h - Relation and Its Associated Hyperstructures.

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Abstract

An h-relation on a non-empty set S is in fact a relation from S to the power set P(S) of S. Here, in this paper, it is shown that the semigroup of binary relations on a set S can be embedded as a subsemigroup into the semigroup of h-relations on S (with respect to a suitably defined operation). We have introduced and studied here two types of reflexivities, symmetries and transivities of h-relations. The conditions imposed on an h-relation R, under which the hyperstructure $H_R$ (induced by the h-relation R on a non-empty set H) can necessarily and sufficiently be a semihypergroup or a hypergroup, have been obtained. We have established further the necessary and sufficient conditions for which a given semihypergroup $(H, \circ)$ can be induced by a specified h-relation $R$.

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1 Preliminaries

The hyperstructure theory was surfaced up in 1934 when Marty defined hypergroups ([1]), started up analysing their properties and applied them to groups, rational algebraic functions. Since then many researchers have studied in this field and developed it, for example see [4]. In this paper we have studied hyper relation. A hyper relation (h-relation) on a non-empty set $S$ is in fact a relation from $S$ to the power set $P(S)$ of $S$. In [7], Dasgupta introduced the notion of h-relation and studied some of its properties, in the name of $p$-relation.

Definition 1.1. An h-relation on a non-empty set $S$ is a subset of $S \times P(S)$. 
Remark 1.1. (a) Any mapping $f$ from a non-empty set $S$ into itself induces a binary relation of $S$, such as $\rho = \{(a, b) \in S \times S : f(a) = f(b)\}$. The inverse $f^{-1}$ of $f$ may not be a mapping from $S$ to $S$, but as a natural phenomenon, we observe that $f^{-1}$ induces an $h$-relation on $S$ such as $R = \{(a, A) \in S \times P(S) : A = f^{-1}(a)\}$.

(b) For a non-empty set $S$, $\mathcal{R}(S) = S \times P(S)$ is called the universal $h$-relation.

Theorem 1.2. [7] Let $S$ be a non-empty set and $\mathcal{B}_h(S)$ be the set of all $h$-relations on $S$.

For non-null $R_1, R_2 \in \mathcal{B}_h(S), R_1 \circ R_2 = \{(x, A) : \exists B \in P^*(S) \text{ and } A_b \in P(S) \} \cup \{(x, \phi) : (x, \phi) \in R_1\}$

and $R_1 \circ \Phi = \Phi = \Phi \circ R_1$. Then $\left(\mathcal{B}_h(S), \circ\right)$ is a semigroup. [Here, $P^*(S) = P(S) - \{\emptyset\}$].

Proposition 1.3. For any non-empty set $S$, there is an embedding from $\mathcal{B}(S)$ to $\mathcal{B}_h(S)$.

Proof. Let $h : \mathcal{B}(S) \rightarrow \mathcal{B}_h(S)$ be defined as $h(\rho) = \rho_h$ for any $\rho \in \mathcal{B}(S)$ where $\rho_h = \{(x, \{y\}) : (x, y) \in \rho\} \in \mathcal{B}_h(S)$. Suppose $\rho, \sigma \in \mathcal{B}(S)$ with $\rho \neq \sigma$. Without any loss of generality, we can write $(x, y) \in \rho$ such that $(x, y) \not\in \sigma$. Then, $(x, \{y\}) \in \rho_h$ while $(x, \{y\}) \not\in \sigma_h$, whence $\rho_h \neq \sigma_h$. i.e., $\rho \neq \sigma \Rightarrow h(\rho) \neq h(\sigma)$. Now, $(x, \{y\}) \in (\rho \circ \sigma)_h \Leftrightarrow (x, y) \in \rho \circ \sigma \Leftrightarrow (x, z) \in \rho$ and $(z, y) \in \sigma$ for some $z \in S \Leftrightarrow (x, \{z\}) \in \rho_h$ and $(z, \{y\}) \in \sigma_h \Leftrightarrow (x, \{y\}) \in \rho_h \circ \sigma_h \text{ whence } (\rho \circ \sigma)_h = \rho_h \circ \sigma_h \text{ i.e., } h(\rho \circ \sigma) = h(\rho) \circ h(\sigma)$.

Thus, $h$ is an embedding from $\mathcal{B}(S)$ into $\mathcal{B}_h(S)$.

Definition 1.4. Let $R$ be an $h$-relation on a non-empty set $S$. The domain of $R$ is the set $\text{Dom}(R) = \left\{x \in S : (x, A) \in R \text{ for some } A \in P(S)\right\}$ and codomain of $R$ is the set $\text{Cod}(R) = \left\{A \in P(S) : (x, A) \in R \text{ for some } x \in S\right\}$.

Remark 1.5. Let $R$ be a non-null $h$-relation on a non-empty set $S$. Then $R \circ \mathcal{R}(S) = \text{Dom}(R) \times P(S)$ and $\mathcal{R}(S) \circ R = S \times \text{Cod}(R)$.

In [2] I.G.Rosenberg constructed the hypergroupoid $H_R$ corresponding to every binary relation $R$ ( with full domain ) on a nonempty set $H$ by defining the hyperoperation as $xy = \left\{z \in H : (x, z) \in R \text{ or } (y, z) \in R\right\}$ for all $x, y \in H$. He characterised all $R$ such that the hypergroupoid $H_R = (H, \circ)$ is a semihypergroup, hypergroup and join space.

2 Reflexivities, Symmetries, Transitivities

Definition 2.1. An $h$-relation $R$ on a non-empty set $S$ is (i) reflexive of type 1 if $(x, \{x\}) \in R$ for all $x \in S$, and (ii) reflexive of type 2 if for any $x \in S, \exists A_x \in P(S)$ such that $(x, A_x) \in R$.
and \( x \in A_x \).

**Notation 2.2.** For any \( h \)-relation \( R \) on \( S \) and \( A \in P(S) \), we write \( A_R = \{ x \in S : (x, A) \in R \} \). Naturally, \( A_R \subseteq \text{Dom}(R) \) for any \( A \in P(S) \). In this notation of \( A_R \), we can say that \( R \) is reflexive of type 1 if \( x \in \{ x \}_R \) for all \( x \in S \) and reflexive of type 2 if for any \( x \in S, \exists A_x \in P(S) \) such that \( x \in A_x \cap A_R \).

**Remark 2.3.** Every type 1 reflexive \( h \)-relation on a non-empty \( S \) set is a type 2 reflexive \( h \)-relation on the same set.

**Examples 2.4.**

1. Belongingness \((\in)\) of an element to a set is a type 1 (hence type 2) reflexive \( h \)-relation on any non-empty set.

2. Let \((H, \cdot)\) be a hypergroup (i.e. a non-empty set \( H \) endowed with a hyperoperation \( \cdot \) such that for all \( x \in H \), \( x \cdot H = H = H \cdot x \) (reproducibility) and for all \( x, y, z \in H \), \((x \cdot y) \cdot z = x \cdot (y \cdot z) \) (associativity)). On \( H \), we define an \( h \)-relation as \( R = \{ (x, A) : \exists y \in H \text{ such that } x \cdot y = A \} \). Now, consider following two hypergroups.

   (a) Let \((H, \cdot)\) be a hypergroup, which is a polygroup [6] (i.e. a system \( \mathcal{M} = \langle M, \cdot, e, -1 \rangle \) where \( M \) is a non-empty set, \( e \in M \), \(-1 \) is a unary operation on \( M \), \( \cdot \) is a hyper operation on \( M \) satisfying (1) \((x \cdot y) \cdot z = x \cdot (y \cdot z) \) (2) \( e \cdot x = x = x \cdot e \) (3) \( x \in y \cdot z \Rightarrow y \in x \cdot z^{-1} \) and \( z \in y^{-1} \cdot x \) where for any non-empty \( A \subseteq M \) and \( x \in M, x \cdot A = \bigcup_{a \in A} x \cdot a \) and \( A \cdot x = \bigcup_{a \in A} a \cdot x \). Then above defined \( h \)-relation \( R \) on \( H \) is type 1 reflexive, since \( x \cdot e = \{ x \} \) for all \( x \in H \).

   (b) Let \( H = \{ a, b, c, d \} \). On \( H \), a hyperoperation \( \cdot \) is defined as stated in the following table (set braces have been omitted)

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & a, b & a, b & c, d & c, d \\
b & a, b & a, b & c, d & c, d \\
c & c, d & c, d & a & b \\
d & c, d & c, d & b & a \\
\end{array}
\]

\((H, \cdot)\) is a hypergroup [1]. The above defined \( h \)-relation \( R \) on hypergroup \((H, \cdot)\) is type 2 reflexive but not a type 1 reflexive \( h \)-relation.

In fact, here \( H = x \cdot H = \bigcup \{ x \cdot y : y \in H \} \) for all \( x \in H \). So, for any \( x \in H \), \( \exists y \in H \) such that \( x \in x \cdot y = A \) (say). Thus, \( \exists A \in P(H) \) such that \( x \in A \) and \((x, A) \in R\) for all \( x \in H \).
whence \( R \) is type 2 reflexive. But we see that for any \( x \in H, a \cdot x \neq \{a\} \); thus \((a, \{a\}) \notin R.\) So, \( R \) is not type 1 reflexive.

3. Let \( M \) be a monoid. Then a non-deterministic \( M \)-automaton [3] \( \mathcal{V} = (S, f) \) is a pair of a set \( S \) and a mapping \( f \) from \( S \times M \) into \( P(S) \), satisfying \( f(s, 1) = \{s\} \) and \( f(f(s, m), m') = f(s, mm') \) for all \( s \in S \) and \( m, m' \in M \). Now \( R = \{(s, A) \in S \times P(S) : \exists m \in M \text{ such that } f(s, m) = A\} \) is a type 1 reflexive \( h \)-relation on \( S \) [because, for any \( s \in S, f(s, 1) = \{s\} \) where 1 is the identity element of \( M \), so \((s, \{s\}) \in R\) for all \( s \in S\)].

4. Let \((G, \cdot)\) be a group and \( H \) be a sub-group of the group \( G \). Then \( R = \{(x, A) : A \subseteq x \cdot H\} \) is a type 1 reflexive \( h \)-relation on \( G \).

5. Let \( S = [0, 1] \). On \( S \), an \( h \)-relation \( R \) is defined as \( R = \{(x, \{x\}) : x = 0, 1\} \cup \{(x, A) : x \in (0, 1) \text{ and } A(\subseteq S) \text{ is an open set}\} \). Then, \( R \) is a type 2 reflexive \( h \)-relation, which is not type 1 reflexive.

**Definition 2.5.** An \( h \)-relation \( R \) (with \( \phi_R = \phi \)) on a non-empty set \( S \) is said to be (i) a symmetry of type 1 if \((a, X) \in R, x \in X \Rightarrow (x, A) \in R \) for any \( A \in P(S) \) with \( a \in A \subseteq X_R \) and \(|A| \leq |X| \), and (ii) a symmetry of type 2 if for any \( A, B \in P^*(S), A \subseteq B_R \Rightarrow B \subseteq A_R \).

**Proposition 2.6.** Every type 2 symmetric \( h \)-relation on \( S \) is a type 1 symmetry.

**Proof.** Let \( R \) be a type 2 symmetric \( h \)-relation on \( S \). Suppose \((a, X) \in R \) be arbitrary. Let \( x \in X \) and \( A \in P(S) \) be such that \( a \in A \subseteq X_R \) and \(|A| \leq |X| \). Then \( X \subseteq A_R \) (since, \( R \) is a type 2 symmetry) whence \((x, A) \in R \) (since, \( x \in X \)). Thus, \( R \) is a symmetry of type 1.

**Examples 2.7.**

1. Let \( S \) be an infinite set . \( R = \{(x, A) : A \text{ is finite}\}\) is a type 1 symmetry on \( S \), but not a type 2 symmetry. This is also to be noted that \( R \) is a type 1 (and hence type 2) reflexive on \( S \).

2. The \( h \)-relation \( R \) as defined in example 2.4(4), is a symmetry of type 2 (and hence a symmetry of type 1). In fact, for any \( A, B \in P^*(G), A \subseteq B_R \Rightarrow B \subseteq b \cdot H \) for all \( a \in A \Rightarrow a \cdot H = b \cdot H \) for all \( a \in A \) and \( b \in B \) (since, \( H \) is a subgroup of \( G \)) \Rightarrow A \subseteq b \cdot H \) for all \( b \in B \)

\[(a \in a \cdot H) \Rightarrow (b, A) \in R \text{ for all } b \in B \Rightarrow B \subseteq A_R.\]

**Proposition 2.8.** If two type 2 symmetric \( h \)-relations \( R_1 \) and \( R_2 \) on \( S \) be such that \( R_1 \circ R_2 \) and \( R_2 \circ R_1 \) both are type 2 symmetries, then \( R_1 \circ R_2 = R_2 \circ R_1 \).

**Proof** Let \((x, A) \in R_1 \circ R_2 \). Then \( A \neq \phi \), since \( R_1 \circ R_2 \) is a symmetry. Also, there exists
\(B \in P^*(S)\) and \(A_b \in P(S)\) for each \(b \in B\) such that \((x, B) \in R_1, (b, A_b) \in R_2\) and \(\bigcup_b A_b = A\).

Now, here \(\{x\} \subseteq B_{R_1}, \{b\} \subseteq (A_b)_{R_2}\) for all \(b \in B\) \(\Rightarrow\) \(B \subseteq \{x\}_{R_1}, A_b \subseteq \{b\}_{R_2}\) for all \(b \in B\) (because, \(R_1\) and \(R_2\) both are symmetries of type 2) \(\Rightarrow\) \(A_b \subseteq \{b\}_{R_2}, \{b\} \subseteq \{x\}_{R_1}\) for all \(b \in B\) \(\Rightarrow\) \(A_b \subseteq \{x\}_{R_2 \circ R_1}\) for all \(b \in B\) \(\Rightarrow\) \(A = \bigcup_b A_b \subseteq \{x\}_{R_2 \circ R_1} \Rightarrow \{x\} \subseteq A_{R_2 \circ R_1}\) \((R_2 \circ R_1\) is a symmetry of type 2) \(\because\) \(R_2 \circ R_1\) is transitive of type 1 (and hence of type 2).

**Definition 2.9.** An \(h\)-relation \(R\) on a non-empty set \(S\) is called (i) transitive of type 1 if \((x, A) \in R, (a, B_a) \in R\) corresponding to each \(a \in A\) \(\Rightarrow\) \((x, \bigcup_a B_a) \in R\) and (ii) transitive of type 2 if for any \(A, B, C \in P^*(S), A \subseteq B_R, B \subseteq C_R\) \(\Rightarrow\) \(A \subseteq C_R\).

**Remark 2.10.** Let \(R\) be an \(h\)-relation on a set \(S\) such that \(R \circ R \subseteq R\); then \(R\) is a type 2 transitive \(h\)-relation.

**Proposition 2.11.** An \(h\)-relation \(R\) is type 1 transitive if and only if \(R \circ R \subseteq R\).

**Proof.** Let \(R \circ R \subseteq R\). Suppose \((x, A) \in R\) and \((a, B_a) \in R\) for each \(a \in A\). Then by definition of composition \(\circ\), \((x, \bigcup_a B_a) \in R \circ R \subseteq R\). Hence \(R\) is a type 1 transitive \(h\)-relation.

Conversely, let \(R\) be transitive of type 1. Suppose \(x \in S\) be such that \((x, \phi) \not\in R\). Then \((x, \phi) \not\in R \circ R\). In fact, \((x, \phi) \in R \circ R\) and \((x, \phi) \not\in R\) \(\Rightarrow\) \(\exists B \in P^*(S)\) and \(A_b \in P(S)\) corresponding to each \(b \in B\) such that \((x, B) \in R\) and \((b, A_b) \in R\) and \(\bigcup_b A_b = \phi\)
\(\Rightarrow\) \((x, \phi) = (x, \bigcup_b A_b) \in R\) (because \(R\) is a type 1 transitive \(h\)-relation) which is contradictory. Hence \((x, \phi) \in R \circ R\) \(\Rightarrow\) \((x, \phi) \in R\). Suppose \((x, A) \in R \circ R\) where \(A \in P^*(S)\). Then \(\exists B \in P^*(S)\) and \(A_b \in P(S)\) corresponding to each \(b \in B\) such that \((x, B) \in R\), \((b, A_b) \in R\) and \(\bigcup_b A_b = A\). Then \((x, A) = (x, \bigcup_b A_b) \in R\) \((\because R\) is transitive of type 1). Thus, \(R \circ R \subseteq R\).

**Proposition 2.12.** Every type 1 transitive \(h\)-relation is a type 2 transitive \(h\)-relation.

**Proof.** It follows from observation 2.10 and proposition 2.11.

**Examples 2.13.**

1. Since arbitrary union of open sets is open, so the \(h\)-relation defined in the example 2.4(5) is transitive of type 1 (and hence of type 2).

2. Let \(S\) be an infinite set \(\). For each \(n \in \mathbb{N}, R_n = \{(x, A) : |A| \leq n\}\) is a \(h\)-relation on \(S, R_n(n > 1)\) is not a type 1 transitive \(h\)-relation. In fact, let \(A = \{x_1, x_2, \cdots, x_n\} \subseteq S\) and for each \(i \in \{1, 2, \cdots, n\}, A_i = \{x_{i1}, x_{i2}, \cdots, x_{in}\} \subseteq S\). Since \(S\) is infinite, one can assume
that for each \( i, A \cap A_i = \phi \) and for any \( i, j \) with \( i \neq j, A_i \cap A_j = \phi \). Then, for any \( x \in S \), we have by definition that \( (x, A) \in R_n, (x_i, A_i) \in R_n \) for each \( x_i \in A \). But \( (x, \bigcup A_i) \notin R \), since here \( \bigcup_{i=1}^{n} A_i = n^2 > n (\because n > 1) \).

On the other hand, for \( B, C \in P^*(S) \), if \( B \subseteq C_{R_n} \), then of course \( |C| \leq n \). Thus clearly for any \( A, B, C \in P^*(S) \), \( A \subseteq B_{R_n}, B \subseteq C_{R_n} \Rightarrow A \subseteq C_{R_n} \) whence \( R_n(n > 1) \) is transitive of type 2. This is to be noted that \( R_n(n \in \mathbb{N}) \) is a type 1 (and hence type 2) reflexive and type 1 symmetric \( h \)-relation, but \( R_n(n \in \mathbb{N}) \) is not a symmetry of type 2. Observe that \( R_1 \subset R_2 \subset R_3 \subset \cdots \subset R_n \subset R_{n+1} \cdots \) is a non-terminating increasing sequence of \( h \)-relations on \( S \) (since \( S \) is infinite) and \( \bigcup_{i=1}^{\infty} R_i = R \) is the \( h \)-relation as defined in the example 2.7(1), which is reflexive (of both the types) symmetric (of type 1 but not type 2) and transitive (of both the types).

**Proposition 2.14.** A type 2 reflexive \( h \)-relation \( R \) is type 1 reflexive if it is a type 1 symmetric \( h \)-relation.

**Proof.** Let \( R \) be a type 2 reflexive as well as a type 1 symmetric \( h \)-relation on \( S \). Then for any \( x \in S \), there exists \( A \in P(S) \) such that \( x \in A \) and \( (x, A) \in R \) i.e., \( x \in A_R \). So, \( x \in \{x\} \subseteq A_R \) and \( |\{x\}| \leq |A| \) (here \( |A| \geq 1 \), since \( x \in A \)). Thus, \( (a, \{x\}) \in R \) for all \( a \in A \) (since \( R \) is a type 1 symmetry). But \( x \in A \); so \( (x, \{x\}) \in R \) for all \( x \in S \), whence \( R \) is reflexive of type 1.

**Corollary 2.15.** A type 2 reflexive \( h \)-relation \( R \) is type 1 reflexive if it is a type 2 symmetric \( h \)-relation.

**Proof.** It follows from proposition 2.6 and 2.14.

**Proposition 2.16.** A type 2 transitive \( h \)-relation \( R \) is a type 1 transitive \( h \)-relation if it is a type 2 symmetric \( h \)-relation.

**Proof.** Let \( R \) be a type 2 transitive and type 2 symmetric \( h \)-relation on a non-empty set \( S \). Suppose \( (x, A) \in R \) and \( (a, B_a) \in R \) for each \( a \in A \). Then, \( A \subseteq \{x\}_R (\because R \) is a type 2 symmetry); i.e., \( \{a\} \subseteq \{x\}_R \) for all \( a \in A \). Hence \( \{x\} \subseteq \{a\}_R \) for all \( a \in A \). So, \( \{a\} \subseteq \{x\} \subseteq \{a\}_R \Rightarrow \{x\} \subseteq (B_a)_R \) for all \( a \in A \) (\( \because R \) is a type 2 transitive \( h \)-relation) whence \( B_a \subseteq \{x\}_R \) for all \( a \in A \). Thus, \( \bigcup_{a \in A} B_a \subseteq \{x\}_R \) implying \( \{x\} \subseteq \bigcup_{a \in A} (B_a)_R \) i.e., \( (x, \bigcup_{a} B_a) \in R \). So, \( R \) is a type 1 transitive \( h \)-relation.
3 Inverses and Invertibility

**Definition 3.1.** An $h$-relation $R$ on a non-empty set $S$ is said to be perfect if for any $A \in P(S)$, $A_R = \phi$ or else $|A_R| \geq |A|$. 

**Examples 3.2.**

1. For a positive integer $n$, let $S = \bigcup_{i=1}^{n} (i-1, i)$, $R = \{(x, A) : x \in (i-1, i) \text{ and } A \text{ is a countably infinite subset of } (i-1, i) \}$ for some $i = 1, 2, \cdots, n$ is a perfect $h$-relation on $S$.

2. Let $S = [0, 1]$. Then $R = \{(x, A) : x \in (0, 1) \text{ and } A \text{ is an open subset of } (0, 1) \}$ is a perfect $h$-relation on $S$.

3. Let $S$ be an open interval and $R = \{(x, A) : x \in S \text{ and } A \text{ is a } \delta \text{- neighbourhood of } x \in S \}$. Then $R$ is not a perfect $h$-relation on $S$. In fact, if $S = (a, b)$, then $|S_R| = 1 < |S|$. Note that $S = (a, b) = \left(\frac{a+b}{2} - \delta, \frac{a+b}{2} + \delta\right)$ for $\delta = \frac{b-a}{2} > 0$. Thus, $S_R = \left\{\frac{a+b}{2}\right\}$.

**Proposition 3.3.** The number of perfect $h$-relations on a finite set $S$ having $n(\in \mathbb{N})$ elements is $2^{n(n+1)} \prod_{j=2}^{n} \left[1 + \sum_{i=j}^{n} \binom{n}{i} \right]^{(i)}$. 

**Proof.** Let $S$ be a (non-empty) finite set with $|S| = n$. An $h$-relation $R$ on $S$ is perfect if any $A \in P(S)$ satisfies the condition(a) $|A_R| = 0$ or else $|A_R| \geq |A|$. 

Now, $\phi$ (empty subset of $S$) satisfies the condition(a) in $2^n$ ways. A singleton $A(\subseteq S)$ satisfies condition (a) in $2^n$ ways. There are $n$ number of singleton sets in $P(S)$. So, any number of singleton sets, each satisfying condition(a) can be taken in $(2^n)^n = 2^{n^2}$ ways. Any set $A$ (with $|A| = k > 1$) can satisfy condition(a) in $\left[1 + \sum_{i=k}^{n} \binom{n}{i}\right]$ ways. There are $\binom{n}{k}$ number of sets $A$ (with $|A| = k > 1$). So, any number of sets each having $k$ elements and satisfying condition(a) can be taken in $\left[1 + \sum_{i=k}^{n} \binom{n}{i}\right]^{(i)}$ ways. Thus, any number of sets in $P(S)$, each satisfying condition(a) can be taken in $2^n 2^{n^2} \prod_{j=2}^{n} \left[1 + \sum_{i=j}^{n} \binom{n}{i}\right]^{(i)}$ ways. So, there are $2^{n(n+1)} \prod_{j=2}^{n} \left[1 + \sum_{i=j}^{n} \binom{n}{i}\right]^{(i)}$ number of perfect $h$-relations on $S$ (with $|S| = n$).

**Definition 3.4.** The type 1 inverse of an $h$-relation $R$ on $S$ is an $h$-relation 

$$R^{-1} = \{(x, A) \in S \times P^*(S) : \exists X \in P(S) \text{ such that } x \in X, A \subseteq X_R \text{ and } |A| \leq |X|\}.$$

**Proposition 3.5.** An $h$-relation $R$ is perfect and symmetric of type 1 if and only if $R = R^{-1}$.
Proof. Let $R$ be a perfect and type 1 symmetric $h$-relation on $S$. Suppose $(a, X) \in R$. Then $X \neq \emptyset$. Since $R$ is perfect and $a \in X_R$ (i.e., $X_R \neq \emptyset$), so $|X_R| \geq |X|$. Now, we have $A \subseteq X_R$ such that $a \in A$ and $|A| = |X|$. Then, $(x, A) \in R$ for all $x \in X$ (since $R$ is symmetric of type 1), whence $X \subseteq A_R$. Then, by definition of type 1 inverse of $R$, $(a, X) \in R^{-1}$ (since $|X| = |A|$ and $a \in A$). So, $R \subseteq R^{-1}$.

Conversely, let $(a, X) \in R^{-1}$. Then $X \neq \emptyset$ and $\exists A \in P(S)$ such that $a \in A, X \subseteq A_R$ and $|X| \leq |A|$. Now, $X \subseteq A_R \Rightarrow (x, A) \in R$ for all $x \in X \Rightarrow$ for any $a' \in A, (a', Y) \in R$ whenever $x \in Y \subseteq A_R$ and $|Y| \leq |A|$ (because $R$ is symmetric of type 1). Since, $x \in X \subseteq A_R$, $|X| \leq |A|$ and $a \in A$, so we have $(a, X) \in R$. Hence, $R^{-1} \subseteq R$ whence $R = R^{-1}$.

Now, suppose that $R = R^{-1}$ and $(a, X) \in R, x \in X$. Also, suppose that $A \in P(S)$ be such that $a \in A \subseteq X_R$ and $|A| \leq |X|$. Clearly then $(x, A) \in R^{-1}$ and hence $(x, A) \in R$ [$\because R = R^{-1}$]. So, $R$ is a symmetry of type 1. Let $A \in P(S)$. If $A = \emptyset$, then $A_R = \emptyset$ [$\because R$ is a symmetry of type 1] and thus $|A_R| = |A|$. Suppose $A \neq \emptyset$. Then, $\exists x \in S$ such that $(x, A) \in R$. So, $(x, A) \in R^{-1}$ [$\because R = R^{-1}$]. Then, $\exists X \in P(S)$ such that $x \in X, A \subseteq X_R$ and $|A| \leq |X|$ (by definition of $R^{-1}$). Now, $A \subseteq X_R \Rightarrow (a, X) \in R$ for all $a \in A$. Since, $a \in A \subseteq X_R$ and $|A| \leq |X|$, we have $(x, A) \in R$ for all $x \in X$ whence $X \subseteq A_R$. Thus $|A| \leq |X| \leq |A_R|$ implying that $R$ is perfect.

Examples 3.6.

1. Let $S = \left\{ \frac{n}{m+1} : n \in \mathbb{N} \right\}$. On $S$, an $h$-relation $R$ is defined as follows:

$$(\frac{n}{m+1}, A) \in R \iff A \subseteq \left\{ \frac{k}{k+1} : k \in I_{2n+1} \right\}$$

where for any $m \in \mathbb{N}, I_m = \left\{ 1, 2, \ldots, m \right\}$. Then, $R$ is a perfect $h$-relation but not a symmetry of type 1 (and hence not a symmetry of type 2). In fact, for any infinite subset $A$ of $S, A_R = \emptyset$. When $A$ is a finite subset of $S$; there exists $m_i \in \mathbb{N}$($i = 1, 2, \ldots, k$) such that $m_1 < m_2 < \cdots < m_k$ and $A = \left\{ \frac{m_i}{m_{i+1}} : i \in I_k \right\}$. Then, $A \subseteq \left\{ \frac{k}{k+1} / k \in I_{2m_i+1} \right\}$ for any $m_i \geq m_k$. So clearly, $(\frac{m}{m+1}, A) \in R$ for any $m \geq m_k$ i.e., $B = \left\{ \frac{m}{m+1} / m \geq m_k \right\} \subseteq A_R$ i.e., $|B| \leq |A_R|$. But, $B$ is an infinite set, whereas $|A| = k(\in \mathbb{N})$. Hence, $|A| < |B|$ whereas $|A| \leq |A_R|$. Now, we see that $(\frac{4}{5}, A) \in R$ for $A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \right\} \subseteq \left\{ \frac{k}{k+1} : k \in I_9 \right\}$. Thus, $\left\{ \frac{4}{5} \right\} \subseteq A_R$ with $|\left\{ \frac{4}{5} \right\}| \leq |A|$, but $(\frac{1}{2}, \left\{ \frac{4}{5} \right\}) \notin R$ (note that $\frac{1}{2} \in A$).

2. Let $A$ and $B$ be two sets such that $|A| < |B|$. Suppose $S = A \cup B$. Then $R = (A \times P(B)) \cup (B \times P(A))$ be a $p$-relation on $S$. $R$ is not perfect. In fact $B \in P(S)$ for which $B_R = A$ and thus $|B_R| = |A| < |B|$. Moreover $R$ is a type 2 (and hence a type 1) symmetry on $S$.

3. The $h$-relation $R_n(n \in \mathbb{N})$ on an infinite set $S$ (as defined in example 2.13(2)) is a perfect, type 1 symmetric $h$-relation.
Proposition 3.7. A perfect, type 1 symmetric h-relation $R$ on a (non-empty) set $S$ is universal if and only if $S_R \neq \emptyset$.

**Proof.** $S_R \neq \emptyset \Rightarrow S_R = S$ (since $R$ is perfect) $\Rightarrow$ For any $x \in S$ and any $A \in P^*(S)$, $(x, A) \in R$ (since, $R$ is a type 1 symmetry) $\Rightarrow$ $R$ is universal. Converse part is immediate.

**Definition 3.8.** An h-relation $R$ on a non-empty set $S$ is called a type 2 invertible h-relation if there exists an h-relation $R'$ on $S$ such that, $\phi_{R'} = \phi$ and for any $A, B \in P^*(S)$, $B \subseteq A_R \Leftrightarrow A \subseteq B_{R'}$. In this case, $R'$ is called type 2 inverse of $R$.

**Proposition 3.9.** (1) Type 2 inverse of a type 2 invertible h-relation is unique. (2) If $R'$ is the type 2 inverse of a h-relation $R$, then $R$ is the type 2 inverse of $R'$ i.e., $(R')' = R$.

**Example 3.10.**

Let $S = \{1, 2, 3, 4\}$. On $S$, we take two h-relations $R_1 = \{(1, \{2, 3\}), (1, \{2, 4\}), (1, \{2\}), (1, \{4\}), (1, \{3\})\}$ and $R_2 = \{(1, \{2, 3\}), (1, \{2, 4\}), (1, \{2\}), (1, \{3\}), (1, \{4\})\}$. Here, $R_1$ is not a type 2 invertible h-relation. In fact, if possible let $R'_1$ exist. Then, $\{1\} \subseteq \{2, 3\} \cap R_1, \{1\} \subseteq \{2, 4\} \cap R_1 \Rightarrow \{2, 3\} \subseteq \{1\} \cap R'_1, \{2, 4\} \subseteq \{1\} \cap \{2, 3\}; \{2, 3, 4\} \subseteq \{1\} \cap \{2, 3, 4\} \cap R'_1 \Rightarrow \{1\} \subseteq \{2, 3, 4\} \cap R'_1$; but $(1, \{2, 3, 4\}) \not\in R_1$. On the other hand, $R_2$ is a type 2 invertible h-relation with $R'_2 = \{(2, \{1\}), (3, \{1\}), (4, \{1\})\}$.

The above example (3.10) leads us to propose the following.

**Proposition 3.11.** An h-relation $R$ on a non-empty set $S$ is type 2 invertible if and only if, for any $A_i \in P^*(S)(i \in \Lambda), (x, A_i) \in R \Rightarrow (x, P) \in R$ where $P$ is any non-empty subset of $\bigcup_{i \in \Lambda} A_i$.

**Proof.** Let the type 2 inverse $R'$ of an h-relation $R$ on $S$ exist. Then, for any $A_i \in P^*(S)(i \in \Lambda), (x, A_i) \in R \Rightarrow \{x\} \subseteq (A_i)_R \Rightarrow A_i \subseteq \{x\}_R \Rightarrow \bigcup_{i \in \Lambda} A_i \subseteq \{x\}_R \Rightarrow P \subseteq \{x\}_R$ (where $P$ is any non-empty subset of $\bigcup_{i \in \Lambda} A_i$) $\Rightarrow \{x\} \subseteq P_R \Rightarrow (x, P) \in R$.

Conversely let the condition hold true. We write $\rho = \{(x, A) : \exists X \in P(S) \text{ such that } x \in X \text{ and } A \subseteq X_R\}$. Then, for any $A, B \in P^*(S), A \subseteq B_\rho \Rightarrow$ for each $a \in A, \exists A_a \in P(S)$ such that $a \in A_a$ and $B \subseteq (A_a)_R$. Now, $B \subseteq (A_a)_R$ for all $a \in A \Rightarrow B \subseteq P_R$ where $P$ is any
non-empty subset of $\bigcup A_a$ (by the given condition) \( \Rightarrow \) $B \subseteq A_R$ (since, $A \subseteq \bigcup A_a$).

So, for any $A, B \in P^*(S), A \subseteq B_{\rho} \Rightarrow B \subseteq A_R$. Again, by definition of $\rho$, for any $A, B \in P^*(S), B \subseteq A_R \Rightarrow (a, B) \in \rho$ for all $a \in A \Rightarrow A \subseteq B_{\rho}$. Hence, for any $A, B \in P^*(S), B \subseteq A_R \iff A \subseteq B_{\rho}$ i.e., $\rho = R'$ and thus $R$ is a type 2 invertible $h$-relation.

**Corollary 3.12.** If $R$ is a type 2 invertible $h$-relation, then for any $A, B \in P^*(S)$

1. $(A \cup B)_R = A_R \cap B_R$ and
2. $A_R \cup B_R \subseteq (A \cap B)_R$ (provided $A \cap B \neq \phi$).

**Proof.** Let $A, B \in P^*(S)$ and $R$ is a type 2 invertible $h$-relation on $S$. Then, $x \in (A \cup B)_R \Rightarrow x \in A_R \cap B_R$ (since, $A \subseteq A \cup B, B \subseteq A \cup B$) $\Rightarrow x \in A_R \cap B_R$ and conversely, $x \in A_R \cap B_R \Rightarrow x \in A_R$ and $x \in B_R \Rightarrow x \in (A \cup B)_R$. Thus, $(A \cup B)_R = A_R \cap B_R$. Again, $x \in A_R \cup B_R \Rightarrow x \in A_R$ or $x \in B_R \Rightarrow x \in (A \cap B)_R$ (since, $A \cap B \subseteq A, A \cap B \subseteq B$). Hence, $A_R \cup B_R \subseteq (A \cap B)_R$.

**Proposition 3.13.** For any type 2 invertible $h$-relation $R$ on $S, R^{-1} \subseteq R'$. Also, $R^{-1} = R'$ if and only if $R'$ is perfect.

**Proof.** Let $(x, A) \in R^{-1}$. Then $A \neq \phi$ and there exists $X \in P^*(S)$ such that $x \in X, A \subseteq X_R$ and $|A| \leq |X|$. Now, $x \in X$ (i.e., $X \neq \phi$) and $A \subseteq X_R \Rightarrow X \subseteq A_{\rho}$ (since, $A \neq \phi$) $\Rightarrow (x, A) \in R'$. Hence, $R^{-1} \subseteq R'$.

Suppose, $R'$ is perfect and $(x, A) \in R'$. Then, $A \neq \phi$ and $A_{R'} \neq \phi$. So, $|A_{R'}| \geq |A|$ (since, $R'$ is perfect). Let $X = A_{R'}$. Then $A \subseteq X_R$. Now, $x \in X(=A_{R'})$, $A \subseteq X_R$, $|A| \leq |X| \Rightarrow (x, A) \in R^{-1}$ (since, $A \neq \phi$). Therefore, $R' \subseteq R^{-1}$ whence $R^{-1} = R'$.

Conversely, let $R'$ be not perfect. Then, there exists $A \in P^*(S)$ such that $|A_{R'}| < |A|$. Let $X = A_{R'}$. Then, $A \subseteq X_R$. If possible let, $(x, A) \in R^{-1}$ for all $x \in X$. Then by definition of $R^{-1}$, there exists $A_x \in P(S)$ for each $x \in X$ such that $x \in A_x, A \subseteq (A_x)_R$ and $|A| \leq |A_x|$. Now, $A \subseteq (A_x)_R$ for all $x \in X \Rightarrow A \subseteq \bigcup_{x \in X} A_x$ (since, $R$ is type 2 invertible) $\Rightarrow \bigcup_{x \in X} A_x \subseteq A_{R'} \Rightarrow \bigcup_{x \in X} A_x \leq |A_{R'}|$. But, $|A| \leq |A_x|$ for each $x \in X$ and so, $|A| \leq \bigcup_{x \in X} A_x$ (by proposition 3.11). Hence, $|A| \leq |A_{R'}|$ - a contradiction. Hence, there exists $x \in X$ such that $(x, A) \notin R^{-1}$ whereas $(x, A) \in R'$ for all $x \in X$ (since, $X = A_{R'}$). Thus, $R^{-1} \neq R'$.

**Proposition 3.14.** A type 2 invertible type 2 reflexive $h$-relation $R$ on $S$ is a type 1 reflexive $h$-relation.

**Proof.** Since $R$ is a type 2 reflexive $h$-relation on $S$, so, for any $x \in S$, there exists $A \in P(S)$ such that $x \in A$ and $(x, A) \in R$ which implies by proposition 3.11 that $(x, \{x\}) \in R$ (since, $x \in A$ and $R$ is type 2 invertible) whence $R$ is type 1 reflexive.
Proposition 3.15. An $h$-relation $R$ on a non-empty set $S$ is a type 2 symmetry if and only if $R'$ exists and $R = R'$.

Proof. Suppose $R'$ exists and $R = R'$. Then, for any $A, B \in P^*(S), A \subseteq B_R \Rightarrow B \subseteq A_{R'} \Rightarrow B \subseteq A_R$ (since, $R' = R$) whence $R$ is a symmetry of type 2.

Conversely, let $R$ be a type 2 symmetry. Then, for any $A, B \in P^*(S)$ and $x \in S, (x, A) \in R, (x, B) \in R \Rightarrow \{x\} \subseteq A_R, \{x\} \subseteq B_R \Rightarrow A \subseteq \{x\}_R, B \subseteq \{x\}_R$ (since, $R$ is a type 2 symmetry) $\Rightarrow A \cup B \subseteq \{x\}_R \Rightarrow P \subseteq \{x\}_R$ (where $P$ is any non-empty subset of $A \cup B$) $\Rightarrow \{x\} \subseteq P_R \Rightarrow (x, P) \in R$. So by proposition 3.11, $R$ is type 2 invertible i.e., $R'$ exists.

4 Associated Hyperstructures

Let $H$ be a non-empty set and $R$ be an $h$-relation on $H$. We define a hyperoperation $\circ_R$ on $H$ as follows : For any $x, y \in H$, (i) $x \circ_R x = \bigcup\{A \in P(H) : (x, A) \in R\}$ and (ii) $x \circ_R y = (x \circ_R x) \cup (y \circ_R y)$. The hypergropuid $(H, \circ_R)$ is denoted by $H_R$. Unless otherwise stated, henceforth we shall use $\circ$ in place of $\circ_R$.

Lemma 4.1. Let $R$ be an $h$-relation on a non-empty set $H$, with full domain. Then $(a, C) \in R$, $(c, X) \in R$ for some $c \in C \Rightarrow \exists Y \in P(H)$ with $X \subseteq Y$ such that $(a, Y) \in R^2$.

Proof. Let $(a, C) \in R$, $(c, X) \in R$ for some $c \in C$. Suppose $C_1 = C \setminus \{c\}$. Since $R$ has full domain, so for each $c_1 \in C_1$, $\exists X_{c_1} \in P(H)$ such that $(c_1, X_{c_1}) \in R$. Now, suppose $Y = X \bigcup \left( \bigcup_{c_1 \in C_1} X_{c_1} \right)$. Then, $(a, C) \in R$, $(c, X) \in R$, $(c_1, X_{c_1}) \in R$ for each $c_1 \in C_1$ imply that $(a, Y) \in R^2$ where $X \subseteq Y$.

Lemma 4.2. Let $R$ be an $h$-relation on $H$. Then $(a, X) \in R^2 \Rightarrow X \subseteq (a^2)^2$ (where $a^2 = a \circ a$) and for any $A \in P(H)$, $A^2 = A \circ A$.

Proof. Let $(a, X) \in R^2$. If $X = \phi$, then clearly $X \subseteq (a^2)^2$. Suppose $X \neq \phi$. Then, $\exists B \in P^*(H)$ and $X_b \in P(H)$ for each $b \in B$ such that $(a, B) \in R, (b, X_b) \in R$ and $X = \bigcup_{b \in B} X_b$. Then, for any $x \in X, \exists b \in B$ such that $x \in X_b$. So, $(a, B) \in R$ and $(b, X_b) \in R$ imply that $b \in a \circ a$ and $x \in b \circ b$. Hence $x \in (a \circ a) \circ (a \circ a)$ whence $X \subseteq (a^2)^2$.

Lemma 4.3. Let $R$ be an $h$-relation on $H$ and have full domain. Then, on the hypergropuid $H_R$, one can assert that, for any $a, b, c \in H$, $(a) \ x \in a \circ (b \circ c) \iff \exists X, Y \in P(H)$ with $x \in X \cap Y$ such that $(a, X) \in R$ or $(b, Y) \in R^2$ or $(c, Y) \in R^2$, and $(\beta) \ x \in (a \circ b) \circ c \iff \exists X', Y' \in P(H)$ with $x \in X' \cap Y'$ such that $(a, X') \in R^2$ or $(b, Y') \in R^2$ or $(c, Y') \in R$.
Proof. (a) Clearly here \( a \circ (b \circ c) = (a \circ a) \cup \bigcup_{u \in (b \circ b) \cup (c \circ c)} u \circ u \). So, \( x \in a \circ (b \circ c) \Rightarrow x \in a \circ a \) or \( x \in u \circ u \) for some \( u \in (b \circ b) \cup (c \circ c) \) \( \Rightarrow \exists X \in P(H), Y_o \in P(H) \) with \( x \in X \cap Y_o \) such that \((a, X) \in R \) or \((u, Y_o) \in R \). Now, \( u \in (b \circ b) \) or \( u \in (c \circ c) \) \( \Rightarrow \exists U \in P(H) \) with \( u \in U \) such that \((b, U) \in R \) or \((c, U) \in R \). Again, \((b, U) \in R, (u, Y_o) \in R \) for some \( u \in U \) \( \Rightarrow \exists Y \in P(H) \) with \( x \in Y_o \subseteq Y \) such that \((b, Y) \in R^2 \) [by lemma 4.1, since \( R \) has full domain].

Similarly \((c, U) \in R, (u, Y_o) \in R \) for some \( u \in U \) \( \Rightarrow \exists Y \in P(H) \) with \( x \in Y_o \subseteq Y \) such that \((c, Y) \in R^2 \) [by lemma 4.1]. Thus, \( x \in a \circ (b \circ c) \Rightarrow \exists X, Y \in P(H) \) with \( x \in X \cap Y \) such that \((a, X) \in R \) or \((b, Y) \in R^2 \).

Conversely, let \( \exists X, Y \in P(H) \) with \( x \in X \cap Y \) such that \((a, X) \in R \) or \((b, Y) \in R^2 \) or \((c, Y) \in R^2 \). Then, \((a, X) \in R, x \in X \Rightarrow x \in a \circ a \Rightarrow x \in a \circ (b \circ c) \). Again, \((b, Y) \in R^2 \), \( x \in Y \Rightarrow \exists Z \in P^*(H) \) and \( Y_z \in P(H) \) corresponding to each \( z \in Z \) such that \((b, Z) \in R, (z, Y_z) \in R \) and \( Y = \bigcup_{z \in Z} Y_z \Rightarrow \exists z_1 \in Z \) such that \( x \in Y z_1 \) (since \( x \in Y \)), \((b, Z) \in R, (z_1, Y_{z_1}) \in R \Rightarrow z_1 \in b \circ b \) and \( x \in z_1 \circ z_1 \Rightarrow z_1 \in (b \circ b) \cup (c \circ c) \) such that \( x \in z_1 \circ z_1 \Rightarrow x \in \left( \bigcup_{u \in (b \circ b) \cup (c \circ c)} u \circ u \right) \Rightarrow x \in a \circ (b \circ c) \). Similar is the case when \((c, Y) \in R^2 \) with \( x \in Y \). (b) can be proved in an essentially same manner.

Definition 4.4. Let \( R \) be an \( h \)-relation on a non-empty set \( H \). Then, \( X \in P^*(H) \) is said to be an outer set of \( R \) if for any \( A \in P(H) \) with \( A \cap X \neq \phi \), \( \exists h \in H \) such that \((h, A) \notin R^2 \).

Theorem 4.5. Let \( R \) be an \( h \)-relation with full domain defined on a non-empty set \( H \). Then, \( H_R \) is a semihypergroup if and only if the \( h \)-relation \( R \) satisfies the following conditions: 

\( \gamma \) \((a, X) \in R, x \in X \Rightarrow \exists Y_x \in P(H) \) with \( x \in Y_x \) such that \((a, Y_x) \in R^2 \).

\( \delta \) For any \( X \in P(H) \) and \( A \in P(H) \) with \( A \cap X \neq \phi \), \( \exists B \in P(H) \) with \( B \cap X \neq \phi \) such that \((a, A) \in R^2 \) \( \Rightarrow (a, B) \in R \) whenever \( X \) is an outer set of \( R \).

Proof. Suppose \( H_R \) be a semihypergroup. To prove \( \gamma \), we suppose on contrary that \( a \in H \) and \( X \in P^*(H) \) such that \((a, X) \in R \) but there exists \( x \in X \) for which \((a, Y) \notin R^2 \) for any \( Y \in P(H) \) with \( x \in Y \). Then by \( \alpha \) (Lemma 4.3 ), we have \((a, X) \in R, x \in X \Rightarrow x \in (x \circ a) \circ a \Rightarrow x \in x \circ (a \circ a) \) [since \( H_R \) is a semihypergroup]. So, by \( \beta \) (lemma 4.3), \( \exists X', Y' \in P(H) \) with \( x \in X' \cap Y' \) such that \((a, Y') \in R^2 \) or \((x, X') \in R \). But \((a, Y') \notin R^2 \), since \( x \in Y' \), we have \((x, X') \in R \), which together with \((a, X) \in R \) and \( x \in X \) imply that \( \exists Y'' \in P(H) \) with \( X' \subseteq Y'' \) such that \((a, Y'') \in R^2 \) [by lemma 4.1]. But this is also not true, since \( x \in Y'' \). Hence \( \gamma \) is satisfied by \( R \).

To prove \( \delta \), we suppose on contrary that, there exists an outer set \( X \) of \( R \) and \( A \in P(H) \) with \( A \cap X \neq \phi \) such that \((a, A) \in R^2 \) but \((a, B) \notin R \) for any \( B \in P(H) \).
with \( B \cap X \neq \emptyset \). Then, since \( X \) is an outer set of \( R \), \( \exists b \in H \) such that \( b, B \notin R^2 \) for any \( B \in P(H) \) with \( B \cap X \neq \emptyset \). Let \( x \in A \cap X \). Then, by (\( \beta \)) (lemma 4.3), \((a, A) \in R^2 \), \( x \in A \models x \in (a \circ b) \circ b \Rightarrow x \in a \circ (b \circ b) \) (since \( H_R \) is a semihypergroup) \( \Rightarrow \exists Y, Z \in P(H) \) with \( x \in Y \cap Z \) such that \((a, Y) \in R \) or \((b, Z) \in R^2 \) [by (\( \alpha \)) lemma 4.3]. But here \( Y \cap X \neq \emptyset \). So, \((a, Y) \notin R \). Hence, \((b, Z) \in R^2 \) which is contradictory to our assumption (since \( Z \cap X \neq \emptyset \)). Hence \( \delta \) is satisfied. Conversely, suppose that both the conditions are satisfied. For any \((a, b, c) \in H \), let \( x \in a \circ (b \circ c) \). If \((b, Y) \in R^2 \) for some \( Y \in P(H) \) with \( x \in Y \), then clearly \( x \in (a \circ b) \circ b \) [by (\( \beta \)) lemma 4.3].

Now, \((a, X) \in R \), \( x \in X \Rightarrow \exists Y_x \in P(H) \) with \( x \in Y_x \) such that \((a, Y_x) \in R^2 \) [by (\( \gamma \))] \( \Rightarrow x \in (a \circ b) \circ c \) [by (\( \beta \))]. Again, \((c, Y) \in R^2 \), \( x \in Y \) and \( \{x\} \) is an outer set of \( R \Rightarrow \exists Z \in P(H) \) with \( x \in Z \) such that \((c, Z) \in R \) [by (\( \delta \))] \( \Rightarrow x \in (a \circ b) \circ c \) [by (\( \beta \))].

Hence, \( a \circ (b \circ c) \subseteq a \circ (b \circ c) \) whenever \((a, b, c) \in H \) i.e., \( H_R \) is a semihypergroup.

**Theorem 4.6.** Let \( R \) be an \( h \)-relation on a non-empty set \( H \). Then, the hypergroupoid \( H_R \) is a hypergroup if and only if

1. \( R \) has full domain,
2. \( \bigcup_{A \in \text{Cod}(R)} A = H \),
3. \((a, X) \in R \), \( x \in X \Rightarrow \exists Y_x \in P(H) \) with \( x \in Y_x \) such that \((a, Y_x) \in R^2 \) [i.e. the condition (\( \gamma \)) in theorem 4.5.]
4. for any \( X \in P(H) \) and \( A \in P(H) \) with \( A \cap X \neq \emptyset \), there exists \( B \in P(H) \) with \( B \cap X \neq \emptyset \) such that \((a, A) \in R^2 \Rightarrow (a, B) \in R \) whenever \( X \) is an outer set of \( R \) [i.e., the condition (\( \delta \)) in theorem 4.5].

**Proof.** Suppose \( H_R \) be a hypergroup. Then for any \( a \in H \), \( \exists b \in H \) such that \( a \in a \circ b \). So, \( \exists A \in P(H) \) with \( a \in A \) such that \((a, A) \in R \) or \((b, A) \in R \). Hence \( A \in \text{Cod}(R) \) whence \( a \in \bigcup_{A \in \text{Cod}(R)} A \). Thus, \( H \subseteq \bigcup_{A \in \text{Cod}(R)} A \), whence \( \bigcup_{A \in \text{Cod}(R)} A = H \).

Again, for any \( a \in H \), \( a \circ a \neq \emptyset \) (since, \( H_R \) is a hypergroup). So, \( \exists A \in P^*(H) \) such that \( a \in A \) implying that \( a \in \text{Dom}(R) \) whence \( H \subseteq \text{Dom}(R) \) i.e., \( R \) has full domain.

Since \( H_R \) is a semihypergroup (being hypergroup) for an \( h \)-relation \( R \) having full domain, (3) and (4) are satisfied by virtue of theorem 4.5.

Conversely, the validity of the conditions (1), (3) and (4) for \( H_R \) implies that \( H_R \) is a semihypergroup (by theorem 4.5). Let \( a, k \in H \) be arbitrary. By condition (2), then \( \exists K \in \text{Cod}(R) \) such that \( k \in K \). So, \( \exists h \in H \) such that \((h, K) \in R \) implying that \( k \in h \circ h \) (since, \( k \in K \)). So, \( k \in h \circ h \cup a \circ a = a \circ h \subseteq a \circ H \). Thus, \( H \subseteq a \circ H \).
Hence, \( a \circ H = H \circ a = H \) (since, \( \circ \) is commutative on \( H_R \)). i.e., \( H \) is a hypergroup.

**Theorem 4.7.** A hypergroupoid \((H, \circ)\) is a semihypergroup with \( a \circ b = a^2 \cup b^2 \) for any \( a, b \in H \) if and only if

1) \((H, \circ) = H_R\) for some \( h \)-relation \( R \) on \( H \), with full domain,

2) \( a^2 \subseteq (a^2)^2 \) for any \( a \in H \),

3) for any \( X \in P^\ast(H) \) and \( a, b \in H \), \( X \subseteq (a^2)^2 \cap H \setminus (b^2)^2 \Rightarrow \exists Y \in P(H) \) with \( Y \cap X \neq \phi \) such that \( Y \subseteq a^2 \).

**Proof.** Suppose \((H, \circ)\) be a hypergroupoid which satisfies the conditions (1), (2) and (3). Then, by (1) there exists an \( h \)-relation on \( H \) with full domain such that \( a \circ_R b = a \circ b \) for any \( a, b \in H \). Then, \( a \circ b = a \circ a \cup b \circ b = a^2 \cup b^2 \). We claim that (2) \( \Rightarrow (\gamma) \) [ of theorem 4.5 ]. In fact, let \((a, X) \in R, x \in X \). Then, \( X \subseteq a^2 \Rightarrow X \subseteq (a^2)^2 \) [ by (2) ] \( \Rightarrow x \in (a^2)^2 \Rightarrow \exists u \in a \circ a \) such that \( x \in u \circ u \Rightarrow \exists U \in P(H), Y' \subseteq P(H) \) with \( u \in U \) and \( x \in Y' \) such that \((a, U) \in R, (u, Y') \in R \Rightarrow \exists Y \in P(H) \) with \( Y' \subseteq Y \) such that \((a, Y) \in R^2 \Rightarrow \exists Y \in P(H) \) with \( x \in Y \) (since \( x \in Y' \)) such that \((a, Y) \in R^2 \). Hence \((\gamma)\) is satisfied by \( R \).

Now, we assert that (3) \( \Rightarrow (\delta) \) [ of theorem 4.5 ]. In fact, let \( X \) be an outer set of \( R \) and \( A \in P(H) \) with \( A \cap X \neq \phi \) such that \((a, A) \in R^2 \). Then by lemma 4.2, \( A \subseteq (a^2)^2 \) whence \( B \subseteq (a^2)^2 \) ( where \( B = A \cap X \) ). Now, since \( X \) is an outer set of \( R \), \( \exists b \in H \) such that \((b, Z) \notin R^2 \) for any \( Z \in P(H) \) with \( Z \cap X \neq \phi \). Then \( B \subseteq Z \setminus (b^2)^2 \). In fact, \( B \cap (b^2)^2 \neq \phi \Rightarrow \exists z \in X \) such that \( x \in (b^2)^2 \) [ since \( B = A \cap X \) ] \( \Rightarrow x \in c \circ c \) for some \( c \in b \circ b \Rightarrow \exists C, Z' \in P(H) \) with \( c \in C, x \in Z' \) such that \((b, C) \in R, (c, Z') \in R \Rightarrow (b, Z) \in R^2 \) for some \( Z \in P(H) \) with \( Z' \subseteq Z \) [ by lemma 4.1 ] \( \Rightarrow \exists Z \in P(H) \) with \( Z \cap X \neq \phi \) such that \((b, Z) \in R^2 \) [ since \( x \in Z' \) ]. This is contradictory to the fact that \( X \) is an outer set of \( R \). Hence \( B \subseteq (a^2)^2 \cap H \setminus (b^2)^2 \). Thus by (3), there exists \( Y \in P(H) \) with \( B \cap Y \neq \phi \) such that \( Y \subseteq a^2 \). So, for any \( x \in B \cap Y, x \in a^2 \) which implies that there exists \( C \in P(H) \) with \( x \in C \) such that \((a, C) \in R \). Now, \( x \in (B \cap Y) \cap C \Rightarrow (B \cap Y) \cap C \neq \phi \Rightarrow (A \cap X \cap Y) \cap C \neq \phi \Rightarrow (C \cap X) \cap (A \cap Y) \neq \phi \Rightarrow C \cap X \neq \phi \). Hence, \((\delta)\) is satisfied by \( R \). So, \( H_R \) is a semihypergroup [ since, \( R \) has full domain ] and thus \((H, \circ)\) is a semihypergroup.

Conversely, let \((H, \circ)\) be a semihypergroup with \( a \circ b = a^2 \cup b^2 \) for any \( a, b \in H \). We define \( R = \{(a, a \circ a) : a \in H \} \). Clearly, \( R \) is an \( h \)-relation on \( H \), with full domain. By definition, \( a \circ_R b = a \circ b \) for any \( a \in H \). Moreover, \( a \circ_R b = a \circ_R a \cup b \circ_R b = a \circ a \cup b \circ b = a \circ b \). Thus, \((H, \circ) = H_R \) is a semihypergroup with \( h \)-relation \( R \) having full domain. So, by theorem (4.5), the conditions \((\gamma)\) and \((\delta)\) both are satisfied by \( R \).

Now, let \( a, x \in H \) such that \( x \in a^2 \). But \((a, a^2) \in R \). Thus, by \((\gamma)\), \( \exists Y_x \in P(H) \)
with \( x \in Y_x \) such that \((a, Y_x) \in R^2\). By lemma 4.2, then \( Y_x \subseteq (a^2)^2 \). So, \( x \in (a^2)^2 \). Consequently, \( a^2 \subseteq (a^2)^2 \).

Let, \( X \subseteq (a^2)^2 \cap H \setminus (b^2)^2 \). Then, \( X \subseteq (a^2)^2 \) and \( X \cap (b^2)^2 = \emptyset \). We claim that \( X \) is an outer set of \( R \). In fact, if we suppose that \( \exists A \in P(H) \) with \( A \subseteq X \neq \emptyset \) such that \((b, A) \in R^2\), then \( A \subseteq (b^2)^2 \) implying \( X \cap (b^2)^2 \neq \emptyset \) - a contradiction. Thus, for any \( A \in P(H) \) with \( A \cap X \neq \emptyset \), \((b, A) \notin R^2\) whence \( X \) is an outer set of \( R \). Now, \( X \subseteq (a^2)^2 \Rightarrow \) for each \( x \in X \), \( \exists c_x \in a \circ a \) such that \( x \in c_x \circ c_x \). So, by lemma 4.1, \( \exists Y_x \in P(H) \) with \( c_x \circ c_x \subseteq Y_x \) such that \((a, Y_x) \in R^2\). Now, \( Y_x \cap X \neq \emptyset \), \((a, Y_x) \in R^2\) and \( X \) is an outer set of \( R \) together imply \([ \) by \((\delta) \) \] that, \( \exists Z \in P(H) \) with \( Z \cap X \neq \emptyset \) such that \((a, Z) \in R\). Hence, \( Z \subseteq a \circ a \) (in fact, \( Z = a \circ a \)). Thus, (3) is satisfied.

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