Robust stability of uncertain fuzzy BAM neural networks of neutral-type with Markovian jumping parameters and impulses

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A B S T R A C T

In this paper, the problem of neutral-type impulsive bidirectional associative memory neural networks (NIBAMNNs) with time delays are first established by a Takagi–Sugeno (T-S) fuzzy model in which the consequent parts are composed of a set of NIBAMNNs with interval delays and Markovian jumping parameters (MJPs). Sufficient conditions to check the robust exponential stability of the derived model are based on the Lyapunov–Krasovskii functionals (LKFs) containing some novel triple integral terms, Lyapunov stability theory and employing the free-weighting matrix method. The delay-dependent stability conditions are established in terms of linear matrix inequalities (LMIs), which can be very efficiently solved using Matlab LMI control toolbox. Finally, numerical examples and remarks are given to illustrate the effectiveness and usefulness of the derived results.

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1. Introduction

Studying artificial neural networks (ANNs) has been the central focus of intensive research activities during the last decades, since these NNs have found a variety of applications in areas like associative memory, pattern classification, reconstruction of moving images, signal processing, solving optimization problems, etc., see [1,2]. Due to these applications different models of NNs have been extensively investigated in the literature. As is well known, in both biological and ANNs, the interactions between neurons are generally asynchronous, which gives rise to the inevitable signal transmission delay, and the delay may result in oscillation and instability. Therefore, the stability analysis for NNs with time delays has received great attention during the past years and a number of remarkable results have been proposed [3–8].

BAMNNs was first proposed and researched by Kosko [9], which generalize the single-layer auto-associative Hebbian correlator to a two-layer pattern matched hetero-associative circuit. It possesses many nice properties due to its special structure of connection weights and its practical applications in storing paired patterns in both forward and backward directions. These applications rely on the dynamical behaviors of the networks heavily, see [10–12]. For instance, some criteria guaranteeing the existence of a unique equilibrium point and its global asymptotic stability of BAMNNs with constant time-delays were established in [13–15]. Moreover, the exponential stability property is particularly important when the exponential convergence rate is used to determine the speed of neural computations. Considering this, the exponential stable analysis problem for delayed BAMNNs has been investigated in [16–18]. In addition, since NNs usually have a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, it is desired to model them by introducing distributed delays. Therefore, both discrete and distributed delays should be taken into account when modelling realistic NNs and since then the stability analysis for NNs with distributed delays has been investigated in [19,20]. In particular, some sufficient conditions have been derived for BAMNNs with discrete and distributed time-delays [21,22].

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Table 1.1

<table>
<thead>
<tr>
<th>Notation</th>
<th>Denotes</th>
</tr>
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<tbody>
<tr>
<td>$\mathbb{R}^n$</td>
<td>n-dimensional Euclidean space</td>
</tr>
<tr>
<td>$\mathbb{R}^{n \times n}$</td>
<td>Set of all $n \times n$ real matrices</td>
</tr>
<tr>
<td>$X &gt; Y$</td>
<td>$X - Y$ is positive definite</td>
</tr>
<tr>
<td>$X \geq Y$</td>
<td>$X - Y$ is positive semi-definite</td>
</tr>
<tr>
<td>$X^T$</td>
<td>Transpose of matrix $X$</td>
</tr>
<tr>
<td>$| \cdot |$</td>
<td>Euclidean norm in $\mathbb{R}^n$</td>
</tr>
<tr>
<td>$\lambda_{\text{max}}(X)$</td>
<td>The largest eigen value of $X$</td>
</tr>
<tr>
<td>$\lambda_{\text{min}}(X)$</td>
<td>The smallest eigen value of $X$</td>
</tr>
<tr>
<td>$(\Omega, \mathcal{F}, \mathcal{P})$</td>
<td>A complete probability space with a filtration ${\mathcal{F}<em>t}</em>{t \geq 0}$ satisfying the usual conditions that is the filtration contains all $\mathcal{P}$-null sets and is right continuous</td>
</tr>
<tr>
<td>$\mathbb{E}(\cdot)$</td>
<td>The expectation operator with respect to the given probability measure $\mathcal{P}$</td>
</tr>
<tr>
<td>$*$</td>
<td>Symmetric block in one symmetric matrix</td>
</tr>
</tbody>
</table>

On the other hand, parameter uncertainty which can be commonly encountered because of the inaccuracies and changes in the environment of the model will break the stability of the systems. Therefore, it is of great significance to ensure that the system is stable with respect to the uncertainties and recently the robust stability analysis of BAMNNs has gained much research attention [13,18,23,24]. Moreover, authors in [25] dealt with the problem of a partial element equivalent circuit model of neutral type due to its increase in high performance of VLSI systems since the delay circuits become very important. In bio-chemistry experiments, neural information may transfer across chemical reactivity, which results in a neutral-type process [26–28]. However, in many cases the existing NN models cannot characterise the properties of a neural reaction process precisely. Considering this, the delay-dependent asymptotic stability criteria for neutral-type BAMNNs with discrete and distributed time-delays were derived in [29–32].

Dynamical systems are often classified into two categories such as continuous-time and discrete-time systems. These two dynamic systems are widely studied in population models and NNs, yet there is a somewhat new category of dynamical system, which is neither continuous-time nor purely discrete-time; these are called dynamical systems with impulses. This third category of dynamical systems displays a combination of characteristics of both continuous-time and discrete-time systems [33,34]. Though the non-impulsive systems have been well studied in theory and in practice, the theory of impulsive differential equations is now being recognised to be not only richer than the corresponding theory of differential equations without impulse, but also represents a more natural framework for mathematical modelling of many real-world phenomena, such as population dynamics and the NNs and the related issues that have been reported in the literature [27,35–40].

It is well known that the NNs also incorporate abrupt changes in their structure, thus the Markovian jump linear system is very appropriate to describe its dynamics [41]. This class of systems is a special class of hybrid systems, which is specified by two components in the state. The first one refers to the mode, which is described by a continuous-time finite-state Markovian process, and the second one refers to the state which is represented by a system of differential equations. Recently, some pioneering works on stability analysis for MJPs with time delay have emerged [15,28,39,41–46].

In the past few decades, fuzzy logic theory has shown to be an appealing and efficient approach to deal with the analysis and synthesis problems for complex nonlinear systems. The well known T–S fuzzy model [47] is a popular and convenient tool in functional approximations. In contrast to pure NNs [3–8] and Fuzzy Systems (FSs) [48–50], the concept of fuzzy neural networks (FNNs) possesses both of its advantages. It combines the capability of fuzzy reasoning in handling uncertain information and the capability of ANNs in learning from process. In the last decade, the concept of incorporating fuzzy logic into NNs has grown into a popular research topic, see [51,52,23,53,54,40]. The T–S fuzzy model approach has been used to investigate nonlinear MJSS in [55,56]. The T–S fuzzy model approach has been widely used to investigate both nonlinear MJSS and NNs with MJPs in [57,24,58]. Up to now, to the best of our knowledge, the robust exponential stability results for uncertain fuzzy NIBAMNNs (UFNIBAMNNs) with MJPs and interval time-varying delays has not been addressed in the previous literatures.

Motivated by the ideas discussed above, in this paper, we will address the problem of robust exponential stability for delayed NIBAMNNs with MJPs by T–S fuzzy model. Mixed time delays that include discrete delay, distributed delay and neutral delay are assumed to be time-varying and belong to the given intervals. By constructing LKFs including some novel triple integral terms involving both upper and lower bounds of the delay and utilizing the free-weighting matrix method, some delay-range-dependent robust exponential stability conditions are obtained in terms of LMIs. These LMI conditions can be checked easily by using some standard numerical packages. In the absence of neutral delay, the derivative of the time-varying delay need not be differentiable and its derivative need not be less than one. Numerical examples are given to illustrate the effectiveness of the derived methods.

The notations used in this paper are summarised in the following Table 1.1.

2. Problem description and preliminaries

Given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, $(\eta_t, t \geq 0)$ is a homogeneous finite-state Markovian process with right continuous trajectories and taking values in a finite set $S = \{1, 2, \ldots, s\}$ with the initial model $\eta_0$. Let $IT = [\pi_{pp'}]$, $p, p' \in S$ denote the
transition rate matrix with transition probability

\[ \Pr(\eta_{t+\Delta t} = p'|\eta_t = p) = \begin{cases} \pi_{pp'} \Delta t + o(\Delta t), & p \neq p' \\ 1 + \pi_{pp} \Delta t + o(\Delta t), & p = p' \end{cases} \]

where \( \Delta t > 0 \), \( \lim_{\Delta t \to 0} (o(\Delta t)/\Delta t) = 0 \), and \( \pi_{pp'} \) is the transition rate from mode \( p \) to mode \( p' \), satisfying \( \pi_{pp'} \geq 0 \) for \( p \neq p' \) with \( \pi_{pp} = -\sum_{p'=1}^{5} \pi_{pp'} \), \( p, p' \in S \).

Consider the following NIBAMNNs with MJPs and mixed interval time-varying delays

\[ \begin{align*}
\dot{u}_i(t) &= -a_i(\eta_t)u_i(t) + \sum_{j=1}^{m} b_{ij}(\eta_t)f_j(u_j(t)) + \sum_{j=1}^{m} c_{ij}(\eta_t)f_j(u_j(t - \rho(t))) \\
&+ \sum_{j=1}^{m} d_{ij}(\eta_t) \int_{t-r(t)}^{t} f_j(v(s))ds + \sum_{j=1}^{n} e_{ij}(\eta_t)\dot{u}_i(t - \tau(t)) + l_i, \quad t > 0, \quad t \neq t_k, \\
u(t_k) &= i_k(\eta_t)(u(t_k^-)), \quad k \in \mathbb{Z}_+, \\
\dot{v}_j(t) &= -a_{2j}(\eta_t)v_j(t) + \sum_{i=1}^{n} b_{2ij}(\eta_t)g_i(u_i(t)) + \sum_{i=1}^{n} c_{2ij}(\eta_t)g_i(u_i(t - \tau(t))) \\
&+ \sum_{i=1}^{n} d_{2ij}(\eta_t) \int_{t-r(t)}^{t} g_i(u(s))ds + \sum_{i=1}^{m} e_{2ij}(\eta_t)\dot{v}_j(t - \rho(t)) + j_j, \quad t > 0, \quad t \neq t_k, \\
v(t_k) &= j_k(\eta_t)(v(t_k^-)), \quad k \in \mathbb{Z}_+, \\
\end{align*} \]

where \( i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, u_i(t) \) and \( v_j(t) \) are the state of the \( i \)-th and \( j \)-th neuron, respectively. \( a_i(\eta_t) > 0 \) and \( d_{2j}(\eta_t) > 0 \) denote the rate with which the cell \( i \) and \( j \) reset their potential to the resting states when isolated from the other cells and inputs. \( b_{ij}(\eta_t), b_{2ij}(\eta_t), c_{ij}(\eta_t), c_{2ij}(\eta_t), d_{ij}(\eta_t), d_{2ij}(\eta_t), e_{ij}(\eta_t) \) and \( e_{2ij}(\eta_t) \) are the connection weights at the time \( t \). \( l_i \) and \( j_j \) denote the constant external inputs. \( i_k(\eta_t)(\cdot), j_k(\eta_t)(\cdot) \) are constant real matrices at the moments of time \( t_k \). \( f_j(\cdot) \) and \( g_i(\cdot) \) stand for the sigmoidal functions of the \( i \)-th and \( j \)-th neurons, respectively. In addition, we are considering that the time-varying delays \( \tau(t), \rho(t) \) and \( \gamma(t) \) are differentiable functions satisfying that for \( t \geq 0 \) and given scalars \( 0 \leq r_1 < r_2, 0 \leq \rho_1 < \rho_2, \tau(t) \geq 0, \gamma(t) \geq 0, \mu_1 \geq 0 \) and \( \mu_2 \geq 0 \), they hold \( 0 \leq \tau_1 \leq \tau(t) < \tau_2, \gamma(t) \leq \gamma_1, 0 \leq \rho_1 \leq \rho_2, 0 \leq \rho(t) \leq \rho_2, 0 \leq \gamma(t) \leq \gamma_2 \).

We shall consider model (1) with the initial conditions

\[ \begin{align*}
u(s) &= \psi(s), \quad s \in [-\kappa_1, 0], \\
\phi(s) &= \varphi(s), \quad s \in [-\kappa_2, 0],
\end{align*} \]

where \( \kappa_1 = \max\{\tau_2, \gamma_2\}, \kappa_2 = \{\rho_2, \tau_2\} \).

In this paper, we make the following assumptions:

(A1) The impulsive times \( t_k \) satisfy \( 0 = t_0 < t_1 < \cdots < t_k \to \infty \) and \( \inf_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} > 0; \)

(A2) \( f_j(\cdot) \) and \( g_i(\cdot) \) are bounded on \( \mathbb{R}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m; \)

(A3) \( f_j(\cdot) \) and \( g_i(\cdot) \) are Lipschitz continuous, that is there exist real scalars \( w_{2j} > 0 \) and \( w_{2i} > 0 \), such that \( |f_j(u) - f_j(v)| \leq w_{2j}|u - v|, |g_i(u) - g_i(v)| \leq w_{2i}|u - v|, \) for any \( u, v \in \mathbb{R}, u \neq v, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m. \)

The system (1) is equivalent to the vector form

\[ \begin{align*}
\dot{u}(t) &= -A_l(\eta_t)u(t) + B_l(\eta_t)f(v(t)) + C_l(\eta_t)f(v(t - \rho(t))) + D_l(\eta_t) \int_{t-r(t)}^{t} f(v(s))ds \\
&+ E_l(\eta_t)\dot{u}(t - \tau(t)), \quad t > 0, \quad t \neq t_k, \\
u(t_k) &= i_l(\eta_t)(u(t_k^-)), \quad k \in \mathbb{Z}_+, \\
\dot{v}(t) &= -A_r(\eta_t)v(t) + B_r(\eta_t)g(u(t)) + C_r(\eta_t)g(u(t - \tau(t))) + D_r(\eta_t) \int_{t-r(t)}^{t} g(u(s))ds \\
&+ E_r(\eta_t)\dot{v}(t - \rho(t)), \quad t > 0, \quad t \neq t_k, \\
v(t_k) &= j_r(\eta_t)(v(t_k^-)), \quad k \in \mathbb{Z}_+,
\end{align*} \]

where \( u(t) = (u_1(t), u_2(t), \ldots, u_m(t))^T, v(t) = (v_1(t), v_2(t), \ldots, v_n(t))^T, A_l = \text{diag}(a_{11}(\eta_t), \ldots, a_{1n}(\eta_t)), A_r = \text{diag}(a_{21}(\eta_t), \ldots, a_{2m}(\eta_t)), B_l = (b_{11}(\eta_t))_{n \times m}, B_r = (b_{21}(\eta_t))_{m \times n}, C_l = (c_{11}(\eta_t))_{n \times m}, C_r = (c_{21}(\eta_t))_{m \times n}, D_l = (d_{1l}(\eta_t))_{n \times m}, D_r = (d_{2r}(\eta_t))_{m \times n}, E_l = (e_{1l}(\eta_t))_{n \times m}, E_r = (e_{2r}(\eta_t))_{m \times n}, f(\cdot) = (f_1(\cdot), f_2(\cdot), \ldots, f_m(\cdot))^T, g(\cdot) = (g_1(\cdot), g_2(\cdot), \ldots, g_n(\cdot))^T. \)

Assume \( u^* = (u^*_1, u^*_2, \ldots, u^*_m), v^* = (v^*_1, v^*_2, \ldots, v^*_n) \) are equilibrium points of system (3). One can derive from (3) that the transformations \( x(\cdot) = u(\cdot) - u^*, y(\cdot) = v(\cdot) - v^* \) and \( g(x(t)) = G(x(t) + u^*) - G(u^*), f(y(t)) = F(y(t) + v^*) - F(v^*) \)
transform system (3) into the following system:

\[
\begin{aligned}
\dot{x}(t) &= -A_1(\eta_1)x(t) + B_1(\eta_1)F(y(t)) + C_1(\eta_1)F(y(t - \rho(t))) + D_1(\eta_1) \int_{t-r(t)}^{t} F(y(s))ds \\
&\quad + E_1(\eta_1)\dot{x}(t - \tau(t)), \quad t > 0, \quad t \neq t_k, \\
x(t_k) &= l_k(\eta_1)(x(t_k^-)), \quad k \in \mathbb{Z}_+, \quad x(s) = \psi(s) - u^*, \quad s \in [-\kappa_1, 0], \\
\dot{y}(t) &= -A_2(\eta_1)y(t) + B_2(\eta_1)G(x(t)) + C_2(\eta_1)G(x(t - \tau(t))) + D_2(\eta_1) \int_{t-r(t)}^{t} G(x(s))ds \\
&\quad + E_2(\eta_1)y(t - \rho(t)), \quad t > 0, \quad t \neq t_k, \\
y(t_k) &= l_k(\eta_1)(y(t_k^-)), \quad k \in \mathbb{Z}_+, \quad y(s) = \varphi(s) - v^*, \quad s \in [-\kappa_2, 0],
\end{aligned}
\]

(4)

where \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T\), \(y(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T\), \(F() = (F_1(), F_2(), \ldots, F_m())^T\), \(G() = (G_1(), G_2(), \ldots, G_n())^T\). It can be easily verified that \(F()\) and \(G()\) satisfy

(H1) \(F()\) and \(G()\) are bounded on \(\mathbb{R}\);

(H2) \(F()\) and \(G()\) are Lipschitz continuous, that is there exist real scalars \(w_{2j} > 0\) and \(w_{1j} > 0\), such that \(|F(x) - F(y)| \leq w_{2j}|x - y|\), \(|G(x) - G(y)| \leq w_{1j}|x - y|\), for any \(x, y \in \mathbb{R}, x \neq y\);

(H3) \(F(0) = 0\) and \(G(0) = 0\).

Then, from (H2), one can obtain

\[
\begin{aligned}
F^T(y(t - \rho(t)))F(y(t - \rho(t))) &\leq y^T(t - \tau(t))W_2y(t - \rho(t)) \\
G^T(x(t - \tau(t)))G(x(t - \tau(t))) &\leq x^T(t - \tau(t))W_1x(t - \tau(t))
\end{aligned}
\]

(5)

where \(W_1 = \text{diag}(w_{11}, w_{12}, \ldots, w_{1m}), W_2 = \text{diag}(w_{21}, w_{22}, \ldots, w_{2m})\).

In the following section, the uncertain NIBAMNNs with MJPs will be represented by a T–S fuzzy model. The ith rule of this T–S fuzzy model is of the following form:

**Plant rule i:**

**IF** \([u_1(t) \text{ and } v_1(t) \text{ are } \eta_{1j}^1], [u_2(t) \text{ and } v_2(t) \text{ are } \eta_{2j}^1], \ldots, [u_p(t) \text{ and } v_p(t) \text{ are } \eta_{pj}^1}\),

**THEN**

\[
\begin{aligned}
\dot{x}(t) &= -[A_{11}(\eta_1) + \Delta A_{11}(t, \eta_1)]x(t) + [B_{11}(\eta_1) + \Delta B_{11}(t, \eta_1)]F(y(t)) \\
&\quad + [C_{11}(\eta_1) + \Delta C_{11}(t, \eta_1)]F(y(t - \rho(t))) + [D_{11}(\eta_1) + \Delta D_{11}(t, \eta_1)] \int_{t-r(t)}^{t} F(y(s))ds \\
&\quad + [E_{11}(\eta_1) + \Delta E_{11}(t, \eta_1)]\dot{x}(t - \tau(t)), \quad t > 0, \quad t \neq t_k, \\
x(t_k) &= l_k(\eta_1)(x(t_k^-)), \quad k \in \mathbb{Z}_+, \\
\dot{y}(t) &= -[A_{21}(\eta_1) + \Delta A_{21}(t, \eta_1)]y(t) + [B_{21}(\eta_1) + \Delta B_{21}(t, \eta_1)]G(x(t)) \\
&\quad + [C_{21}(\eta_1) + \Delta C_{21}(t, \eta_1)]G(x(t - \tau(t))) + [D_{21}(\eta_1) + \Delta D_{21}(t, \eta_1)] \int_{t-r(t)}^{t} G(x(s))ds \\
&\quad + [E_{21}(\eta_1) + \Delta E_{21}(t, \eta_1)]\dot{y}(t - \rho(t)), \quad t > 0, \quad t \neq t_k, \\
y(t_k) &= l_k(\eta_1)(y(t_k^-)), \quad k \in \mathbb{Z}_+,
\end{aligned}
\]

(6)

where \(\eta_{jj}, j = 1, 2, \ldots, p\) are fuzzy sets, \((u_1(t), u_2(t), \ldots, u_p(t), v_1(t), v_2(t), \ldots, v_p(t))^T\) is the premise variable vector, \(x(t)\) and \(y(t)\) are the state variables, \(r\) is the number of IF-THEN rules. It is known that system (6) has a unique global solution on \(t \geq 0\) with the initial values \(x_0 \in C([-\kappa, 0]; \mathbb{R}^n)\) and \(y_0 \in C([-\kappa, 0]; \mathbb{R}^m)\).

For each possible value of \(\eta_i = p, \) in the succeeding discussion, we will denote the matrices associated with the pth mode by

\[
\begin{aligned}
A_{11,p} &= A_{11}(\eta_i), & B_{11,p} &= B_{11}(\eta_i), & C_{11,p} &= C_{11}(\eta_i), & D_{11,p} &= D_{11}(\eta_i), & E_{11,p} &= E_{11}(\eta_i), \\
A_{21,p} &= A_{21}(\eta_i), & B_{21,p} &= B_{21}(\eta_i), & C_{21,p} &= C_{21}(\eta_i), & D_{21,p} &= D_{21}(\eta_i), & E_{21,p} &= E_{21}(\eta_i), \\
\Delta A_{11,p} &= \Delta A_{11}(t, \eta_i), & \Delta B_{11,p} &= \Delta B_{11}(t, \eta_i), & \Delta C_{11,p} &= \Delta C_{11}(t, \eta_i), & \Delta D_{11,p} &= \Delta D_{11}(t, \eta_i), \\
\Delta A_{21,p} &= \Delta A_{21}(t, \eta_i), & \Delta B_{21,p} &= \Delta B_{21}(t, \eta_i), & \Delta C_{21,p} &= \Delta C_{21}(t, \eta_i), & \Delta D_{21,p} &= \Delta D_{21}(t, \eta_i),
\end{aligned}
\]

where \(A_{11,p}, A_{21,p}, B_{11,p}, B_{21,p}, C_{11,p}, C_{21,p}, D_{11,p}, D_{21,p}, E_{11,p}, E_{21,p}\), for \(p \in S\), are known constant matrices of appropriate dimensions that describe the nominal system and \(\Delta A_{11,p}, \Delta A_{21,p}, \Delta B_{11,p}, \Delta B_{21,p}, \Delta C_{11,p}, \Delta C_{21,p}, \Delta D_{11,p}, \Delta D_{21,p}, \Delta E_{11,p}, \Delta E_{21,p}\) are unknown matrices that represent the time-varying parameter uncertainties and are assumed to be of the form:

\[
\begin{aligned}
[\Delta A_{11,p}^T, \Delta B_{11,p}^T, \Delta C_{11,p}^T, \Delta D_{11,p}^T, \Delta E_{11,p}^T] &= H_{F1}(t) [G_{11,p}, G_{21,p}, G_{31,p}, G_{41,p}, G_{51,p}] \\
[\Delta A_{21,p}^T, \Delta B_{21,p}^T, \Delta C_{21,p}^T, \Delta D_{21,p}^T, \Delta E_{21,p}^T] &= H_{F1}(t) [G_{51,p}, G_{71,p}, G_{81,p}, G_{91,p}, G_{101,p}]
\end{aligned}
\]

(7)
\[
F_p^T(t)F_p(t) \leq I \quad \forall p \in S.
\]

It is assumed that all elements of \( F_p \) are Lebesgue measurable, and \( \Delta A_{1l,p}, \Delta B_{1l,p}, \Delta C_{1l,p}, \Delta D_{1l,p}, \Delta E_{1l,p}, \Delta A_{2l,p}, \Delta B_{2l,p}, \Delta C_{2l,p}, \Delta D_{2l,p}, \Delta E_{2l,p} \) are said to be admissible if both (7) and (8) hold. Now define

\[
A_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))A_{1i,p}, \quad B_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))B_{1i,p}, \quad C_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))C_{1i,p},
\]

\[
D_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))D_{1i,p}, \quad E_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))E_{1i,p}, \quad A_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))A_{2i,p},
\]

\[
B_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))B_{2i,p}, \quad C_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))C_{2i,p}, \quad D_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))D_{2i,p},
\]

\[
E_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))E_{2i,p}, \quad \Delta A_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta A_{1i,p}, \quad \Delta B_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta B_{1i,p},
\]

\[
\Delta C_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta C_{1i,p}, \quad \Delta D_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta D_{1i,p}, \quad \Delta E_{1p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta E_{1i,p},
\]

\[
\Delta A_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta A_{2i,p}, \quad \Delta B_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta B_{2i,p}, \quad \Delta C_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta C_{2i,p},
\]

\[
\Delta D_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta D_{2i,p}, \quad \Delta E_{2p}(t) := \sum_{i=1}^{r} h_i(\theta(t))\Delta E_{2i,p}.
\]

Then the defuzzified output of system (6) is inferred as follows:

\[
\dot{x}(t) = -[A_{1p}(t) + \Delta A_{1p}(t)]x(t) + [B_{1p}(t) + \Delta B_{1p}(t)]y(t) + [C_{1p}(t) + \Delta C_{1p}(t)]F(y(t)) + [D_{1p}(t) + \Delta D_{1p}(t)] \int_{t-r(t)}^{t} F(y(s))ds
\]

\[
+ [E_{1p}(t) + \Delta E_{1p}(t)]\dot{x}(t) \quad t > 0, \ t \neq t_k,
\]

\[
x(t_k) = I_{p_0}(x(t^-_k)), \quad k \in \mathbb{Z}^+,
\]

\[
\dot{y}(t) = -[A_{2p}(t) + \Delta A_{2p}(t)]y(t) + [B_{2p}(t) + \Delta B_{2p}(t)]G(x(t)) + [C_{2p}(t) + \Delta C_{2p}(t)]G(x(t) - \tau(t)) + [D_{2p}(t) + \Delta D_{2p}(t)] \int_{t-\gamma(t)}^{t} G(x(s))ds
\]

\[
+ [E_{2p}(t) + \Delta E_{2p}(t)]\dot{y}(t) \quad t > 0, \ t \neq t_k,
\]

\[
y(t_k) = J_{p_0}(y(t^-_k)), \quad k \in \mathbb{Z}^+,
\]

with \( h_i(\theta(t)) = M_i(\theta(t))/\sum_{j=1}^{r} M_j(\theta(t)) \), \( M_i(\theta(t)) = \prod_{j=1}^{r} \eta^1_j(\theta(t), \eta^2_j(\theta(t)) \) is the grade of membership of \( \theta(t) \) in \( \eta_j \),. It is assumed that \( M_i(\theta(t)) \geq 0, \ i = 1, 2, \ldots, r, \sum_{i=1}^{r} M_i(\theta(t)) > 0 \) for all \( t \). Therefore, \( h_i(\theta(t)) \geq 0 \) (for \( i = 1, 2, \ldots, r \)) and \( \sum_{i=1}^{r} h_i(\theta(t)) = 1 \).

For the sake of convenience, the following definition and lemmas are introduced as follows:

**Definition 2.1.** For the UFNABMNN (9) and every initial condition \( \psi \in C\{[-\kappa_1, 0]; \mathbb{R}^n\}, \varphi \in C\{[-\kappa_2, 0]; \mathbb{R}^m\} \), the equilibrium point is robustly exponentially stable in the mean square if, for every network mode, there exist scalars \( \alpha > 0 \) and \( M_1 > 0, M_2 > 0 \) such that

\[
\mathcal{E}\left(\|x(t, \psi)\|^2 + \|y(t, \varphi)\|^2\right) \leq e^{-\alpha t}\left(M_1\mathcal{E}\|\psi\|^2 + M_2\mathcal{E}\|\varphi\|^2\right),
\]

for all admissible uncertainties satisfying (7)-(8).

**Lemma 2.2** (Schur Complement [59]). Given constant matrices \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) with appropriate dimensions, where \( \Omega_1^T = \Omega_1 \) and \( \Omega_2^T = \Omega_2 \), then

\[
\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0.
\]

if and only if

\[
\begin{bmatrix}
\Omega_1 & \Omega_3^T \\
\Omega_3 & -\Omega_2
\end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix}
-\Omega_2 & \Omega_3 \\
\Omega_3^T & \Omega_1
\end{bmatrix} < 0.
\]
Lemma 3.1. Let $M = M^T$, $H$ and $E$ be real matrices of appropriate dimensions, which satisfy $F^T(t)F(t) \leq I$ then
$$M + HF(t)E + E^TF(t)H^T < 0,$$
if and only if there exists a positive scalar $\epsilon > 0$ such that
$$M + \frac{1}{\epsilon}HH^T + \epsilon E^TE < 0.$$
\[\Omega_{1,1}^{i,p} = P_{12p} + \sum_{p' = 1}^{i} \pi_{pp'} P_{2p'} + \frac{1}{\tau_2 + \tau_1} (S_2 + S_2^T), \quad \Omega_{2,2}^{i,p} = -Q_2 - R_1 - \frac{1}{\tau_2} R_2 - R_3,\]

\[\Omega_{2,3}^{i,p} = R_1 + \frac{1}{\tau_2} R_2, \quad \Omega_{2,4}^{i,p} = P_{3p}^T,\]

\[\Omega_{2,9}^{i,p} = P_{12p}^T, \quad \Omega_{2,10}^{i,p} = P_{14p}^T, \quad \Omega_{2,11}^{i,p} = P_{15p}^T, \quad \Omega_{3,3}^{i,p} = -Q_3 - R_1 - \frac{1}{\tau_2} R_2 - \frac{1}{\tau_2} R_4,\]

\[\Omega_{3,4}^{i,p} = -P_{8p}^T + \frac{1}{\tau_2} R_4,\]

\[\Omega_{3,9}^{i,p} = -P_{11p}^T, \quad \Omega_{3,10}^{i,p} = -P_{13p}^T, \quad \Omega_{3,11}^{i,p} = -P_{14p}^T,\]

\[\Omega_{4,4}^{i,p} = -(1 - \mu_1)P_{7p} - (1 - \mu_1)P_{7p}^T + (1 - \mu_1)P_{8p} + (1 - \mu_1)P_{8p}^T - (1 - \mu_1)P_{9p} - (1 - \mu_1)P_{9p}^T + \sum_{p' = 1}^{s} \pi_{pp'} P_{6p'}, \quad (1 - \mu_1)Q_1 - \frac{1}{\tau_2} R_4 - \frac{1}{\tau_2} R_4 + a_2 W_1^T W_1,\]

\[\Omega_{4,5}^{i,p} = P_{2p}^T, \quad \Omega_{4,6}^{i,p} = P_{6p},\]

\[\Omega_{4,9}^{i,p} = -(1 - \mu_1)P_{10p} + (1 - \mu_1)P_{11p}^T - (1 - \mu_1)P_{12p}^T + \sum_{p' = 1}^{s} \pi_{pp'} P_{7p'},\]

\[\Omega_{4,10}^{i,p} = -(1 - \mu_1)P_{11p} + (1 - \mu_1)P_{13p}^T - (1 - \mu_1)P_{14p}^T + \sum_{p' = 1}^{s} \pi_{pp'} P_{8p'},\]

\[\Omega_{4,11}^{i,p} = -(1 - \mu_1)P_{12p} + (1 - \mu_1)P_{14p}^T - (1 - \mu_1)P_{15p}^T + \sum_{p' = 1}^{s} \pi_{pp'} P_{9p'},\]

\[\Omega_{5,5}^{i,p} = \tau_2 R_1 + \tau_2 R_2 + \tau_2 R_3 + \tau_2 R_4 + \frac{\tau_2}{2} S_1 + \frac{\tau_2 - \tau_1}{2} S_2 - M - M^T + T_1,\]

\[\Omega_{5,6}^{i,p} = ME_{1p} + K^T E_{1p},\]

\[\Omega_{5,7}^{i,p} = MB_{1p}, \quad \Omega_{5,8}^{i,p} = MC_{1p}, \quad \Omega_{5,9}^{i,p} = P_{3p}, \quad \Omega_{5,10}^{i,p} = P_{4p}, \quad \Omega_{5,11}^{i,p} = P_{5p},\]

\[\Omega_{5,12}^{i,p} = MD_{1p},\]

\[\Omega_{6,6}^{i,p} = -E_{1p}^T K E_{1p} - E_{1p}^T K E_{1p} - (1 - \mu_1)T_1, \quad \Omega_{6,7}^{i,p} = -E_{1p}^T K B_{1p}, \quad \Omega_{6,8}^{i,p} = -E_{1p}^T K C_{1p},\]

\[\Omega_{6,9}^{i,p} = P_{7p}, \quad \Omega_{6,10}^{i,p} = P_{8p}, \quad \Omega_{6,11}^{i,p} = P_{9p}, \quad \Omega_{6,12}^{i,p} = -E_{1p}^T K D_{1p}, \quad \Omega_{7,7}^{i,p} = \tau_2 Z_2 - b_1 I,\]

\[\Omega_{8,8}^{i,p} = -b_2 I,\]

\[\Omega_{9,9}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'} P_{10p'} - \frac{1}{\tau_2} Q_4 - \frac{1}{\tau_2} S_1 + S_1^T, \quad \Omega_{9,10}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'} P_{11p'} + \frac{1}{\tau_2} (S_1 + S_1^T),\]

\[\Omega_{9,11}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'} P_{12p'}, \quad \Omega_{10,10}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'} P_{13p'} - \frac{1}{\tau_2} Q_5 - \frac{1}{\tau_2} (S_1 + S_1^T) - \frac{1}{\tau_2} (S_2 + S_2^T),\]

\[\Omega_{10,11}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'} P_{14p'} - \frac{1}{\tau_2} (S_2 + S_2^T), \quad \Omega_{11,11}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'} P_{15p'} - \frac{1}{\tau_2} (S_2 + S_2^T) - \frac{1}{\tau_2} Q_5,\]

\[\Omega_{12,12}^{i,p} = -Z_2,\]

\[A_{1,1}^{i,p} = Q_{3p} + Q_{3p}^T + \sum_{p' = 1}^{s} \pi_{pp'} Q_{1p'} + Q_6 + Q_7 + Q_8 + \rho_2 Q_9 + \rho_2 Q_{10} - R_7 - \frac{1}{\rho_2} R_8 - S_3 - S_3^T - \frac{\rho_{21}}{\rho_2 + \rho_1} (S_4 + S_4^T) + b_1 W_1^T W_2,\]

\[A_{1,2}^{i,p} = Q_{5p} + R_7, \quad A_{1,3}^{i,p} = -Q_4.\]
\[ A_{1,4}^{i,p} = -\left(1 - \mu_2\right)Q_{23p} + \left(1 - \mu_2\right)Q_{4p} - \left(1 - \mu_2\right)Q_{5p} + Q_{7p}^T + \sum_{p' = 1}^{s} \rho_{pp'} Q_{2p'} + \frac{1}{\rho_2} R_{8}, \]
\[ A_{1,5}^{i,p} = Q_{1p} - A_{2ip}^T N^T, \]
\[ A_{1,6}^{i,p} = Q_{2p} + A_{2ip}^T T E_{2ip}, \]
\[ A_{1,9}^{i,p} = Q_{10p} + \sum_{p' = 1}^{s} \rho_{pp'} Q_{3p'} + \frac{1}{\rho_2} \left(S_3 + S_3^T\right), \]
\[ A_{1,10}^{i,p} = \sum_{p' = 1}^{s} \rho_{pp'} Q_{3p'} + \frac{1}{\rho_2} \left(S_3 + S_3^T\right), \]
\[ A_{2,2}^{i,p} = -Q_2 - R_5 - \frac{1}{\rho_{21}} R_6 - R_7, \]
\[ A_{2,3}^{i,p} = R_5 + \frac{1}{\rho_{21}} R_5, \]
\[ A_{2,4}^{i,p} = Q_{8p}^T, \]
\[ A_{2,9}^{i,p} = Q_{14p}, \]
\[ A_{2,11}^{i,p} = Q_{15p}^T, \]
\[ A_{2,15}^{i,p} = Q_{2p}, \]
\[ A_{3,10}^{i,p} = -Q_{13p}, \]
\[ A_{3,11}^{i,p} = -Q_{14p}, \]
\[ A_{4,4}^{i,p} = -\left(1 - \mu_2\right)Q_{7p} - \left(1 - \mu_2\right)Q_{7p}^T + \left(1 - \mu_2\right)Q_{8p} + \left(1 - \mu_2\right)Q_{8p}^T + \left(1 - \mu_2\right)Q_{9p}^T \]
\[ + \sum_{p' = 1}^{s} \rho_{pp'} Q_{2p'}^T - \left(1 - \mu_2\right)Q_6 - \frac{1}{\rho_2} R_8 - \frac{1}{\rho_{21}} R_8 + b_2 W_{21}^T W_2, \]
\[ A_{5,5}^{i,p} = R_5 + \frac{1}{\rho_{21}} R_5, \]
\[ A_{5,6}^{i,p} = N E_{2ip} + L^T E_{2ip}, \]
\[ A_{5,7}^{i,p} = N B_{2ip}, \]
\[ A_{5,8}^{i,p} = N C_{2ip}, \]
\[ A_{5,9}^{i,p} = Q_{3p}, \]
\[ A_{5,10}^{i,p} = Q_{4p}, \]
\[ A_{5,11}^{i,p} = Q_{5p}, \]
\[ A_{6,6}^{i,p} = -E_{2ip}^T L E_{2ip} - E_{2ip}^T L^T E_{2ip} - \left(1 - \mu_2\right)T_2, \]
\[ A_{6,7}^{i,p} = -E_{2ip}^T L B_{2ip}, \]
\[ A_{6,8}^{i,p} = -E_{2ip}^T L C_{2ip}, \]
\[ A_{6,9}^{i,p} = Q_{7p}, \]
\[ A_{6,10}^{i,p} = Q_{8p}, \]
\[ A_{6,11}^{i,p} = Q_{9p}, \]
\[ A_{6,12}^{i,p} = -E_{2ip}^T L D_{2ip}, \]
\[ A_{7,7}^{i,p} = p^2 Z_1 - a_1 T, \]
\[ A_{8,8}^{i,p} = -a_2 T, \]
\[ A_{9,9}^{i,p} = \sum_{p' = 1}^{s} \rho_{pp'} Q_{10p'}, - \frac{1}{\rho_2} Q_{9} - \frac{1}{\rho_2} \left(S_3 + S_3^T\right), \]
\[ A_{9,10}^{i,p} = \sum_{p' = 1}^{s} \rho_{pp'} Q_{10p'}, \]
\[ A_{10,10}^{i,p} = \sum_{p' = 1}^{s} \rho_{pp'} Q_{10p'}, \]
\[ A_{11,11}^{i,p} = \sum_{p' = 1}^{s} \rho_{pp'} Q_{10p'}, \]
\[
\begin{align*}
\Lambda_{12,12} = -Z_1, \quad \tau_{21} = \tau_2 - \tau_1, \quad \rho_{21} = \rho_2 - \rho_1,
\end{align*}
\]
and the other parameters are defined to be zero.

**Proof.** For each \( p \in S \) consider the following LKFs,
\[
V(t, x(t), y(t), k) = \sum_{m=1}^{5} V_m(t, x(t), y(t), p),
\]
(14)
where
\[
\begin{align*}
V_1(t, x(t), y(t), p) &= \xi_1^T(t) P_1 \xi_1(t) + \xi_2^T(t) Q_2 \xi_2(t), \\
V_2(t, x(t), y(t), p) &= \int_{t-\tau(t)}^{t} x^T(s) Q_1 x(s) ds + \int_{t-\tau_1}^{t} x^T(s) Q_2 x(s) ds + \int_{t-\tau_2}^{t} x^T(s) Q_3 x(s) ds \\
&\quad + \int_{t-\tau_3}^{t} x^T(s) Q_4 x(s) ds + \int_{t-\tau_3}^{t} x^T(s) Q_5 x(s) ds, \\
V_3(t, x(t), y(t), p) &= \tau_1 \int_{t-\tau_1}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds + \int_{t-\tau_1}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds \\
&\quad + \tau_1 \int_{t-\tau_1}^{t} \dot{x}^T(s) R_4 \dot{x}(s) ds + \int_{t-\tau_1}^{t} \dot{x}^T(s) R_5 \dot{x}(s) ds, \\
V_4(t, x(t), y(t), p) &= \bar{y} \int_{-\tau(t)}^{0} C^T(s) \xi_1(s) ds + \bar{y} \int_{-\tau(t)}^{0} F^T(y(s)) Z_2 F(y(s)) ds, \\
V_5(t, x(t), y(t), p) &= \int_{t}^{\tau(t)} \dot{y}^T(s) S_1 \dot{y}(s) ds + \int_{t}^{\tau(t)} \dot{y}^T(s) S_2 \dot{y}(s) ds \\
&\quad + \int_{t}^{\tau(t)} \dot{y}^T(s) S_3 \dot{y}(s) ds + \int_{t}^{\tau(t)} \dot{y}^T(s) S_4 \dot{y}(s) ds,
\end{align*}
\]
with
\[
\begin{align*}
\xi_1^T(t) &= \begin{bmatrix} x^T(t), x^T(t - \tau(t)), \left( \int_{t-\tau(t)}^{t} x(s) ds \right)^T, \left( \int_{t-\tau(t)}^{t} x(s) ds \right)^T, \left( \int_{t-\tau(t)}^{t} x(s) ds \right)^T \end{bmatrix}, \\
\xi_2^T(t) &= \begin{bmatrix} y^T(t), y^T(t - \rho(t)), \left( \int_{t-\rho(t)}^{t} y(s) ds \right)^T, \left( \int_{t-\rho(t)}^{t} y(s) ds \right)^T, \left( \int_{t-\rho(t)}^{t} y(s) ds \right)^T \end{bmatrix}.
\end{align*}
\]
For \( t = t_k \) and using the techniques in [39], we have
\[
\begin{align*}
V_1(t_k, x(t), y(t), q) - V_1(t_k^-, x(t), y(t), p) &= \xi_1^T(t_k) P_q \xi_1(t_k) + \xi_2^T(t_k) Q_2 \xi_2(t_k) \\
&\quad - \xi_1^T(t_k^-) P_q \xi_1(t_k^-) - \xi_2^T(t_k^-) Q_2 \xi_2(t_k^-) \\
&= x^T(t_k^-) (I_{p_k}^- P_{k_k^-} - P_{k_k^-} \lambda(t_k^-)) \\
&\quad + y^T(t_k^-) (I_{p_k}^- Q_{k_k^-} - Q_{k_k^-}) y(t_k^-) < 0.
\end{align*}
\]
For $t \in [t_{k-1}, t_k]$, the weak infinitesimal generator $\mathcal{L}V$ along with (15), we can obtain that

$$\mathcal{L}V_1(t, x(t), y(t), p) = 2\xi_x^T(t)p_x\dot{\xi}_1(t) + \xi_y^T(t)\sum_{p' = 1}^n \pi_{pp'}\xi^{(p')}y^{(p')} \xi_1(t) + 2\xi_z^2(t)Q_0\dot{\xi}_2(t) + \xi_z^2(t)\sum_{p' = 1}^n \pi_{pp'}\xi^{(p')}z(t),$$  \hspace{1cm} (16)

$$\mathcal{L}V_2(t, x(t), y(t), p) \leq x^T(t)\left(\sum_{i=1}^3 Q_i + \tau_2 Q_4 + \tau_2 Q_5\right)x(t) - (1 - \mu_1)x^T(t - \tau(t))Q_3x(t - \tau(t))$$

$$- x^T(t - \tau_1)Q_2x(t - \tau_1) - x^T(t - \tau_2)Q_3x(t - \tau_2) - \frac{1}{\tau_2} \left(\int_{t-\tau(t)}^t x(s)ds\right)^TQ_4$$

$$\times \left(\int_{t-\tau(t)}^t x(s)ds\right) - \frac{1}{\tau_2} \left(\int_{t-\tau_1}^t x(s)ds\right)^TQ_5 \left(\int_{t-\tau_1}^t x(s)ds\right)$$

$$- \frac{1}{\tau_2} \left(\int_{t-\tau(t)}^t x(s)ds\right)^TQ_6 \left(\int_{t-\tau(t)}^t x(s)ds\right) + \dot{x}^T(t)T_1\dot{x}(t)$$

$$- (1 - \mu_1)\dot{x}^T(t - \tau(t))T_1\dot{x}(t - \tau(t)) + y^T(t)\left(\sum_{i=6}^8 Q_i + \rho_2 Q_9 + \rho_2 Q_{10}\right)y(t)$$

$$- (1 - \mu_2)y^T(t - \rho(t))Q_5y(t - \rho(t)) - y^T(t - \rho_1)Q_2y(t - \rho_1)$$

$$- y^T(t - \rho_2)Q_3y(t - \rho_2) - \frac{1}{\rho_2} \left(\int_{t-\rho(t)}^t y(s)ds\right)^TQ_9 \left(\int_{t-\rho(t)}^t y(s)ds\right)$$

$$- \frac{1}{\rho_2} \left(\int_{t-\rho_1}^t y(s)ds\right)^TQ_{10} \left(\int_{t-\rho_1}^t y(s)ds\right)$$

$$\times \left(\int_{t-\rho_1}^t y(s)ds\right) + \dot{y}^T(t)T_2\dot{y}(t) - (1 - \mu_2)\dot{y}^T(t - \rho(t))T_2\dot{y}(t - \rho(t)), \hspace{1cm} (17)$$

$$\mathcal{L}V_3(t, x(t), y(t), p) \leq \dot{x}^T(t)\left(\tau_2^2R_1 + \tau_2^2R_2 + \tau_2^2R_3 + \tau_2^2R_4\right)\dot{x}(t) - x^T(t - \tau_1)R_1x(t - \tau_1)$$

$$+ 2x^T(t - \tau_1)R_1x(t - \tau_2) - x^T(t - \tau_2)R_2x(t - \tau_2) - \frac{1}{\tau_2} x^T(t - \tau_1)R_2x(t - \tau_1)$$

$$+ 2x^T(t - \tau_1)R_2x(t - \tau_2) - \frac{1}{\tau_2} x^T(t - \tau_2)R_3x(t - \tau_2) - x^T(t)R_3x(t)$$

$$+ 2x^T(t)R_3x(t - \tau_1) - x^T(t - \tau_1)R_3x(t - \tau_1) - \frac{1}{\tau_2} x^T(t)R_4x(t) + \frac{2}{\tau_2} x^T(t)$$

$$\times R_4x(t - \tau(t)) - \frac{1}{\tau_2} x^T(t - \tau(t))R_4x(t - \tau(t)) - \frac{1}{\tau_2} x^T(t - \tau(t))R_4x(t - \tau(t))$$

$$+ 2x^T(t - \tau(t))R_4x(t - \tau_2) - \frac{1}{\tau_2} x^T(t - \tau_2)R_4x(t - \tau_2)$$

$$+ \dot{y}^T(t)\left(\rho_2^2R_5 + \rho_2^2R_6 + \rho_2^2R_7 + \rho_2^2R_8\right)\dot{y}(t) - y^T(t - \rho_1)R_3y(t - \rho_1)$$

$$+ 2y^T(t - \rho_1)R_3y(t - \rho_2) - y^T(t - \rho_2)R_3y(t - \rho_2) - \frac{1}{\rho_2} y^T(t - \rho_1)R_3y(t - \rho_1)$$

$$+ 2y^T(t - \rho_1)R_3y(t - \rho_2) - \frac{1}{\rho_2} y^T(t - \rho_2)R_3y(t - \rho_2) - \frac{1}{\rho_2} y^T(t - \rho_1)R_3y(t - \rho_1)$$

$$+ 2y^T(t - \rho_1)R_3y(t - \rho_2) - y^T(t - \rho_2)R_3y(t - \rho_2) - \frac{1}{\rho_2} y^T(t - \rho_1)R_3y(t - \rho_1)$$

$$+ 2y^T(t - \rho_1)R_3y(t - \rho_2) - \frac{1}{\rho_2} y^T(t - \rho_2)R_3y(t - \rho_2) - \frac{1}{\rho_2} y^T(t - \rho_1)R_3y(t - \rho_1)$$

$$\times R_3y(t - \rho(t)) - \frac{1}{\rho_2} y^T(t - \rho(t))R_3y(t - \rho(t)) - \frac{1}{\rho_2} y^T(t - \rho(t))R_3y(t - \rho(t))$$

$$+ 2y^T(t - \rho(t))R_3y(t - \rho_2) - \frac{1}{\rho_2} y^T(t - \rho_2)R_3y(t - \rho_2), \hspace{1cm} (18)$$
\[ \mathcal{L}V_4(t, x(t), y(t), p) \leq G^T(x(t))Y^2Z_1G(x(t)) - \left( \int_{t-r(t)}^{t} G(x(s))ds \right)^T Z_1 \left( \int_{t-r(t)}^{t} G(x(s))ds \right) \]

\[ + F^T(y(t))Y^2Z_2F(y(t)) - \left( \int_{t-r(t)}^{t} F(y(s))ds \right)^T Z_2 \left( \int_{t-r(t)}^{t} F(y(s))ds \right), \]

\[ \mathcal{L}V_5(t, x(t), y(t), p) = \frac{\bar{\tau}_1^2}{2} \dot{x}^T(\tau(t))S_1\dot{x}(\tau(t)) - \int_{-\tau_2}^{\tau_2} \int_{t+\theta}^{t} \dot{x}^T(s)S_1\dot{x}(s)dsd\theta + \frac{\bar{\tau}_1^2}{2} \dot{x}^T(t)S_2\dot{x}(t) - \int_{-\tau_2}^{\tau_2} \int_{t+\theta}^{t} \dot{x}^T(s)S_2\dot{x}(s)dsd\theta \]

\[ - \int_{-\tau_2}^{\tau_2} \int_{t+\theta}^{t} \dot{y}^T(s)S_2\dot{y}(s)dsd\theta + \frac{\rho_1^2 - \rho_1^2}{2} \dot{y}^T(t)S_2\dot{y}(t) - \int_{-\rho_2}^{\rho_2} \int_{t+\theta}^{t} \dot{y}^T(s)S_2\dot{y}(s)dsd\theta. \]

For any matrices \( M, N, K, L \) with appropriate dimensions, we can obtain that

\[ 2\dot{x}^T(t)M \left[ -A_{1p}(t)x(t) + B_{1p}(t)F(y(t)) + C_{1p}(t)F(y(t) - \rho(t)) \right] \]

\[ + D_{1p}(t) \int_{t-r(t)}^{t} F(y(s))ds + E_{1p}(t)\dot{x}(t - \tau(t)) - \dot{x}(t) \right] = 0, \]

\[ 2\dot{y}^T(t)N \left[ -A_{2p}(t)y(t) + B_{2p}(t)G(x(t)) + C_{2p}(t)G(x(t) - \tau(t)) \right] \]

\[ + D_{2p}(t) \int_{t-r(t)}^{t} G(x(s))ds + E_{2p}(t)\dot{y}(t - \rho(t)) - \dot{y}(t) \right] = 0, \]

\[ 2\dot{x}^T(t - \tau(t))E_{1p}(t)K \left[ \dot{x}(t) + A_{1p}(t)x(t) - B_{1p}(t)F(y(t)) - C_{1p}(t)F(y(t) - \rho(t)) \right] \]

\[ - D_{1p}(t) \int_{t-r(t)}^{t} F(y(s))ds - E_{1p}(t)\dot{x}(t - \tau(t)) \right] = 0, \]

\[ 2\dot{y}^T(t - \rho(t))E_{2p}(t)L \left[ \dot{y}(t) + A_{2p}(t)y(t) - B_{2p}(t)G(x(t)) - C_{2p}(t)G(x(t) - \tau(t)) \right] \]

\[ - D_{2p}(t) \int_{t-r(t)}^{t} G(x(s))ds - E_{2p}(t)\dot{y}(t - \rho(t)) \right] = 0. \]

From (5), it can be verified easily that

\[ a_1x^T(t)W_1^TW_1x(t) - a_1G^T(x(t))G(x(t)) \geq 0, \]

\[ a_2x^T(t - \tau(t))W_1^TW_1x(t - \tau(t)) - a_2G^T(x(t - \tau(t)))G(x(t - \tau(t))) \geq 0, \]

\[ b_1y^T(t)W_2^TW_2y(t) - b_1F^T(y(t))F(y(t)) \geq 0, \]

\[ b_2y^T(t - \rho(t))W_2^TW_2y(t - \rho(t)) - b_2F^T(y(t - \rho(t)))F(y(t - \rho(t))) \geq 0. \]

Then, combining (16)-(28) together and using Lemma 2.3, we can obtain

\[ \mathcal{L}V(t, x(t), y(t), p) \leq \xi_1^T(t) \sum_{i=1}^{r} h_i(\theta(t))\Omega^{i,p} \xi_1(t) + \xi_2^T(t) \sum_{i=1}^{r} h_i(\theta(t))\Lambda^{i,p} \xi_2(t) \]
where
\[
\xi_1^2(t) = \left[ x^T(t), x^T(t - \tau_1), x^T(t - \tau_2), x^T(t - \tau(t)), \dot{x}^T(t), \dot{x}^T(t - \tau(t)), F^T(y(t)), F^T(y(t - \rho(t))) \right]^T \left( \int_{t - \tau(t)}^{t} x(s)ds \right) + \left( \int_{t - \rho(t)}^{t} \dot{y}(s)ds \right) + \left( \int_{t - \gamma(t)}^{t} G(y(s))ds \right)
\]
\[
\xi_2^2(t) = \left[ y^T(t), y^T(t - \rho_1), y^T(t - \rho_2), y^T(t - \rho(t)), \dot{y}^T(t), \dot{y}^T(t - \rho(t)), G^T(x(t)), C^T(x(t - \tau(t))) \right]^T \left( \int_{t - \rho(t)}^{t} y(s)ds \right) + \left( \int_{t - \gamma(t)}^{t} \dot{y}(s)ds \right) + \left( \int_{t - \gamma(t)}^{t} G(y(s))ds \right)
\]

From the conditions of Theorem 3.1, if \(\xi_1^2(t) \neq 0\) and \(\xi_2^2(t) \neq 0\) we can obtain
\[
\mathcal{L}V(t, x(t), y(t), p) < 0.
\]  
(29)

For \(t \in [t_{k-1}, t_k]\), in view of (15) and (29), we have
\[
V(t_k, x(t), y(t), q) < V(t_{k-1}, x(t), y(t), p) \leq V(t_{k-1}, x(t), y(t), p).
\]  
(30)

By the similar proof and mathematical induction, we can derive that (30) is true for all \(p, q, r(0) = p_0 \in S, k \in \mathbb{Z_+}\).

\[
V(t_k, x(t), y(t), q) < V(t_{k-1}, x(t), y(t), p) < \cdots < V(t_0, x(0), y(0), p_0).
\]

From (12)–(13), we can obtain
\[
\mathcal{L}V(t, x(t), y(t), p) \leq -M_1 \left( \|x(t)\|^2 + \|x(t - \tau(t))\|^2 + \|x(t - \gamma(t))\|^2 \right) + M_2 \left( \|y(t)\|^2 + \|y(t - \rho(t))\|^2 + \|y(t - \gamma(t))\|^2 \right)
\]  
(31)

where \(M_1 = \lambda_{\min}\{A^2(t)\} > 0, M_2 = \lambda_{\min}\{A(t)\} > 0\). The remaining part of the proof follows from [15]. Consequently, by the proof of Lyapunov stability theory and Definition 2.1, we know that the equilibrium solution of the fuzzy NIBAMNNs with MJPs (9) without uncertainty is stochastically exponentially stable in mean square for any time delays satisfying 0 ≤ \(\tau_1, \gamma_1, \tau_2, \gamma_2 \leq \tau_1, 0 \leq \rho_1, \rho_2 \leq \rho(t) \leq \mu_2, 0 \leq r(t) \leq \bar{r}, 0 \leq \gamma(t) \leq \bar{\gamma}\). The proof is completed. □

Now consider the model (9) without the neutral delays and uncertainties. Then the system (9) becomes as follows:
\[
\begin{cases}
\dot{x}(t) = -A_{1p}(t)x(t) + B_{1p}(t)F(y(t)) + C_{1p}(t)F(y(t - \rho(t))) + D_{1p}(t) \int_{t - \tau(t)}^{t} F(y(s))ds, & t > 0, t \neq t_k, \\
x(t_k) = l_{pk}(x(t_k^*)) , & k \in \mathbb{Z_+}, \\
\dot{y}(t) = -A_{2p}(t)y(t) + B_{2p}(t)G(x(t)) + C_{2p}(t)G(x(t - \tau(t))) + D_{2p}(t) \int_{t - \gamma(t)}^{t} G(x(s))ds, & t > 0, t \neq t_k, \\
y(t_k) = f_{pk}(y(t_k^*)), & k \in \mathbb{Z_+}.
\end{cases}
\]  
(32)

**Corollary 3.2.** Under assumptions (A1) and (H1)–(H3), for given scalars \(\tau_2 > \tau_1 \geq 0, \rho_2 > \rho_1 \geq 0, \bar{r} \geq 0, \bar{\gamma} \geq 0, 0 \leq \rho_1 < \infty, 0 \leq \mu_2 < \infty,\) the equilibrium solution of FNIBAMNNs with MJPs (32) without uncertainty is exponentially stable in the mean square if there exist symmetric positive definite matrices
\[
P_p = \begin{bmatrix}
P_{1p} & P_{2p} & P_{3p} & P_{4p} & P_{5p} \\
* & P_{6p} & P_{7p} & P_{8p} & P_{9p} \\
* & * & P_{10p} & P_{11p} & P_{12p} \\
* & * & * & P_{13p} & P_{14p} \\
* & * & * & * & P_{15p}
\end{bmatrix} > 0, \quad Q_p = \begin{bmatrix}
Q_{1p} & Q_{2p} & Q_{3p} & Q_{4p} & Q_{5p} \\
* & Q_{6p} & Q_{7p} & Q_{8p} & Q_{9p} \\
* & * & Q_{10p} & Q_{11p} & Q_{12p} \\
* & * & * & Q_{13p} & Q_{14p} \\
* & * & * & * & Q_{15p}
\end{bmatrix} > 0,
\]
\[
Q_w > 0, R > 0, Z > 0, S > 0, (w = 1, 2, \ldots, 10; r = 1, 2, \ldots, 8; z = 1, 2; s = 1, 2, 3, 4), \text{ for any matrices } M, N, \text{ as well as constants } a > 0, b > 0 \text{ and } k \in \mathbb{Z_+} \text{ satisfying the following LMIs:}
\]
\[
j_{pk}^T P_{1q} l_{pk} - P_{1p} < 0, \quad (33)
\]
\[
j_{pk}^T Q_{1q} l_{pk} - Q_{1p} < 0, \quad (34)
\]
\( \Omega_{j,l}^{i,p} < 0, \)  
\( \Lambda_{j,l}^{i,p} < 0, \)

where \( j, l \) varies from 1, 2, \ldots, 11 with

\[
\Omega_{1,1}^{i,p} = P_{3p} + P_{7p}^T + \sum_{p' = 1}^{s} \pi_{pp'}P_{1p'} + Q_1 + Q_2 + Q_3 + r_2Q_4 + r_{21}Q_5 - R_3 - \frac{1}{r_2}R_4 - S_1 - S_1^T \\
- \frac{\tau_{21}}{r_2 + \tau_1} (S_2 + S_2^T) + a_1W_1^T W_1, \\
\Omega_{1,2}^{i,p} = P_{5p} + R_3, \quad \Omega_{1,3}^{i,p} = -P_{4p}, \\
\Omega_{1,4}^{i,p} = -(1 - \mu_1)P_{3p} + (1 - \mu_1)P_{4p} - (1 - \mu_1)P_{5p} + P_{7p}^T + \sum_{p' = 1}^{s} \pi_{pp'}P_{2p'} + \frac{1}{r_2}R_4, \quad \Omega_{1,5}^{i,p} = P_{1p} - A_{1p}^T M^T, \\
\Omega_{1,6}^{i,p} = P_{10p}^T + \sum_{p' = 1}^{s} \pi_{pp'}P_{3p'} + \frac{1}{r_2}(S_1 + S_1^T), \\
\Omega_{1,7}^{i,p} = P_{11p} + \sum_{p' = 1}^{s} \pi_{pp'}P_{4p'} + \frac{1}{r_2}(S_1 + S_1^T) + \frac{1}{r_2 + \tau_1}(S_2 + S_2^T), \\
\Omega_{1,10}^{i,p} = P_{12p} + \sum_{p' = 1}^{s} \pi_{pp'}P_{5p'} + \frac{1}{r_2 + \tau_1}(S_2 + S_2^T), \\
\Omega_{2,2}^{i,p} = -Q_2 - R_1 - \frac{1}{r_{21}}R_2 - R_3, \quad \Omega_{2,3}^{i,p} = R_1 + \frac{1}{r_{21}}R_2, \quad \Omega_{2,4}^{i,p} = P_{9p}^T, \quad \Omega_{2,8}^{i,p} = P_{12p}^T, \\
\Omega_{2,9}^{i,p} = P_{14p}^T, \\
\Omega_{2,10}^{i,p} = P_{15p}^T, \quad \Omega_{2,3}^{i,p} = -Q_3 - R_1 - \frac{1}{r_{21}}R_2 - \frac{1}{r_{21}}R_4, \quad \Omega_{2,4}^{i,p} = -P_{8p}^T + \frac{1}{r_{21}}R_4^T, \quad \Omega_{2,8}^{i,p} = -P_{11p}^T, \\
\Omega_{3,3}^{i,p} = -P_{13p}^T, \quad \Omega_{3,10}^{i,p} = -P_{14p}^T, \\
\Omega_{4,4}^{i,p} = -(1 - \mu_1)P_{7p} + (1 - \mu_1)P_{7p}^T + (1 - \mu_1)P_{8p} + (1 - \mu_1)P_{9p}^T - (1 - \mu_1)P_{10p} - (1 - \mu_1)P_{11p}^T + \sum_{p' = 1}^{s} \pi_{pp'}P_{6p'}, \\
\Omega_{4,8}^{i,p} = -(1 - \mu_1)P_{10p} + (1 - \mu_1)P_{11p}^T - (1 - \mu_1)P_{12p}^T + \sum_{p' = 1}^{s} \pi_{pp'}P_{2p'}, \\
\Omega_{4,9}^{i,p} = -(1 - \mu_1)P_{11p} + (1 - \mu_1)P_{13p}^T - (1 - \mu_1)P_{14p}^T + \sum_{p' = 1}^{s} \pi_{pp'}P_{4p'}, \\
\Omega_{4,10}^{i,p} = -(1 - \mu_1)P_{12p} + (1 - \mu_1)P_{15p}^T - (1 - \mu_1)P_{15p}^T + \sum_{p' = 1}^{s} \pi_{pp'}P_{5p'}, \\
\Omega_{5,5}^{i,p} = \tau_2^2R_1 + \tau_2R_2 + \tau_2^2R_3 + \tau_2R_4 + \frac{\tau_2^2}{2}S_1 + \frac{\tau_2^2}{2}S_2 - M - M^T, \quad \Omega_{5,6}^{i,p} = MB_{1tp}, \\
\Omega_{5,7}^{i,p} = MC_{1tp}, \quad \Omega_{5,8}^{i,p} = P_{4p}, \quad \Omega_{5,10}^{i,p} = P_{5p}, \quad \Omega_{5,11}^{i,p} = MD_{1tp}, \quad \Omega_{6,6}^{i,p} = \hat{T}_2^2Z_2 - b_1I, \quad \Omega_{7,7}^{i,p} = -b_2I, \\
\Omega_{8,8}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'}P_{10p'} - \frac{1}{r_2}Q_4 - \frac{1}{r_2}(S_1 + S_1^T), \quad \Omega_{8,9}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'}P_{11p'} + \frac{1}{r_2}(S_1 + S_1^T), \\
\Omega_{8,10}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'}P_{12p'}, \\
\Omega_{9,9}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'}P_{13p'} - \frac{1}{r_2}Q_5 - \frac{1}{r_2}(S_1 + S_1^T) - \frac{1}{r_{21}}(S_2 + S_2^T), \quad \Omega_{9,10}^{i,p} = \sum_{p' = 1}^{s} \pi_{pp'}P_{14p'},
$$\Omega_{1,10}^p = \sum_{p'=1}^s \pi_{pp'}P_{14p'} - \frac{1}{\tau_{21}} (S_2 + S_2^T),$$

$$\Omega_{10,10}^p = \sum_{p'=1}^s \pi_{pp'}P_{15p'} - \frac{1}{\tau_{21}} (S_2 + S_2^T) - \frac{1}{\tau_{21}} Q_5, \quad \Omega_{1,11}^p = -Z_2,$$

$$A_{1,1}^p = Q_{3p} + Q_{3p}^T + \frac{s}{\rho_2 + \rho_1} (S_2 + S_2^T) + Q_5 + Q_6 + Q_8 + \rho_2 Q_9 + \rho_2 Q_{10} - R_T - \frac{1}{\rho_2} R_5 - S_3 - S_3^T$$

$$- \frac{\rho_2}{\rho_2 + \rho_1} (S_4 + S_4^T) + b_1 W_1 W_2,$$

$$A_{1,2}^p = Q_{2p} + R_T, \quad A_{1,3}^p = -Q_{4p},$$

$$A_{1,4}^p = -(1 - \mu_2) Q_{3p} + (1 - \mu_2) Q_{4p} - (1 - \mu_2) Q_{5p} + Q_T^T + \frac{s}{\rho_2} Q_{2p} + \frac{1}{\rho_2} R_5,$$

$$A_{1,5}^p = Q_{1p} - A_{2p}^T N^T, \quad A_{1,8}^p = Q_{10p}^T + \frac{1}{\rho_2} (S_3 + S_3^T),$$

$$A_{1,9}^p = Q_{11p} + \frac{s}{\rho_2} Q_{4p} + \frac{1}{\rho_2} (S_3 + S_3^T) + \frac{1}{\rho_2 + \rho_1} (S_4 + S_4^T),$$

$$A_{1,10}^p = Q_{12p} + \frac{s}{\rho_2} Q_{25p} + \frac{1}{\rho_2 + \rho_1} (S_4 + S_4^T), \quad A_{2,2}^p = -Q_2 - R_5 - \frac{1}{\rho_{21}} R_5 - R_7,$$

$$A_{2,3}^p = R_5 + \frac{1}{\rho_{21}} R_5,$$

$$A_{2,4}^p = Q_{3p}^T, \quad A_{2,9}^p = Q_{12p}^T, \quad A_{2,10}^p = Q_{14p}^T,$$

$$A_{3,4}^p = -Q_8 - R_5 - \frac{1}{\rho_{21}} R_5 - \frac{1}{\rho_{21}} R_8,$$

$$A_{3,8}^p = -Q_{11p}, \quad A_{3,10}^p = -Q_{13p}, \quad A_{3,11}^p = -Q_{14p},$$

$$A_{4,4}^p = -(1 - \mu_2) Q_{6p} - (1 - \mu_2) Q_{7p} + (1 - \mu_2) Q_{8p} + (1 - \mu_2) Q_{9p} - (1 - \mu_2) Q_{10p} - (1 - \mu_2) Q_{11p}$$

$$+ \frac{s}{\rho_2} Q_{6p} - (1 - \mu_2) Q_5 - \frac{1}{\rho_2} R_8 - \frac{1}{\rho_{21}} R_8 + b_2 W_1 W_2,$$

$$A_{4,5}^p = Q_{2p}^T, \quad A_{4,8}^p = -(1 - \mu_2) Q_{10p} + (1 - \mu_2) Q_{11p} - (1 - \mu_2) Q_{12p} + \frac{s}{\rho_2} Q_{7p}^T,$$

$$A_{4,9}^p = -(1 - \mu_2) Q_{11p} + (1 - \mu_2) Q_{13p} - (1 - \mu_2) Q_{14p} + \frac{s}{\rho_{21}} Q_{5p}^T,$$

$$A_{4,10}^p = -(1 - \mu_2) Q_{12p} + (1 - \mu_2) Q_{14p} - (1 - \mu_2) Q_{15p} + \frac{s}{\rho_{21}} Q_{9p}^T,$$

$$A_{5,5}^p = \rho_{21}^2 R_5 + \rho_{21} R_5 + \rho_2 R_5 + \rho_2^2 S_3 + \rho_2^2 S_4 - N - N^T, \quad A_{5,6}^p = N B_{2ip},$$

$$A_{5,7}^p = -RC_{2ip}, \quad A_{5,8}^p = Q_{3p}, \quad A_{5,9}^p = Q_{4p}, \quad A_{5,10}^p = Q_{5p}, \quad A_{5,11}^p = ND_{2ip}, \quad A_{6,6}^p = \rho_{21}^2 Z_1 - a_{11},$$

$$A_{1,7}^p = -a_{21},$$

$$A_{8,8}^p = \sum_{p'=1}^s \pi_{pp'} Q_{10p'} - \frac{1}{\rho_2} Q_S - \frac{1}{\rho_2} (S_3 + S_3^T), \quad A_{8,9}^p = \sum_{p'=1}^s \pi_{pp'} Q_{13p'} + \frac{1}{\rho_2} (S_3 + S_3^T),$$

$$A_{8,10}^p = \sum_{p'=1}^s \pi_{pp'} Q_{12p'}, \quad A_{8,11}^p = \sum_{p'=1}^s \pi_{pp'} Q_{14p'} + \frac{1}{\rho_2} (S_3 + S_3^T) - \frac{1}{\rho_{21}} (S_4 + S_4^T).$$
The proof immediately follows from the similar method of the proof of Theorem 3.1.

Very recently, to derive much less conservative delay-dependent criteria, great efforts have been made in seeking new LKF’s. By utilizing augmented LKF’s discussed in [24], for analyzing time-delay systems and to provide larger feasible region than the existing stability criterions. Such a stability criterion provides additional inclusion of states that may lead to less conservative results. Meanwhile, authors in [6], derived LKF’s containing novel triple integral terms for establishing less conservative results in delayed Hopfield neural networks (HNNs). Although the results in [6] made a new novel stability criterion on the time-delay systems, to the best of the authors’ knowledge there are no results involving such novel triple integral terms in the LKF’s for deriving the FNIBAMNNs with MJP’s and time delays, which makes another novel stability criterion on BAMNNs.

Some stability conditions were derived for the FBAMNNs with time delays [53]. In that paper, the fuzzy AND and fuzzy OR operation makes the analysis complex, which is very restrictive and limits their applications. Moreover, our stability criterion can take into account the sign of entries in the connected matrix, which illustrates the differences between the excitatory and inhibitory effects. Hence, the result obtained in this paper are less restrictive than the previous works available in the literature.

4. Robust exponential stability results

The motivation for this section containing uncertainties stems from the fact that, in practice, it is almost impossible to get an exact mathematical model of a dynamical system due to modelling errors, measurement errors, liberalization approximation and so on. Indeed, it is reasonable and practical to assume that the model of the system to be controlled almost always contains some type of uncertainty. In the following, we investigate the problem of delay-dependent robust exponential stability analysis for model (9).

Theorem 4.1. Under assumptions (A1) and (H1)–(H3), for given scalars \( \tau_2 \geq \tau_1 \geq 0, \rho_2 > \rho_1 \geq 0, \bar{r} \geq 0, \bar{y} \geq 0, 0 \leq \mu_1 < 1, 0 \leq \mu_2 < 1 \), the equilibrium solution of UFBNIBAMNNs with MJP’s (9) is robustly exponentially stable in the mean square if there exist symmetric positive definite matrices

\[
P_p = \begin{bmatrix}
P_{1p} & P_{2p} & P_{3p} & P_{4p} & P_{5p} \\
* & P_{6p} & P_{7p} & P_{8p} & P_{9p} \\
* & * & P_{10p} & P_{11p} & P_{12p} \\
* & * & * & P_{13p} & P_{14p} \\
* & * & * & * & P_{15p}
\end{bmatrix} > 0, \quad Q_p = \begin{bmatrix}
Q_{1p} & Q_{2p} & Q_{3p} & Q_{4p} & Q_{5p} \\
* & Q_{6p} & Q_{7p} & Q_{8p} & Q_{9p} \\
* & * & Q_{10p} & Q_{11p} & Q_{12p} \\
* & * & * & Q_{13p} & Q_{14p} \\
* & * & * & * & Q_{15p}
\end{bmatrix} > 0,
\]

\( Q_{pi} > 0, \quad R_i > 0, \quad T_2 > 0, \quad Z_i > 0, \quad S_i > 0, \quad (w = 1, 2, \ldots, 10; \ r = 1, 2, \ldots, 8; \ z = 1, 2; \ s' = 1, 2, 3, 4), \) for any matrices \( M, N, K, L \), as well as constants \( a_2 > 0, b_2 > 0, \epsilon_{2p} > 0 \) and \( k \in \mathbb{Z}_+ \) satisfying the following LMI’s:

\[
I_{p_1}^T P_{1p} I_{p_1} - P_{1p} < 0, \quad (37)
\]

\[
I_{p_2}^T Q_{1p} I_{p_2} - Q_{1p} < 0, \quad (38)
\]

\[
\begin{bmatrix}
\hat{\Omega}_i^{1p} & \chi' \\
\chi & -\epsilon_{1p}\end{bmatrix} < 0, \quad (39)
\]

\[
\begin{bmatrix}
\hat{\Lambda}_i^{1p} & \gamma' \\
\gamma & -\epsilon_{2p}\end{bmatrix} < 0, \quad (40)
\]

where

\[
\hat{\Omega}_i^{1p} = \Omega_i^{1p} + \epsilon_{1p}G^{T}_{i1p}G_{i1p}, \quad \hat{\Omega}_6^{1p} = \Omega_6^{1p} + \epsilon_{1p}G^{T}_{6i1p}G_{6i1p}, \quad \hat{\Omega}^{1p}_{12} = \Omega^{1p}_{12} + \epsilon_{1p}G^{T}_{12i1p}G_{12i1p},
\]

\[
\hat{\Omega}_i^{8p} = \Omega_i^{8p} + \epsilon_{1p}G^{T}_{8i1p}G_{8i1p}, \quad \hat{\Omega}^{1p}_{812} = \Omega^{1p}_{812} + \epsilon_{1p}G^{T}_{812i1p}G_{812i1p},
\]

\[
A_i^{1p} = \Lambda_i^{1p} + \epsilon_{2p}G^{T}_{i6i1p}G_{i6i1p}, \quad \hat{A}_i^{1p} = \Lambda_i^{1p} + \epsilon_{2p}G^{T}_{i10i1p}G_{i10i1p}, \quad \hat{A}^{1p}_{i11} = \Lambda^{1p}_{i11} + \epsilon_{2p}G^{T}_{i11i1p}G_{i11i1p},
\]

\[
\hat{A}^{1p}_{i11} = -Z_1,
\]

and the other parameters are defined to be zero.

Proof. The proof immediately follows from the similar method of the proof of Theorem 3.1, hence it is omitted. This completes the proof. □
\[ \hat{\lambda}_{k,8}^{p} = \lambda_{k,8}^{p} + \varepsilon_{2p}C_{8i,p}^{T}G_{8i,p}, \quad \hat{\lambda}_{12,12}^{p} = \lambda_{12,12}^{p} + \varepsilon_{2p}G_{8i,p}^{T}C_{8i,p}, \]
\[ \chi' = \begin{bmatrix} 0_{n \times 4n} \quad (H_{p}^{T}M^{T}) \quad 0_{n \times 7n} \end{bmatrix}^{T}, \quad \chi'' = \begin{bmatrix} 0_{n \times 5n} \quad (H_{p}^{T}K^{T}E_{11p}) \quad 0_{n \times 6n} \end{bmatrix}^{T}, \]
\[ y' = \begin{bmatrix} 0_{m \times 4m} \quad (H_{p}^{T}N^{T}) \quad 0_{m \times 7m} \end{bmatrix}^{T}, \quad y'' = \begin{bmatrix} 0_{m \times 5m} \quad (H_{p}^{T}E_{22p}) \quad 0_{m \times 6m} \end{bmatrix}^{T}, \]
and the other parameters are defined as in \textit{Theorem 3.1}.

\textbf{Proof.} Replacing \( A_{11,p}, B_{11,p}, C_{11,p}, D_{11,p}, E_{11,p} \) with \( A_{11,p} + \Delta A_{11,p}, B_{11,p} + \Delta B_{11,p}, C_{11,p} + \Delta C_{11,p}, D_{11,p} + \Delta D_{11,p}, E_{11,p} + \Delta E_{11,p} \) and \( A_{21,p}, B_{21,p}, C_{21,p}, D_{21,p}, E_{21,p} \) by \( A_{21,p} + \Delta A_{21,p}, B_{21,p} + \Delta B_{21,p}, C_{21,p} + \Delta C_{21,p}, D_{21,p} + \Delta D_{21,p}, E_{21,p} + \Delta E_{21,p} \) respectively in \textit{Theorem 3.1} and applying \textit{Lemma 2.2}, we obtain the results which are equivalent to (37)–(40). This shows that system (9) is robustly exponentially stable in the mean square for all admissible parameter uncertainties satisfying (7) and (8). This completes the proof of \textit{Theorem 4.1}. \( \square \)

Now consider the model (9) without the neutral delays. Then the system (9) becomes as follows:
\[
\begin{cases}
\dot{x}(t) = -[A_{1p}(t) + \Delta A_{1p}(t)]x(t) + [B_{1p}(t) + \Delta B_{1p}(t)]y(t) + [C_{1p}(t) + \Delta C_{1p}(t)]F(y(t - \rho(t))) \\
+ [D_{1p}(t) + \Delta D_{1p}(t)] \int_{t-\tau(t)}^{t} F(y(s))ds, \quad t > 0, \quad t \neq t_k, \\
x(t_k) = p_k(x(t_k^-)), \quad k \in Z_+, \\
y(t) = -[A_{2p}(t) + \Delta A_{2p}(t)]y(t) + [B_{2p}(t) + \Delta B_{2p}(t)]G(x(t)) + [C_{2p}(t) + \Delta C_{2p}(t)]G(x(t - \tau(t))) \\
+ [D_{2p}(t) + \Delta D_{2p}(t)] \int_{t-\gamma(t)}^{t} G(x(s))ds, \quad t > 0, \quad t \neq t_k, \\
y(t_k) = p_k(y(t_k^-)), \quad k \in Z_+.
\end{cases}
\] (41)

\textbf{Corollary 4.2.} Under assumptions (A1) and (H1)–(H3), for given scalars \( \tau_2 > \tau_1 \geq 0, \rho_2 > \rho_1 \geq 0, \tilde{r} \geq 0, \tilde{y} \geq 0, 0 \leq \mu_1 < \infty, 0 \leq \mu_2 < \infty \), the equilibrium solution of UFNIBAMNNs with MJPs (41) is robustly exponentially stable in the mean square if there exist symmetric positive definite matrices
\[
P_p = \begin{bmatrix}
P_{1p} & P_{2p} & P_{3p} & P_{4p} & P_{5p} \\
* & P_{6p} & P_{7p} & P_{8p} & P_{9p} \\
* & * & P_{10p} & P_{11p} & P_{12p} \\
* & * & * & P_{13p} & P_{14p} \\
* & * & * & * & P_{15p}
\end{bmatrix} > 0, \quad Q_p = \begin{bmatrix}
Q_{1p} & Q_{2p} & Q_{3p} & Q_{4p} & Q_{5p} \\
* & Q_{6p} & Q_{7p} & Q_{8p} & Q_{9p} \\
* & * & Q_{10p} & Q_{11p} & Q_{12p} \\
* & * & * & Q_{13p} & Q_{14p} \\
* & * & * & * & Q_{15p}
\end{bmatrix} > 0.
\]
\( \omega_w > 0, R > 0, Z_i > 0, S_i > 0, (w = 1, 2, \ldots, 10; r = 1, 2, \ldots, 8; z = 1, 2; s' = 1, 2, 3, 4) \), for any matrices \( M, N \), as well as constants \( a_i > 0, b_i > 0, c_{ex} > 0 \) and \( k \in Z_+ \), satisfying the following LMI:
\[
\begin{bmatrix}
I_T & P_{1p} & -P_{1p} \\
J_T & Q_{1p} & -Q_{1p}
\end{bmatrix} < 0, \quad k \in Z_+.
\] (42)
\[
\begin{bmatrix}
\hat{\Omega}_{ij}^{p} & \chi' \\
* & -\varepsilon_{1p} I
\end{bmatrix} < 0, \quad k \in Z_+.
\] (43)
\[
\begin{bmatrix}
\hat{\Lambda}_{ij}^{p} & y' \\
* & -\varepsilon_{2p} I
\end{bmatrix} < 0, \quad k \in Z_+.
\] (44)

where
\[
\hat{\Omega}_{ij}^{p} = \Omega_{ij}^{p} + \varepsilon_{ip}C_{11p}^{T}G_{11p}, \quad \hat{\Omega}_{6,6}^{p} = \Omega_{6,6}^{p} + \varepsilon_{ip}G_{21p}^{T}C_{21p}, \quad \hat{\Omega}_{7,7}^{p} = \Omega_{7,7}^{p} + \varepsilon_{ip}G_{31p}^{T}C_{31p}, \quad \hat{\Omega}_{11,11}^{p} = \Omega_{11,11}^{p} + \varepsilon_{ip}G_{41p}^{T}C_{41p}, \quad \hat{\Omega}_{11,11}^{p} = \Omega_{11,11}^{p} + \varepsilon_{ip}G_{51p}^{T}C_{51p}, \quad \hat{\Lambda}_{ij}^{p} = \Lambda_{ij}^{p} + \varepsilon_{ip}G_{61p}^{T}C_{61p}, \quad \hat{\Lambda}_{ij}^{p} = \Lambda_{ij}^{p} + \varepsilon_{ip}G_{71p}^{T}C_{71p}, \quad \hat{\Lambda}_{ij}^{p} = \Lambda_{ij}^{p} + \varepsilon_{ip}G_{81p}^{T}C_{81p}, \quad \hat{\Lambda}_{ij}^{p} = \Lambda_{ij}^{p} + \varepsilon_{ip}G_{91p}^{T}C_{91p},
\]
\[
\chi' = \begin{bmatrix} 0_{n \times 4n} \quad (H_{p}^{T}M^{T}) \quad 0_{n \times 7n} \end{bmatrix}^{T}, \quad y' = \begin{bmatrix} 0_{m \times 4m} \quad (H_{p}^{T}N^{T}) \quad 0_{m \times 7m} \end{bmatrix}^{T},
\]
and the other parameters are defined as in \textit{Corollary 3.2}.

\textbf{Proof.} Replacing \( A_{11,p}, B_{11,p}, C_{11,p}, D_{11,p} \) with \( A_{11,p} + \Delta A_{11,p}, B_{11,p} + \Delta B_{11,p}, C_{11,p} + \Delta C_{11,p}, D_{11,p} + \Delta D_{11,p}, E_{11,p} + \Delta E_{11,p} \) and \( A_{21,p}, B_{21,p}, C_{21,p}, D_{21,p}, E_{21,p} \) by \( A_{21,p} + \Delta A_{21,p}, B_{21,p} + \Delta B_{21,p}, C_{21,p} + \Delta C_{21,p}, D_{21,p} + \Delta D_{21,p}, E_{21,p} + \Delta E_{21,p} \) respectively in \textit{Corollary 3.2} and applying \textit{Lemma 2.2}, we obtain the results which are equivalent to (42)–(45). This shows that system (41) is robustly exponentially stable in the mean square for all admissible parameter uncertainties satisfying (7) and (8). This completes the proof of \textit{Corollary 4.2}. \( \square \)
Remark 4.3. When $r = 1$ and if the Markov chain $\{r(t), t \geq 0\}$ is only taking a unique value 1, that is $S = \{1\}$, and in the absence of distributed delay, impulsive effects and uncertainties, the system (9) is simplified to the BAMNNs of neutral-type with time delays similar to the system described in [29–32]. Moreover, when $r = 1$ and in the absence of neutral delay, distributed delay, impulsive effects and uncertainties, the system (9) is reduced to the same as in [15,46]. Further the system (9) without MJPs, neutral delay, distributed delay and impulsive effects is already investigated in [51,52,23,53,54]. Although there have been some papers published on stability analysis problems for robust stability of uncertain fuzzy BAMNNs with MJPs [24], to the best of the authors’ knowledge, there are no results concerning the simultaneous presence of FNNs, MJPs, parameter uncertainties, impulsive effects and mixed time-varying delays (including neutral delay, discrete delay, and distributed delay). Unlike those papers, in this paper novel LKFs with triple integral terms are taken to ensure the robust exponential stability of UFNIBAMNNs with MJPs.

Remark 4.4. The traditional assumption that derivatives of the time-varying delays are less than 1 is no longer required in our analysis. Therefore, Corollaries 3.2 and 4.2 are more relevant, general and practical than the one investigated in papers [18,32]. In many cases, the information of the derivative of time delays is unknown because it is a difficult problem to obtain the precise values (even their boundedness or the boundedness of their derivatives) of time delay systems. Regarding this circumstance, rate-independent criteria for delays $\tau(t)$ and $\rho(t)$ satisfying the conditions $0 \leq \tau_1 < \tau_2$, $0 \leq \rho_1 < \rho_2$, are derived by choosing $P_{xy}$, $Q_{xy}$, $(v$ varies from 2 to 15), $Q_1$, $Q_0$, $T_1$, $T_2$ to be zero in Corollaries 3.2 and 4.2. Unfortunately it still requires that the derivatives of the time-varying delays $\tau(t)$, $\rho(t)$ are less than one in Theorems 3.1 and 4.1 (in the case of neutral terms included). In the future, we will carry out some studies to overcome this shortage.

Remark 4.5. Theorems and corollaries in this paper provide a LMI-based delay-dependent sufficient condition that guarantees the robust exponential stability of the UFNIBAMNNs with MJPs in (9). It is well known that the delay-dependent criteria discussed in [14–16,18,22,29–32,51,24] are generally less conservative than delay-independent criteria discussed in [46,23] when the delay is small. These LMI conditions can be checked very efficiently by using the interior-point algorithms which have been developed recently by solving LMIs see [59]. The obtained results are applicable to both constant delays and time-varying delays.

Remark 4.6. Theorems and corollaries presented in this paper provide a new robust exponential stability criterion for UFNIBAMNNs with MJPs by choosing more general LKFs combined with free weighting matrices. Moreover, the new augmented LKF considers the relationship between the state and the time delay. Also, a new type of Markovian jumping matrix $P_{xy}$ and $Q_{xy}$ (augmented type) are taken into account. These types of augmented matrices have not been taken in any of the existing literatures for the robust exponential stability of UFNIBAMNNs with MJPs and mixed interval time-varying delays.

Remark 4.7. Results in this paper considered the interval time-varying delays, namely, $\tau(t)$, $\rho(t)$. There exist positive constants $\tau_1$, $\tau_2$, $\rho_1$ and $\rho_2$ such that $0 \leq \tau_1 < \tau_2$ and $0 \leq \rho_1 < \rho_2$. It is obvious that when $\tau_1 = 0$ and $\rho_1 = 0$, the routine case for the time-delays, that is $0 \leq \tau(t) < \tau_2$ and $0 \leq \rho(t) < \rho_2$ is covered. Similar results can be derived for this case from Theorems 3.1 and 4.1 and Corollaries 3.2 and 4.2.

Remark 4.8. The parameter uncertainty structure has been widely explored in the problems of robust control and robust filtering of uncertain systems (see for example [55,56]). However so far very little work on robust stability of uncertain BAMNNs with time varying delays (see [13,18,23,24]), which shows the important of the system (9) and (41) including uncertainties, has been carried out.

5. Numerical examples

Example 5.1. Consider 2-neuron model, UFNIBAMNNs with MJPs (9) without uncertainties (that is FNIBAMNNs with MJPs) with the following parameters:

\[
A_{111} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad B_{111} = \begin{bmatrix} 0.45 & 0.12 \\ 0.22 & 0.72 \end{bmatrix}, \quad C_{111} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.1 \end{bmatrix}, \quad D_{111} = \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.
\]

\[
E_{111} = \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.1 \end{bmatrix}, \quad A_{211} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_{211} = \begin{bmatrix} 0.2 & 0.5 \\ 0.4 & 0.4 \end{bmatrix}, \quad C_{211} = \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.4 \end{bmatrix}.
\]

\[
D_{211} = \begin{bmatrix} -0.1 & 0.2 \\ -0.1 & 0.1 \end{bmatrix}, \quad E_{211} = \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.1 \end{bmatrix}, \quad A_{112} = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad B_{112} = \begin{bmatrix} 0.5 & 0.15 \\ 0.1 & 0.6 \end{bmatrix}.
\]

\[
C_{112} = \begin{bmatrix} 0.7 & 0.4 \\ 0.1 & 0.15 \end{bmatrix}, \quad D_{112} = \begin{bmatrix} -0.3 & 0.2 \\ -0.3 & 0.1 \end{bmatrix}, \quad E_{112} = \begin{bmatrix} 0.2 & 0.01 \\ 0.01 & 0.2 \end{bmatrix}, \quad A_{212} = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}.
\]

\[
B_{212} = \begin{bmatrix} 0.21 & 0.52 \\ 0.41 & 0.39 \end{bmatrix}, \quad C_{212} = \begin{bmatrix} 0.3 & 0.39 \\ 0.7 & 0.3 \end{bmatrix}, \quad D_{212} = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}.
\]
Table 5.1
The MAUB for delays $\tau_2 = \rho_2 = \bar{r} = \bar{y}$ with different $\mu (\mu_1 = \mu_2)$.

<table>
<thead>
<tr>
<th>Theorem 3.1</th>
<th>$\mu = 0.0$</th>
<th>$\mu = 0.25$</th>
<th>$\mu = 0.5$</th>
<th>$\mu = 0.75$</th>
<th>$\mu = 0.8$</th>
<th>$\mu = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_1 = \rho_1 = 0.1$</td>
<td>0.4977</td>
<td>0.4681</td>
<td>0.4240</td>
<td>0.3483</td>
<td>0.3260</td>
<td>0.2513</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.2$</td>
<td>0.5082</td>
<td>0.4706</td>
<td>0.4446</td>
<td>0.3687</td>
<td>0.3469</td>
<td>0.2971</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.3$</td>
<td>0.5160</td>
<td>0.4838</td>
<td>0.4516</td>
<td>0.3977</td>
<td>0.3927</td>
<td>0.3647</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.4$</td>
<td>0.5204</td>
<td>0.4994</td>
<td>0.4582</td>
<td>0.4469</td>
<td>0.4351</td>
<td>0.4210</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.5$</td>
<td>0.5224</td>
<td>0.5194</td>
<td>0.4982</td>
<td>0.4569</td>
<td>0.4451</td>
<td>0.4339</td>
</tr>
</tbody>
</table>

The activation functions are chosen to be $f_j(y(t)) = \frac{1}{2}(|y_j(t)| + 1 - |y_j(t) - 1|)$, $g_i(x(t)) = \frac{1}{2}(|x_i(t)| + 1 - |x_i(t) - 1|)$, and the time-varying delays are taken as $\tau(t) = \rho(t) = r(t) = y(t) = 0.2256 + 0.25 \sin(t)$. It is easy to see that $\tau_2 = \rho_2 = \bar{r} = \bar{y} = 0.4756, \mu_1 = \mu_2 = 0.25, W_1 = W_2 = I$. The membership functions for Rules 1 and 2 are $\eta_1 = \frac{1}{2} - \frac{1}{2} \eta^1$.

Then via the Matlab LMI control toolbox, one can see that the LMIs given in Theorem 3.1 are feasible. In Table 5.1, we listed the maximum allowable upper bound (MAUB) for delays $\tau_2, \rho_2, \bar{r}, \bar{y}$ with different values of $\mu_1, \mu_2$. Hence, from Theorem 3.1, system (9) without uncertainties has a unique equilibrium point which is exponentially stable in the mean square.

By using Matlab Simulink Toolbox, simulation results are given as follows: The response of the state dynamics for the system (9) without uncertainties are shown in Fig. 1(a) with the initial condition $\psi(t) = [-1, 2], \varphi(t) = [3, -3]$. The response of the state dynamics for the system (9) without uncertainties and impulsive effects is shown in Fig. 1(b) with the initial condition $\psi(t) = [-1, 2], \varphi(t) = [3, -3]$.

Example 5.2. Consider the 2-neuron model, fuzzy impulsive BAMNNs with MJPAs (32) with the following parameters:

$A_{111} = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}, B_{111} = \begin{bmatrix} 0.54 & 0.21 \\ 0.22 & 0.27 \end{bmatrix}, C_{111} = \begin{bmatrix} 0.5 & 0.1 \\ 0.8 & 0.2 \end{bmatrix}, D_{111} = \begin{bmatrix} -0.2 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}$.

$A_{211} = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}, B_{211} = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.4 \end{bmatrix}, C_{211} = \begin{bmatrix} 0.4 & 0.8 \\ 0.4 & 0.4 \end{bmatrix}, D_{211} = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}$.

$A_{112} = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.5 \end{bmatrix}, B_{112} = \begin{bmatrix} 0.25 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}, C_{112} = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.2 \end{bmatrix}, D_{112} = \begin{bmatrix} -0.4 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$.

$A_{212} = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.5 \end{bmatrix}, B_{212} = \begin{bmatrix} 0.12 & 0.25 \\ 0.14 & 0.93 \end{bmatrix}, C_{212} = \begin{bmatrix} 0.4 & 0.93 \\ 0.4 & 0.5 \end{bmatrix}, D_{212} = \begin{bmatrix} -0.2 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$.

$A_{121} = \begin{bmatrix} 3.2 & 0 \\ 0 & 3.2 \end{bmatrix}, B_{121} = \begin{bmatrix} 0.15 & 0.15 \\ 0.12 & 0.17 \end{bmatrix}, C_{121} = \begin{bmatrix} 0.18 & 0.15 \\ 0.12 & 0.11 \end{bmatrix}$.
The MAUB for delays $\tau_2 = \rho_2 = \bar{\rho} = \gamma$ with different $\mu (\mu_1 = \mu_2)$.

Table 5.2

<table>
<thead>
<tr>
<th>Corollary 3.2</th>
<th>$\mu = 0.0$</th>
<th>$\mu = 0.25$</th>
<th>$\mu = 0.75$</th>
<th>$\mu = 0.9$</th>
<th>$\mu \geq 1$</th>
<th>unknown $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_1 = \rho_1 = 0.1$</td>
<td>0.4989</td>
<td>0.4887</td>
<td>0.4740</td>
<td>0.4643</td>
<td>0.4407</td>
<td>0.4324</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.2$</td>
<td>0.5320</td>
<td>0.5251</td>
<td>0.5170</td>
<td>0.4916</td>
<td>0.4683</td>
<td>0.4453</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.3$</td>
<td>0.5399</td>
<td>0.5313</td>
<td>0.5210</td>
<td>0.5122</td>
<td>0.5058</td>
<td>0.4773</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.4$</td>
<td>0.5490</td>
<td>0.5354</td>
<td>0.5336</td>
<td>0.5284</td>
<td>0.5148</td>
<td>0.5008</td>
</tr>
<tr>
<td>$\tau_1 = \rho_1 = 0.5$</td>
<td>0.5721</td>
<td>0.5633</td>
<td>0.5471</td>
<td>0.5325</td>
<td>0.5282</td>
<td>0.5195</td>
</tr>
</tbody>
</table>

$D_{121} = \begin{bmatrix} -0.14 & 0.11 \\ 0.11 & 0.21 \end{bmatrix}$, $A_{221} = \begin{bmatrix} 3.2 & 0 \\ 0 & 3.2 \end{bmatrix}$, $B_{221} = \begin{bmatrix} 0.15 & 0.25 \\ 0.24 & 0.3 \end{bmatrix}$, $C_{221} = \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.4 \end{bmatrix}$, $D_{221} = \begin{bmatrix} -0.3 & 0.3 \\ 0.3 & 0.2 \end{bmatrix}$, $D_{122} = \begin{bmatrix} -0.13 & 0.12 \\ 0.13 & 0.11 \end{bmatrix}$, $A_{122} = \begin{bmatrix} 2.75 & 0 \\ 0 & 2.75 \end{bmatrix}$, $B_{122} = \begin{bmatrix} 0.84 & 0.71 \\ 0.21 & 0.4 \end{bmatrix}$, $C_{122} = \begin{bmatrix} 0.17 & 0.24 \\ 0.11 & 0.61 \end{bmatrix}$, $A_{222} = \begin{bmatrix} 2.75 & 0 \\ 0 & 2.75 \end{bmatrix}$, $B_{222} = \begin{bmatrix} 0.11 & 0.61 \\ 0.51 & 0.15 \end{bmatrix}$, $C_{222} = \begin{bmatrix} 0.31 & 0.15 \\ 0.91 & 0.15 \end{bmatrix}$, $D_{222} = \begin{bmatrix} -0.21 & 0.12 \\ 0.21 & 0.11 \end{bmatrix}$, $I_{11} = I_{12} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}$, $J_{11} = J_{12} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$, $\Pi = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$, $t_k = 0.05k$, $k \in \mathbb{Z}_+$.

The activation functions are chosen to be $f_i(y(t)) = \frac{1}{2}(|y_i(t)| + 1 - |y_i(t)|)$, $g_i(x(t)) = \frac{1}{2}(|x_i(t)| + 1 - |x_i(t)|)$, and the time-varying delays are taken as $\tau(t) = \rho(t) = \bar{\rho}(t) = \gamma(t) = 0.2785 + 0.25 \sin(t)$, $t_k = 0.05k$, $k \in \mathbb{Z}_+$. It is easy to see that $\tau_2 = \rho_2 = \bar{\rho} = \gamma = 0.5285$, $\mu_1 = \mu_2 = 0.25$, $W_1 = W_2 = I$. The membership functions for Rule 1 and Rule 2 are $\eta^1 = e^{-\frac{\tau(t)}{\eta_2(t)}}$, $\eta^2 = 1 - \eta^1$. Then via the Matlab LMI control toolbox, one can see that the LMs given in Corollary 3.2 are feasible. Moreover, one can obtain the MAUB for delays $\tau_2$, $\rho_2$, $\bar{\rho}$, $\gamma$ with different values of $\mu_1$, $\mu_2$, which are shown in Table 5.2.

Hence, from Corollary 3.2, system (32) has a unique equilibrium point which is exponentially stable in the mean square. By using a Euler numerical simulation scheme, the results are as follows: Let step size $h = 0.001$. The response
of the state dynamics for the system (32) with impulsive effects and without impulsive effects for the initial condition \( \psi(t) = [0.7, 0.5], \varphi(t) = [-0.7, -1] \) is shown in Fig. 2(a) and (b) respectively.

**Example 5.3.** Consider 2-neuron model, UFNIBAMNNs with MJPs (9) with the following parameters:

\[
H_p = 0.2I, \ G_{i,p} = G_{2i,p} = G_{3i,p} = G_{4i,p} = G_{5i,p} = G_{6i,p} = G_{7i,p} = G_{8i,p} = G_{9i,p} = G_{10i,p} = 0.1I, \text{ where } i = 1, 2, p = 1, 2
\]

and the other parameters are defined as in Example 5.1.

Particularly, if we set \( \tau_1 = \rho_3 = 0.3 \), we get \( \tau_2 = \rho_2 = \bar{\tau} = \bar{\varphi} = 0.5784 \) by resorting to the Matlab LMI control toolbox to solve the LMs in Theorem 4.1, we obtain the following feasible solutions matrices:

\[
P_{11} = \begin{bmatrix} 0.0451 & -0.0256 \\ -0.0256 & 0.1016 \end{bmatrix}, \quad P_{61} = \begin{bmatrix} 0.0033 & -0.0010 \\ -0.0010 & 0.0049 \end{bmatrix}, \quad P_{101} = \begin{bmatrix} 0.0169 & -0.0007 \\ -0.0007 & 0.0365 \end{bmatrix},
\]

\[
P_{131} = \begin{bmatrix} 0.0607 & 0.0064 \\ 0.0064 & 0.0510 \end{bmatrix}, \quad P_{151} = \begin{bmatrix} 0.0414 & 0.0024 \\ 0.0024 & 0.0375 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 0.0449 & -0.0256 \\ -0.0256 & 0.1014 \end{bmatrix},
\]

\[
P_{62} = \begin{bmatrix} 0.0030 & -0.0008 \\ -0.0008 & 0.0049 \end{bmatrix}, \quad P_{102} = \begin{bmatrix} 0.0191 & -0.0008 \\ -0.0008 & 0.0383 \end{bmatrix}, \quad P_{132} = \begin{bmatrix} 0.0852 & 0.0164 \\ 0.0164 & 0.0792 \end{bmatrix},
\]

\[
P_{152} = \begin{bmatrix} 0.0712 & 0.0145 \\ 0.0145 & 0.0593 \end{bmatrix}, \quad Q_{11} = \begin{bmatrix} 0.0969 & -0.0483 \\ -0.0483 & 0.0523 \end{bmatrix}, \quad Q_{61} = \begin{bmatrix} 0.0069 & -0.0037 \\ -0.0037 & 0.0042 \end{bmatrix},
\]

\[
Q_{101} = \begin{bmatrix} 0.0236 & -0.0069 \\ -0.0069 & 0.0282 \end{bmatrix}, \quad Q_{131} = \begin{bmatrix} 0.0640 & 0.0066 \\ 0.0066 & 0.0482 \end{bmatrix}, \quad Q_{151} = \begin{bmatrix} 0.0393 & 0.0021 \\ 0.0021 & 0.0353 \end{bmatrix},
\]

\[
Q_{12} = \begin{bmatrix} 0.0957 & -0.0479 \\ -0.0479 & 0.0516 \end{bmatrix}, \quad Q_{62} = \begin{bmatrix} 0.0064 & -0.0035 \\ -0.0035 & 0.0040 \end{bmatrix}, \quad Q_{102} = \begin{bmatrix} 0.0251 & -0.0051 \\ -0.0051 & 0.0255 \end{bmatrix},
\]

\[
Q_{132} = \begin{bmatrix} 0.0756 & 0.0174 \\ 0.0174 & 0.0664 \end{bmatrix}, \quad Q_{152} = \begin{bmatrix} 0.0544 & 0.0160 \\ 0.0160 & 0.0520 \end{bmatrix}, \quad a_1 = 0.0134, \quad a_2 = 0.0136,
\]

\[
b_1 = 0.0185, \quad b_2 = 0.0149, \quad \epsilon_{11} = 0.0298, \quad \epsilon_{12} = 0.0337, \quad \epsilon_{21} = 0.0253, \quad \epsilon_{22} = 0.0329.
\]

Due to the page limit, some of the feasible symmetric positive definite matrices are only listed. Then via the Matlab LMI control toolbox, one can see that the LMs given in Theorem 4.1 are feasible. Hence, from Theorem 4.1, system (9) has a unique equilibrium point which is robustly exponentially stable in the mean square. By using Matlab Simulink Toolbox, simulation results are given as follows: The response of the state dynamics for the system (9) is shown in Fig. 3 with the initial condition \( \psi(t) = [-1, 2], \varphi(t) = [3, -3] \).

**Example 5.4.** Consider 2-neuron model, uncertain fuzzy impulsive BAMNNs with MJPs (41) with the following parameters:

\[
H_p = 0.2I, \ G_{i,p} = G_{2i,p} = G_{3i,p} = G_{4i,p} = G_{6i,p} = G_{7i,p} = G_{8i,p} = G_{9i,p} = G_{10i,p} = 0.1I, \text{ where } i = 1, 2, p = 1, 2 \]

and the other parameters are defined as in Example 5.2.
Corollary 4.2

Theorem 4.1

Remark 4.4

Because of the convexity of the LMI system, one can solve this optimization problem by so-called interior-point methods [59]. In future, the main objective will be obtained for delay-dependent criteria which are feasible to a larger

Fig. 3. State trajectories of system (9) in Example 5.3.

Particularly, if we set \( \tau_1 = \rho_1 = 0.5 \), we get \( \tau_2 = \rho_2 = \bar{\tau} = \bar{\rho} = 0.5418 \) and by resorting to the Matlab LMI control toolbox to solve the LMIs in Theorem 4.1, we obtain the following feasible solutions matrices:

\[
\begin{align*}
P_{11} &= \begin{bmatrix} 0.6003 & -0.3045 \\ -0.3045 & 0.9876 \end{bmatrix}, & P_{61} &= \begin{bmatrix} 0.1412 & 0.0024 \\ 0.0024 & 0.1157 \end{bmatrix}, & P_{101} &= \begin{bmatrix} 0.1210 & -0.0230 \\ -0.0230 & 0.1407 \end{bmatrix}, \\
P_{131} &= \begin{bmatrix} 0.9827 & 0.1098 \\ 0.1098 & 0.8515 \end{bmatrix}, & P_{151} &= \begin{bmatrix} 0.3052 & -0.0026 \\ -0.0026 & 0.3074 \end{bmatrix}, & P_{12} &= \begin{bmatrix} 0.5849 & -0.3012 \\ -0.3012 & 0.9760 \end{bmatrix}, \\
P_{62} &= \begin{bmatrix} 0.1303 & 0.0076 \\ 0.0076 & 0.1143 \end{bmatrix}, & P_{102} &= \begin{bmatrix} 0.1279 & -0.0322 \\ -0.0322 & 0.1536 \end{bmatrix}, & P_{132} &= \begin{bmatrix} 1.1706 & 0.1401 \\ 0.1401 & 0.9894 \end{bmatrix}, \\
P_{152} &= \begin{bmatrix} 0.4806 & 0.0434 \\ 0.0434 & 0.4217 \end{bmatrix}, & Q_{11} &= \begin{bmatrix} 0.6629 & -0.2744 \\ -0.2744 & 0.5634 \end{bmatrix}, & Q_{51} &= \begin{bmatrix} 2.0004 & 0.1777 \\ 0.1777 & 1.9899 \end{bmatrix}, \\
Q_{101} &= \begin{bmatrix} 0.2046 & -0.0177 \\ -0.0177 & 0.1628 \end{bmatrix}, & Q_{131} &= \begin{bmatrix} 0.7511 & 0.0329 \\ 0.0329 & 0.7513 \end{bmatrix}, & Q_{151} &= \begin{bmatrix} 0.2620 & -0.0157 \\ -0.0157 & 0.2570 \end{bmatrix}, \\
Q_{12} &= \begin{bmatrix} 0.6397 & -0.2836 \\ -0.2836 & 0.5540 \end{bmatrix}, & Q_{52} &= \begin{bmatrix} 1.9873 & 0.1770 \\ 0.1770 & 1.9768 \end{bmatrix}, & Q_{102} &= \begin{bmatrix} 0.1590 & -0.0456 \\ -0.0456 & 0.1490 \end{bmatrix}, \\
Q_{132} &= \begin{bmatrix} 0.7980 & 0.0348 \\ 0.0348 & 0.8023 \end{bmatrix}, & Q_{152} &= \begin{bmatrix} 0.2778 & -0.0147 \\ -0.0147 & 0.2765 \end{bmatrix}, & \alpha_1 &= 0.1909, & \alpha_2 &= 0.1801, \\
b_1 &= 0.2306, & b_2 &= 0.0149, & \epsilon_{11} &= 0.5707, & \epsilon_{12} &= 0.4566, & \epsilon_{21} &= 0.4903, & \epsilon_{22} &= 0.4218.
\end{align*}
\]

Due to the page limit, some of the feasible symmetric positive definite matrices are only listed. Now considering the unknown time-varying delay case discussed in Remark 4.4. Let \( \tau_1 = \rho_1 = 0.5 \), we get \( \tau_2 = \rho_2 = \bar{\tau} = \bar{\rho} = 0.5128 \). Then via the Matlab LMI control toolbox, one can see that the LMIs given in Corollary 4.2 are feasible. Hence, from Corollary 4.2, system (41) has a unique equilibrium point which is robustly exponentially stable in the mean square. By using a Euler numerical simulation scheme, the results are as follows: Let step size \( h = 0.01 \). The response of the state dynamics for the system (41) is shown in Fig. 4 with the initial condition \( \psi(t) = [0.7, 0.5], \varphi(t) = [0.7, -1] \).

Remark 5.5. The Examples 5.1–5.4, discussed in this paper are under the condition of the fixed delays. For the given UFNIBAMNNs with the MJP model, we can estimate the maximum delays or the supremum of time-varying delays by solving the following optimization problem:

\[
\max_{\tau_2, \rho_2, \bar{\tau}, \bar{\rho}} \quad \text{such that the conditions in the theorems or corollaries are satisfied.}
\]

Because of the convexity of the LMI system, one can solve this optimization problem by so-called interior-point methods [59]. In future, the main objective will be obtained for delay-dependent criteria which are feasible to a larger
Remark 5.6. From Tables 5.1 and 5.2, one may observe that the MAUBs of $\tau_2$, $\rho_2$, $F$, $\bar{\gamma}$ are small with different $\mu_1$, $\mu_2$, which shows the limited adaptive ranges of the development results in this paper. The sufficient conditions in theorems and corollaries are delay-dependent. The delay-independent ones are irrespective of the size of the time delay. While the delay-dependent ones are concerned with the size of the time delay. Generally speaking, the delay-dependent conditions are regarded to be less conservative than the delay-independent ones, especially when the size of the time delay is small. Much effort has been paid to develop less conservative delay-dependent stability conditions for time-delay systems. Along with the Remark 5.5, in future, the main objective will be obtaining delay-dependent criteria which are feasible for a larger time delay and/or to derive delay-dependent criteria with less decision variables which are still feasible for the same maximal allowable time delay. So there is still some room for us to develop and explore.

Remark 5.7. For the sake of convenience, numerical examples in this paper use a lower-order network (2-neuron model) instead of a higher (but finite) dimensional neuron model, which makes the task more complex here and it can be extended from the Examples 5.1–5.4. If the NNs possess a large number of neurons, the computation time of calculating MAUBs listed in Tables 5.1–5.2 will also automatically increase. In future, we will do some further research on reducing time complexity when the NNs possess a large number of neurons.

Remark 5.8. The results obtained in this paper throw off the usual assumption about the activation function with bounded character, which allow us to use a larger class of activation functions including the usual sigmoid functions and piecewise linear functions.

Remark 5.9. The systems (9), (32) and (41) presented in this paper are new and have not been discussed in any of the previous existing literature, thus we are unable to analyse any comparison results for Examples 5.1–5.4.

6. Conclusion

In this paper, sufficient conditions are established for robust exponential stability of UFNIBAMNNs with MJP s and mixed interval time-varying delays are derived based on the LKFs containing some novel triple integral terms, Lyapunov stability theory and free-weighting matrix method. The delay-dependent stability conditions are derived in terms of LMIs which can be very efficiently solved by using Matlab LMI control toolbox. Moreover, the approaches presented in this paper can be applied to various NNs available in the literature.

In future, the extension of the present results to more general cases to be considered, for example, the case of delay-probability-distribution-dependent stability and the case of mode-dependent stability. Apparently the reduced conservatism of theorems and corollaries in this paper may be reduced from the augmented LKFs constructed along with the main idea of delay partitioning and convex combination techniques, hence the potential conservatism of the LMI conditions in this paper may be reduced. The reduction of the conservatism with the fewer variables remains a future work.
References