and $\varphi(t) = t^p - (\ln(1 + t))^p$ in Corollary 2.8 there exists $x \in X$, a common fixed point of $f$ and $g$, that is, $x$ is a solution for 3.1. \qed

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**References**
