**OUTPUT-FEEDBACK TRACKING WITH PREVIEW OF STATE-MULTIPLICATIVE NOISE SYSTEMS**

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Abstract

The problem of finite-horizon $H_\infty$ output-feedback tracking for linear time-varying systems with stochastic state-multiplicative parameter uncertainties is investigated. We consider three tracking patterns depending on the nature of the reference signal i.e.: whether it is perfectly known in advance, measured on line or previewed in a fixed time-interval ahead. The stochastic uncertainties appear in both the dynamic and measurement matrices of the system. For each of the above three cases a solution is found where, given a specific reference signal, the controller plays against nature which chooses the initial condition and the energy-bounded disturbances. The problems are solved, using an expected value of the standard performance index over the stochastic parameters, based on the state-feedback tracking control solution and a specially devised bounded real lemma for state-multiplicative systems with tracking signal.

1 Introduction

In control theory, tracking stands as one of the main fundamental problems, where the system output is required to be as close as possible to an external reference signal. In the standard setting, the $H_\infty$ tracking control problem has been solved, by [1] where the signal to be tracked is taken as a disturbance that affects both the measurement and the objective function of the problem. This solution does not deal with the case where the signal is known in advance or can be previewed.

A method for solving the continuous and the discrete-time tracking problems, when such preview is available, has been introduced in [2] and [3], respectively. This method processes the information that is gathered on the reference during the system operation and by applying the game-theory approach it derives the optimal tracking strategy.

In [4], control with preview for continuous-time systems has been obtained using a state-space $H_\infty$ theory and introducing time-delay. The theory there resulted in an infinite-dimensional Riccati equation whose solution can not be tackled practically. The discrete-time case was treated in [5] where the disturbance is partially measured with preview.
An important extension of the above mentioned works on $H_{\infty}$ tracking with preview is to allow for uncertainties in the plant parameters. Using the method of [3], both the state-feedback and output-feedback tracking control problems were solved for discrete-time systems with norm-bounded uncertainties in [6]. The norm-bounded uncertainties in [6] may be time-varying with unbounded rates. When the uncertainties can be modeled more precisely, the over-design entailed in the method of [6], can be significantly reduced.

One such case is the case of uncertainties which can be modeled as white noise stochastic processes. The analysis and design of controllers for systems with stochastic uncertainties have received much attention in the past [7], [8] where mainly robust stability has been considered. Recently, a renewed interest in this problem has been encountered and solutions to the stochastic control problem have been derived that ensure a worst case performance bound in the $H_{\infty}$ style ([9]-[13]).

Systems whose parameter uncertainties are modeled as white noise processes in a linear setting have been treated in [11],[9],[10], for the continuous-time case and in the discrete-time case [12],[13] and the references therein.

Such models of uncertainties are encountered in many areas of applications (see [12] and the references therein) such as: nuclear fission and heat transfer, population models and immunology. In control theory such models are encountered in gain scheduling when the scheduling parameters are corrupted with measurement noise. In [14] a discrete-time stochastic estimation for a guidance motivated tracking problem was solved and its results were shown to achieve better results that those achieved by the Kalman-filter. In [10], an example of continuous-time estimation problem was given where a white-noise modeled parameter uncertainty resides in the altitude measurement of a radar altimeter. The white noise appears as a state-multiplicative term in the measurement equation. An $H_{\infty}$ filter which accounts for this multiplicative term was designed in [10] and was shown to achieve better results in comparison with a Kalman-filter, which ignores the multiplicative term.

In [15] the state-feedback tracking control for state-multiplicative systems was solved where the white noise parameter uncertainties are correlated and appear in both the system dynamics and the input matrices. An optimal state-feedback tracking strategy was derived [15] which minimizes the expected value of the standard $H_{\infty}$ performance index with respect to the unknown parameters. The problem was solved using a game theoretic approach where saddle-point controller strategies were found in each of the three cases of preview patterns.

In the present paper, we extend the work of [15] to the output-feedback tracking control problem and we obtain a solution via a min-max strategy arguments, rather than a game approach, as in the state-feedback case [15]. Using the latter solution we re-formulate the problem to a filtering problem which we solve with the aid of a special form of the bounded real lemma for state-multiplicative system with tracking signal.

**Notation:** The notation $P > 0$, (respectively, $P \geq 0$) for $P \in \mathbb{R}^{n \times n}$ means that $P$ is symmetric and positive definite (respectively, semi positive-definite). The space of square summable functions over $[0 \quad N-1]$ is denoted by $l_2[0 \quad N-1]$, and $||d||^2_2$ stands for the standard $l_2$-norm, $||d||^2_2 = (\sum_{k=0}^{N-1} d_k^T d_k)$. Also by $||f||^2_F$ and $||f||^2_R$ we denote the product $f_k^T R f_k$ and the Euclidean norm of $f$, respectively, and by $E\{\cdot\}$ we denote the expectation with respect to $\nu$. By $[Q_k]_+$, $[Q_k]_-$ we denote the causal and non causal parts respectively, of a sequence $\{Q_t, \ i = 1, 2, ..., N\}$. By $\text{Tr}\{\cdot\}$ we denote the trace of a matrix and by $\delta_{ij}$ the Kronecker delta function.

## 2 Problem Formulation

Given the following linear discrete time-varying system:

$$
\begin{align}
  x_{k+1} &= (A_k + F_k v_k) x_k + B_{2,k} u_k + B_{1,k} w_k + B_{3,k} r_k \\
  y_k &= (C_{2,k} + D_k s_k) x_k + D_{21,k} n_k,
\end{align}
$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the measurement vector, $w_k \in \mathbb{R}^p$ is a determinis-
tic exogenous disturbance, \( r_k \in R^r \) is deterministic reference signal which can be measured on line or previewed, \( u_k \in R^l \) is the control input signal and \( x_0 \) is an unknown initial state and where \( \{v_k\} \) and \( \{\zeta_k\} \) are standard random scalar white noise sequences with zero mean that satisfy: \( E\{v_k v_j\} = \delta_{kj} \), \( E\{\zeta_k \zeta_j\} = \delta_{kj} \), \( E\{v_k \zeta_j\} = 0 \). We denote

\[
z_k = C_k x_k + D_{3,k} u_k + D_{3,k} r_k, \quad z_k \in R^q, \quad k \in [0, N]
\]

and we assume, for simplicity that: \([C_k^T D_{3,k}^T D_{2,k}^T]D_{2,k} = [0 \ 0 \ \tilde{R}_k], \ \tilde{R}_k > 0\).

Our objective is to find a control law \( \{u_k\} \) that minimizes the energy of \( E_{v,\zeta} \{z_k\} \) by using the available knowledge on the reference signal, for the worst-case of the process disturbances \( \{w_k\}, \{n_k\} \) and the initial condition \( x_0 \). We, therefore, consider, for a given scalar \( \gamma > 0 \), the following performance index:

\[
J_E(r_k, u_k, w_k, n_k, x_0) = \min_{v, \zeta} \{E_{v,\zeta} \{||C_N x_N + D_{3,N} r_N||^2\} + E_{v,\zeta} \{||z_k||^2 - \gamma^2 [||w_k||^2 + ||n_k||^2]\} - \gamma^2 x_0^T R^{-1} x_0, \quad R^{-1} \geq 0 \}.
\]

Similarly to [15] we consider three tracking problems differing on the information pattern over \( \{r_k\} \):

1) **Stochastic \( H_\infty \)-tracking with full preview of \( \{r_k\} \)**: The tracking signal is perfectly known for the interval \( k \in [0, N] \).

2) **Stochastic \( H_\infty \)-tracking with no preview of \( \{r_k\} \)**: The tracking signal measured at time \( k \) is known for \( i \leq k \).

3) **Stochastic \( H_\infty \)-tracking with fixed-finite preview of \( \{r_k\} \)**: At time \( k \), \( r_i \) is known for \( i \leq \min(N, k+h) \) where \( h \) is the preview length.

In all the above three cases we seek a control law \( \{u_k\} \) of the form

\[
u_k = H_y y_k + H_r r_k
\]

where \( H_y \) is a causal operator and where the causality of \( H_r \) depends on the information pattern of the reference signal. The design objective is to minimize

\[
\max J_E(r_k, u_k, w_k, n_k, x_0) \quad \forall \{w_k\}, \{n_k\}, \quad \{u_k\} \in l_2[0, N - 1], \quad x_0 \in R^n,
\]

where for all of the three tracking problems we derive a controller \( \{u_k\} \) which plays against it’s adversaries \( \{w_k\}, \{n_k\} \) and \( x_0 \).

### 3 Results

#### 3.1 The stochastic state-multiplicative state-feedback tracking

We bring first the result of [15] which constitutes the first step in the solution of the output-feedback control problem. We consider the system of (1,a) with the objective function of (2). We consider the following Riccati difference equation:

\[
Q_k = A_k^T M_{k+1} A_k + C_k^T C_k + F_k^T Q_{k+1} F_k - A_k^T M_{k+1} B_{2,k} \Phi_k^{-1} B_{2,k}^T M_{k+1} A_k, \quad Q(N) = C_N^T C_N, \quad M_{k+1} = \tilde{Q}_{k+1} [I - \gamma^2 B_{1,k} B_{1,k}^T Q_{k+1}]^{-1},
\]

\[
\Phi_k = B_{2,k}^T M_{k+1} B_{2,k} + \tilde{R}_k.
\]

The solution of the state-feedback tracking problem is obtained by the following [15]:

**Theorem 1**: Consider the system of (1,a), (2) and \( J_E \) of (3) with the term of \( n_k \) excluded. Given \( \gamma > 0 \), the state-feedback tracking game possesses a saddle-point equilibrium solution iff there exists \( \gamma > 0, \forall i \in [0, N] \) that solves (4a) and satisfies

\[
R_{k+1} > 0, \quad k \in [0, N - 1], \quad \gamma^2 R^{-1} - Q_0 > 0, \quad R_{k+1} = \tilde{Q}_{k+1} [I - \gamma^2 B_{1,k} B_{1,k}^T Q_{k+1}]^{-1}
\]

When a solution exists, the saddle-point strategies are given by:

\[
x_{0}^* = (\gamma^2 R^{-1} - Q_0)^{-1} \theta_0 \quad \text{and where } \theta_k \text{ satisfies}
\]

\[
\theta_k = A_k^T \theta_{k+1} + \tilde{B}_k r_k, \quad \theta_N = C_N^T D_{3,N} R_N, \quad A_k = Q_{k+1}^{-1} S_{k+1}^{-1} A_k, \quad \tilde{B}_k = A_k^T Q_{k+1} B_{3,k} + C_k^T D_{3,k},
\]

where

\[
T_{k+1} \triangleq \tilde{R}_k, \quad S_{k+1} \triangleq M_{k+1}^{-1} + B_{2,k} T_{k+1}^{-1} B_{2,k}^T.
\]
The game value is then given by:

\[
J_E(r_k, u^*_k, w^*_k, x_0) = \left\| S_{k+1}^{-1/2} (Q_{k+1}^{-1} \theta_{k+1} + B_{3,k} r_k) \right\|_2^2 - \left\| Q_{k+1}^{-1/2} \theta_{k+1} \right\|_2^2 + \left\| D_{3,k} r_k \right\|_2^2 + \left\| D_{3,N} r_N \right\|_2^2 + \frac{1}{2} \theta_0^T (\gamma^2 R^{-1} - Q_0) - \theta_0.
\]

(10)

**Proof:** [15] The proof is based on adapting the standard completing to squares arguments [3] to the stochastic case. We bring in the Appendix the Sufficiency part of the proof, which is needed for the derivation of the bounded real lemma of the next section. We note also, at this point, that the solution of the output-feedback control problem is also based on these derivations.

**Remark 1:** It is important to note that the signal of \( \theta_k \) in (8) is admitted in the above derivation because of the tracking signal which affects the dynamics of (1a) (see [3]). This signal accounts for the nature of the tracking pattern, where its causal part (i.e \( [\theta_{k+1}]_+ \)) appears in the structure of the controller in accordance with the preview patterns.

**Remark 2:** Applying the result of Theorem 1 on the specific pattern of the reference signal it is shown in [15] that the saddle-point controller strategy depends on the causal part of \( \theta_{k+1} \) (i.e \( [\theta_{k+1}]_+ \)), where \( \theta_{k+1} \) is given in (8). The latter dependency on \( \theta_{k+1} \) appears also in the structure of the control signal in the output-feedback case.

### 3.2 The stochastic state-multiplicative output-feedback tracking

The solution of the deterministic counterpart of this problem (i.e with no white noise component) with different preview patterns was obtained, using a game theoretic approach, in [3]. Similarly to the standard dynamic output-feedback control problem solution [16], the solution in [3] involves 2 steps where the latter one is a filtering problem of order \( n \). A second Riccati equation is thus achieved by applying the bounded real lemma [16] to the dynamic equation of the estimation error. In our case we follow the same approach, however, it will be shown that the estimation error contains a state-multiplicative noise component. The latter imposes augmentation of the system to \( 2n \) order. This augmented system contains also a tracking signal component and therefore one needs to apply a special bounded real lemma for state multiplicative system with tracking signal. We thus bring first the following lemma:

#### 3.3 - Bounded Real Lemma (BRL) for stochastic state-multiplicative systems with tracking signal

We consider the following system:

\[
x_{k+1} = (A_k + F_k v_k) x_k + B_{1,k} w_k + B_{3,k} r_k
\]

\[
z_k = C_k x_k + D_{3,k} r_k, \quad z_k \in \mathbb{R}^q, \quad k \in [0, N]
\]

(11a,b)

which is obtained from (1a) and (2) by setting \( B_{2,k} \equiv 0 \) and \( D_{2,k} \equiv 0 \). We consider the following index of performance:

\[
J_B(r_k, u_k, x_0) \triangleq E \left\{ \left| \left| \left| C_N x_N + D_{3,N} r_N \right| \right|_2^2 + \left| \left| z_k \right| \right|_2^2 \right| \right\} - \gamma^2 x_0^T R^{-1} x_0, \quad R^{-1} \geq 0.
\]

(12)

We arrive at the following theorem:

**Theorem 2:** Consider the system of (11a,b) and \( J_B \) of (12). Given \( \gamma > 0 \), \( J_B \) of (12) satisfies

\[
J_B \leq \tilde{J}(r, \epsilon), \quad \forall \{u_k\} \in l_2[0, N - 1], \quad x_0 \in \mathbb{R}^n,
\]

where

\[
\tilde{J}(r, \epsilon) = \sum_{k=0}^{N-1} \theta^T_{k+1} \{B_{1,k} R^{-1} B_{1,k}^T \} \theta_{k+1} + \sum_{k=0}^{N-1} r_k^T (D_{3,k}^T D_{3,k}) r_k + \left| \left| D_{3,N} r_N \right| \right|_2^2 + 2 \sum_{k=0}^{N-1} \theta^T_{k+1} \tilde{Q}^{-1}_{k+1} (M^{-1}_{k+1})^{-1} B_{3,k} r_k + \theta_0^T \epsilon^{-1} \theta,
\]

if there exists \( \tilde{Q}_k \) that solves the following Riccati-type equation

\[
\tilde{Q}_k = A_k^T \tilde{Q}_k A_k + C_k^T \tilde{Q}_k C_k + F_k^T \tilde{Q}_k F_k
\]

(13a,b)

\[
\tilde{Q}_0 = \gamma^2 R^{-1} - \epsilon I.
\]

for some \( \epsilon > 0 \) where

\[
\tilde{\theta}_k = \tilde{A}_k^T \tilde{\theta}_{k+1} + \tilde{B}_k r_k, \quad \tilde{\theta}_N = C_k^T D_{3,N} r_N,
\]

where \( \tilde{A}_k = Q_{k+1}^{-1} M_{k+1} A_k \),

and \( \tilde{B}_k = \tilde{A}_k^T Q_{k+1} B_{3,k} + C_k^T D_{3,k} \).

(14)
Proof:} Unlike the state-feedback tracking control in [15], the solution of the BRL does not acquire saddle-point strategies (Since the input signal \( u_k \) is no longer an adversary). It can, however, be readily derived based on the first part of the sufficiency of Theorem 1 by setting \( B_{2,k} \equiv 0 \) and \( D_{2,k} \equiv 0 \). In the Appendix we bring the proof of the BRL as a derivation of the proof of the state-feedback tracking control solution (which is not included in [15]). The latter proof is also essential for the derivation of the output-feedback tracking control solution.

Remark 3: The choice of \( \epsilon > 0 \) in \( \tilde{Q}(0) \) of (13b) reflects on both, the cost value (i.e \( \tilde{J}(r,\epsilon) \)) of (13b) and the minimum achievable \( \gamma \). If one chooses \( 0 < \epsilon \ll 1 \) then, the cost of \( \tilde{J}(r,\epsilon) \) increases while the solution of (13a) is easier to achieve, which results in a smaller \( \gamma \). The choice of large \( \epsilon \), on the other hand, causes the reverse effect, which leads to a larger \( \gamma \).

3.4 The output-feedback tracking

We consider the system of (1a,b) and (2) where we note that the measurement matrix \( C_{2,k} \) is contaminated with a white noise component of \( D_{k}\xi_k \). Like in the state-feedback case [15] we seek a control law \( \{u_k\} \), based on the information of the reference signal \( \{r_k\} \) that minimizes the tracking error between the system output and the tracking trajectory, for the worst case of the initial condition \( x_0 \), the process disturbances \( \{w_k\} \) and \( \{n_k\} \). We, therefore, consider the performance index of (3) and we assume that (4a) has a solution \( Q_{k+1} > 0 \) over \([0, N]\) where (5a,b) are satisfied. We introduce the following difference linear matrix inequality (DLMI) [17]:

\[
\begin{bmatrix}
\hat{P}_k^{-1} & \tilde{A}_k^T \tilde{P}_{k+1} & \gamma^{-1} \tilde{B}_{1,k} & 0 & 0 & 0 \\
\tilde{A}_k & \tilde{P}_{k+1} & \gamma^{-1} \tilde{B}_{1,k} & 0 & 0 & 0 \\
0 & \gamma^{-1} \tilde{B}_{1,k}^T & I & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{P}_{k+1} & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{P}_{k+1} & 0 \\
\hat{C}_{1,k} & 0 & 0 & 0 & 0 & I
\end{bmatrix} \geq 0,
\]

(15)

where \( \hat{P}_k = \tilde{Q}_k^{-1} \) and where \( \tilde{A}_k, \tilde{B}_{1,k}, \tilde{C}_{1,k} \) and \( \tilde{D}_k \) are defined in (24) and \( \tilde{Q}_k, \tilde{P}_k \) are given in (25) and (26), respectively.

The solution of the output-feedback tracking problem is stated in the following theorem, for the a priori case, where \( u_k \) can use the information on \( \{y_i, 0 \leq i < k\} \):

**Theorem 3:** Consider the system of (1a,b), (2) and \( J_E \) of (3). Given \( \gamma > 0 \), the output-feedback tracking control problem, where \( \{r_k\} \) is known a priori for all \( k \leq N \) (the full preview case) possesses a solution if there exists \( \tilde{P}_k \in \mathbb{R}^{2n \times 2n} > 0, A_{f,k} \in \mathbb{R}^{n \times n}, B_{f,k} \in \mathbb{R}^{n \times z}, C_{f,k} \in \mathbb{R}^{m \times l} \forall i \in [0, N] \) that solves the DLM of (15) with a forward iteration, starting from the following initial condition:

\[
P_0 = \tilde{Q}_0^{-1} = \gamma^{-2} \begin{bmatrix} R & R \\ R & R + \epsilon I_n \end{bmatrix},
\]

(16)

where \( R \) is defined in (3).

**Proof:** Using the expression (E.3) achieved in the Appendix for \( J_E(r_k, u_k, w_k, x_0) \) in the state-feedback case, the index of performance is now given by:

\[
\begin{align*}
J_y(r_k, u_k, & w_k, n_k, x_0) = J_E(r_k, u_k, w_k, x_0) - \gamma^2 \| n_k \|_2^2 \\
&= -\gamma^2 \| x_0 - x_0^* \|_{R^{-1} - \gamma^{-1} Q_0}^2 + \sum_{k=0}^{N-1} \{ \| \tilde{u}_k^2 \|_{C_1,k} + \| \tilde{w}_k \|_{C_1,k} \} + \gamma^2 \| \tilde{w}_k \|_2^2 - \gamma^2 \| n_k \|_2^2 + \tilde{J}(r).
\end{align*}
\]

(17)

where

\[
\begin{align*}
\tilde{u}_k &= \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{1/2} B_{2,k}^T M_{k+1} (B_{3,k} r_k + Q_{k+1} \theta_{k+1}) \\
\tilde{w}_k &= \gamma^{-1} R_{k+1}^{1/2} w_k - \gamma^{-1} R_{k+1}^{1/2} B_{1,k}^T (Q_{k+1} (A_k x_k + B_{2,k} u_k + B_{3,k} r_k) + \theta_{k+1}) \\
\hat{C}_{1,k} &= \Phi_{k+1}^{1/2} B_{2,k}^T M_{k+1} A_k
\end{align*}
\]

(18)

and where we note that in the full preview case \([\theta_{k+1}] = \theta_{k+1} \).

We also note that

\[
\tilde{w}_k = \gamma^{-1} R_{k+1}^{1/2} (w_k - w_k^*),
\]

where \( w_k^* \) is defined in (6b) and that

\[
\tilde{u}_k = \Phi_{k+1}^{1/2} (u_k - u_k^*),
\]
where \( u_k^s \) is defined in (6c) and where we exclude from \( u_k^w \) the terms that are not accessed by the controller (i.e., the terms of \( x_k \)). Considering \( \bar{w}_k \) and \( \bar{u}_k \) we seek a controller of the form

\[
\bar{u}_k = -\hat{C}_{1,k} \hat{x}_k,
\]

where \( \hat{x}_k \) is defined (22). We, therefore, re-formulate the state equation of (1a) adding the additional terms to recover the original equation of (1a).

Considering the above, we obtain the following new state equation:

\[
x_{k+1} = (\hat{A}_k + F_k v_k)x_k + \tilde{B}_{1,k} \bar{w}_k + \tilde{B}_{2,k} \bar{u}_k + \tilde{B}_{3,k} r_k + \tilde{B}_{4,k} \theta_{k+1},
\]

(19)

where

\[
\begin{align*}
\hat{A}_k &= Q_{k+1}^{-1} M_{k+1} A_k, \quad \tilde{B}_{1,k} = \gamma B_{1,k} R_{k+1}^{-1/2} \\
\tilde{B}_{2,k} &= Q_{k+1}^{-1} M_{k+1} B_{2,k} \Phi_{k+1}^{-1/2} \\
\tilde{B}_{3,k} &= B_{3,k} + B_{1,k} R_{k+1}^{-1/2} B_{1,k}^T Q_{k+1} B_{3,k} \\
-\tilde{B}_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k} \\
\tilde{B}_{4,k} &= \tilde{B}_{1,k} R_{k+1}^{-1/2} B_{1,k}^T - \tilde{B}_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} Q_{k+1}^{-1}.
\end{align*}
\]

Replacing for \( \bar{w}_k \) and \( \bar{u}_k \) we obtain:

\[
x_{k+1} = (Q_{k+1}^{-1} M_{k+1} A_k + F_k v_k)x_k + \tilde{B}_{1,k} (\gamma R_{k+1}^{-1/2}(u_k - u_k^s)) + \tilde{B}_{2,k} (\Phi_{k+1}^{-1/2}(u_k - u_k^s)) + \tilde{B}_{3,k} r_k + \tilde{B}_{4,k} \theta_{k+1}.
\]

(20)

Replacing for \( w_k^s \) and \( u_k^s \) we obtain:

\[
x_{k+1} = (Q_{k+1}^{-1} M_{k+1} A_k + F_k v_k)x_k + \gamma B_{1,k} R_{k+1}^{-1/2} (u_k - u_k^s) + Q_{k+1}^{-1} M_{k+1} B_{2,k} \Phi_{k+1}^{-1/2} \Phi_{k+1}^{-1/2} (u_k - u_k^s) + B_{3,k} r_k + B_{4,k} \theta_{k+1}.
\]

We obtain

\[
x_{k+1} = ([I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}^{-1}] A_k - B_{1,k} R_{k+1}^{-1/2} B_{1,k}^T Q_{k+1} A_k] x_k + F_k x_k v_k + \gamma B_{1,k} R_{k+1}^{-1/2} \gamma^{-1} R_{k+1}^{-1/2} \gamma R_{k+1}^{-1/2} (u_k - u_k^s) + B_{3,k} r_k + B_{4,k} \theta_{k+1}.
\]

where we note that:

\[
\dot{A}_k = A_k - B_{1,k} R_{k+1}^{-1} B_{1,k}^T, \quad \dot{B}_{1,k} = (I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}^{-1})^{-1} A_k - B_{1,k} R_{k+1}^{-1} B_{1,k}^T Q_{k+1} A_k, \\
\ddot{B}_{2,k} = B_{2,k} - B_{1,k} R_{k+1}^{-1/2} B_{1,k}^T M_{k+1} B_{3,k} + B_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k} + B_{3,k} \\
B_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} Q_{k+1}^{-1} = Q_{k+1}^{-1} M_{k+1} B_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} Q_{k+1}^{-1}.
\]

We consider the following \textit{a priori}-type state observer:

\[
\hat{x}_{k+1} = A_{f,k} \hat{x}_k + B_{f,k} y_k + d_k, \quad \hat{x}_0 = 0,
\]

(22)

where

\[
d_k = \ddot{B}_{2,k} \ddot{u}_k + \ddot{B}_{3,k} \ddot{r}_k + \ddot{B}_{4,k} \ddot{\theta}_{k+1}.
\]

Denoting \( e_k = x_k - \hat{x}_k \) and using the latter we obtain:

\[
e_{k+1} = (\dot{A}_k - B_{f,k} C_{2,k} - A_{f,k}) x_k + A_{f,k} e_k \\
+ F_k x_k v_k - B_{f,k} D_k x_k \zeta_k + \ddot{B}_{1,k} \ddot{w}_k,
\]

where we define

\[
\ddot{u}_k = [\ddot{w}_k^T \ \ddot{r}_k^T]^T, \quad \ddot{B}_{1,k} = [\ddot{B}_{1,k} - B_{f,k} D_{21,k}].
\]

Defining \( \xi_k = [x_k^T \ e_k^T]^T, \quad \tilde{r}_k = [\theta_{k+1}^T \ \theta_{k+1}^T]^T, \) we obtain

\[
\xi_{k+1} = (\dddot{A}_k + \dddot{F}_k v_k + \dddot{B}_{1,k} \dddot{w}_k + \dddot{B}_{3,k} \dddot{r}_k + \dddot{B}_{4,k} \dddot{\theta}_{k+1}) \xi_k + \ddot{B}_{1,k} \ddot{w}_k + \ddot{B}_{3,k} \ddot{r}_k + \ddot{B}_{4,k} \ddot{\theta}_{k+1},
\]

(23)

where

\[
\dddot{A}_k = \begin{bmatrix} \dot{A}_k - B_{2,k} C_{f,k} & B_{2,k} C_{f,k} \\ \dot{A}_k - B_{f,k} C_{2,k} - A_{f,k} & A_{f,k} \end{bmatrix}, \\
\dddot{B}_{1,k} = \begin{bmatrix} \ddot{B}_{1,k} & 0 \\ 0 & -B_{f,k} D_{21,k} \end{bmatrix}, \\
\dddot{B}_{3,k} = \begin{bmatrix} \dddot{B}_{3,k} \dddot{B}_{4,k} \end{bmatrix}, \\
\dddot{F}_k = \begin{bmatrix} F_k \\ F_k \end{bmatrix}, \\
\dddot{D}_k = \begin{bmatrix} 0 & 0 \\ 0 & -B_{f,k} D_{21,k} \end{bmatrix}, \\
\dddot{C}_1,k = [\dddot{C}_{1,k} - C_{f,k} C_{f,k}].
\]

(24a-f)

Applying the results of Theorem 2 to the system of (23) we obtain the following Riccati-type equation:

\[
\tilde{Q}_{k+1} = A_{f,k}^T \tilde{Q}_{k+1} A_{f,k} + \tilde{F}_k^T \tilde{Q}_{k+1} \tilde{F}_k + \tilde{D}_k^T \tilde{Q}_{k+1} \tilde{D}_k.
\]

(25)
where $\hat{Q}_0$ is given in (16).
Denoting
\[
\hat{P}_k = \hat{Q}_k^{-1}
\]
and using Schur complement we obtain the DLMI of (15). The latter DLMI is initiated with the initial condition of (16) which corresponds to the case where a weighting $\gamma^2 \epsilon^{-1} I_n$ is applied to $\hat{x}_0$ in order to force nature to select $\hat{x}_0 = 0$ in the corresponding differential game [17], [16].

In the case where $\{r_k\}$ is measured on line, or with preview $h > 0$, we note that nature strategy which is not restricted to causality constraints, will be the same as in the case of full preview of $\{r_k\}$, meaning that $\hat{w}_k$ of (18) is unchanged. We obtain the following:

**Lemma 1**  
\textit{H$_\infty$ Output-feedback Tracking with full preview of $\{r_k\}$:} We obtain
\[
\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1}
\]
and using Schur complement we obtain the DLMI
\[
\hat{P}_k = \hat{Q}_k^{-1} + (B_{3,k} r_k + Q_{k+1}^{-1} [\theta_{k+1}^+]_+) = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1}
\]
where we note that in this case $[\theta_{k+1}]_+ = \theta_{k+1}$. Solving (8) we obtain:
\[
[\theta_{k+1}]_+ = \hat{\phi}_{k+1} \theta_N + \sum_{j=1}^{N-k-1} \Psi_{k+1,j} \bar{B}_{N-j} r_{N-j}
\]
where
\[
\hat{\phi}_{k+1} \triangleq \begin{cases} \bar{A}_{N-1}^{T} \bar{A}_{k+1}^{T} \bar{A}_{N-j-1}^{T} & j < N-k-1 \\ I & j = N-k-1 \end{cases} 
\]

(27a-b)

**Proof** Considering (8) and taking $k + 1 = N$ we obtain:
\[
\theta_{N-1} = \bar{A}_{N-1}^{T} \theta_N + \bar{B}_{N-1} r_{N-1},
\]
where $\theta_N$ is given in (8). Similarly we obtain for $N - 2$
\[
\theta_{N-2} = \bar{A}_{N-2}^{T} \theta_{N-1} + \bar{B}_{N-2} r_{N-2} = \bar{A}_{N-2}^{T} \bar{A}_{N-1}^{T} \theta_N + \bar{B}_{N-1} r_{N-1} + \bar{B}_{N-2} r_{N-2} = \bar{B}_{N-2} r_{N-2} + \bar{A}_{N-2}^{T} \theta_N + \bar{A}_{N-2}^{T} \bar{A}_{N-j-1}^{T} \theta_N + \bar{A}_{N-2}^{T} \bar{A}_{N-j-1}^{T} \bar{A}_{N-j-1}^{T} \theta_N.
\]

The above procedure is thus easily iterated to yield (27a,b). Taking, for example $N = 3$ one obtains from (8) the following equation for $\theta_1$:
\[
\theta_1 = \bar{A}_{1}^{T} \bar{A}_{2}^{T} \theta_3 + \bar{A}_{1}^{T} B_{2} r_2 + B_{1} r_1.
\]
The same result is recovered by taking $k = 0$ in (27a,b) where $j = 1, 2$.

**Lemma 2**  
\textit{H$_\infty$ Output-feedback Tracking with no preview of $\{r_k\}$:} In this case $[\theta_{k+1}]_+ = 0$ since at time $k$, $r_i$ is known only for $i \leq k$. We obtain:
\[
\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k} r_k.
\]

**Lemma 3**  
\textit{H$_\infty$ Output-feedback tracking with fixed-finite preview of $\{r_k\}$:} In this case we obtain:
\[
\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k} r_k + \Psi_{k+1} \theta_N + \bar{B}_{4,k} [\theta_{k+1}]_+. \]

where $[\theta_{k+1}]_+$ satisfies:
\[
[\theta_{k+1}]_+ = \begin{cases} \sum_{j=1}^{h} \Psi_{k+1,j} B_{k+1,j} + \bar{B}_{k+1} r_{k+1-j} & k + h \leq N-1 \\ \bar{\Psi}_{k+1} \theta_N + \sum_{j=1}^{h} \Psi_{k+1,j} B_{N-j} r_{N-j} & k + h = N \end{cases}
\]
where $\bar{\Psi}_{k+1,j}$ is obtained from (27b) by replacing $N$ by $k + h + 1$.

**4 Conclusions**

In this paper we solve the problem of tracking signals with preview in the presence of white noise parameter uncertainties in the system state-space model. Unlike the state-feedback case the solution is not obtained by applying a game theory approach, (where a saddle-point tracking strategy is derived) but rather as a min-max optimization problem. The performance index that corresponds to the problem includes averaging over the statistics of the white noise parameters in the system state-space model. Similarly to the stochastic bounded real lemma for
discrete-time state-multiplicative systems derived in [13] we derive a BRL for systems that include a tracking signal. This signal is not modeled as an additional disturbance (namely like \( w_k \)) to allow for the possibility of tracking with preview of the signal. The output-feedback tracking problem is thus solved by applying the latter BRL to a filtering problem which is formulated once the state-feedback solution is obtained. The latter approach follows the standard derivation of the dynamic output-feedback control problem [16]. The parameters of the a priori type state observer used in our proof are recovered by iterative solution of the DLMI of Theorem 3 which can be easily implementend (for detailed use of the above technique see [17]). We note that the white noise uncertainties in both the dynamic and the measurement matrices may be correlated however, we assumed zero correlation for the benefit of clarity. The latter assumption can be readily relaxed.

5 Appendix

In order to comply with a full proof of Theorem 3, we need a preliminary result which deals with the causality issue that is essential for the structure of the control sequence. Note that the following result is also essential for the proof of Theorem 1 in the state-feedback case:

We consider the following cost function:

\[
J_o = \sum_{k=0}^{N-1} ||u_k - \eta_k||^2
\]

where \( \eta_k \) is the output of a linear time-varying system with input \( w_k \in Y_k \) where \( Y_k = \{y_j, j \leq k, \ r_j, j \leq \min(k + h, N - 1)\} \), where we note that for \( h = 0, \{u_k\} \) is obtained using the zero preview on \( \{r_k\} \), finite \( h \) corresponds to finite preview and infinite \( h \) (or larger than \( N - 1 \)) corresponds to full preview on \( \{r_k\} \). We further note that in our case \( y_k = x_k \) (full state-feedback control). In order to obtain \( \{u_k\} \), using the above description, we consider the following:

\[
\eta_k = \sum_{j=0}^{k} \alpha_j y_j + \sum_{j=0}^{k} \beta_j r_j
\]

where \( \alpha_j, \beta_j \) are matrices of the appropriate dimensions. Consider the following realization of \( \{u_k\} \):

\[
u_k = \sum_{j=0}^{k} \alpha_j y_j + \sum_{j=0}^{k} \beta_j r_j,
\]

where \( k_h \triangleq \min(k + h, N - 1) \). We arrive at the following lemma:

**Lemma A1:** Consider the above \( J_o, \eta_k \) and \( u_k \). The matrix parameters \( \alpha_j, \beta_j \) that minimize \( J_o \) are given by

\[
\alpha_j = \alpha_j, \quad \beta_j = \beta_j.
\]

**Proof:** We define

\[
[r_i]_+ = \{ r_i \mid i \leq k_h \}, \quad [r_i]_- = \{ r_i \mid i > k_h \}
\]

Subtracting \( u_k \) from \( \eta_k \) we obtain:

\[
\eta_k - u_k = \sum_{j=0}^{k} (\alpha_j - \bar{\alpha}_j) y_j + \sum_{j=0}^{k} (\beta_j - \bar{\beta}_j) r_j + \sum_{j=k_h+1}^{N-1} \beta_j r_j
\]

By padding by zeros \( \alpha_j, \bar{\alpha}_j \) for \( k + 1 \leq j \leq k_h \), we obtain:

\[
\eta_k - u_k = \sum_{j=0}^{k} (\alpha_j - \bar{\alpha}_j) y_j + (\beta_j - \bar{\beta}_j) r_j + \sum_{j=k_h+1}^{N-1} \beta_j r_j
\]

We further define

\[
\rho_j^T \triangleq [y_j^T \ r_j^T]^T, \quad \epsilon_j \triangleq [\alpha_j - \bar{\alpha}_j, \beta_j - \bar{\beta}_j], \quad \nu_j \triangleq [0 \ \beta_j]
\]

and consider \( [\rho_j]_+ \) and \( [\rho_j]_- \) similarly to (E.2). We have:

\[
\eta_k - u_k = \sum_{j=0}^{k_h} \epsilon_j \rho_j + \sum_{j=k_h+1}^{N-1} \nu_j \rho_j = \sum_{j=0}^{N-1} \epsilon_j [\rho_j]_+ + \sum_{j=0}^{N-1} \nu_j [\rho_j]_-
\]

The result of the Lemma readily follows by calculating \( (\eta_k - u_k)^T (\eta_k - u_k) \) and using the orthogonality of \( [\rho_k]_+ \) and \( [\rho_k]_- \).

**Proof of Theorem 2:** We derive the proof of Theorem 2 from the first part of the proof of Theorem 1 which relates to the state-feedback tracking case.
We follow this line because we derive the output-feedback tracking solution from a certain point of the state-feedback tracking solution as is shown in the proof Section 3.4. We note that the derivation of Theorem 2 is achieved from the following proof by setting $B_{2,k} \equiv 0$ and $D_{2,k} \equiv 0$.

The proof of the state-feedback control follows the standard line of applying a Lyapunov-type quadratic function in order to comply with the index of performance. This is usually done by using two completing to squares operations. However, since the reference signal of $r_k$ is introduced in the dynamics of (1a), we apply a third completing to squares operation with the aid of the fictitious signal of $\theta_k+1$. This latter signal finally affects the controller design through it’s causal part $[\theta_{k+1}]^+$.

Defining $J_k = ||C_k x_k + D_{3,k} r_k||^2 + ||D_{2,k} u_k||^2 - \gamma^2 ||w_k||^2$, we consider (3), excluding the term of $n_k$ and obtain:

$$J_E(r, u, w, x_0) = -\gamma^2 x_0^T R^{-1} x_0 + \mathcal{E} \left\{ ||C_N x_N + D_{3,N} r_N||^2 \right\} + \sum_{k=0}^{N-1} \mathcal{E} \{ J_k \}.$$ 

Denoting $\phi_k = x_k^T Q_{k+1} x_{k+1} + x_k^T Q_k x_k$, and substituting (1a) in the latter, we find that

$$\phi_k = [x_k^T (A_k + F_k v_k)^T + u_k^T B_{2,k}^T + r_k^T B_{1,k}] Q_{k+1} x_{k+1} + (A_k + F_k v_k) x_k + B_{2,k} u_k + B_{3,k} r_k + 2[x_k^T (A_k + F_k v_k)^T + u_k^T B_{2,k}^T + r_k^T B_{1,k}]$$

$$+ [C_k x_k + D_{3,k} r_k]^2 - \gamma^2 w_k^T w_k + ||u||^2 \|_{R_k} - \gamma^2 z_k.$$ 

Taking the expectation with respect to $u_k$ and completing to squares for $u_k$ we get:

$$E \{ \phi_k \} = -\mathcal{E} (w_k - \tilde{w}_k)^T R_{k+1} (w_k - \tilde{w}_k) + (u_k - \tilde{u}_k)^T \Phi_k (u_k - \tilde{u}_k)$$

$$- E \{ J_k \} + x_k^T [A_k^T M_{k+1} A_k - (A_k^T M_{k+1} B_{2,k}) \Phi_k^{-1} (B_{2,k}^T M_{k+1} A_k) + B_{2,k}^T Q_k + B_{3,k}^T C_k - Q_k] x_k + r_k^T [D_{3,k}^T D_{3,k} + B_{3,k}^T M_{k+1} B_{2,k}] r_k + x_k^T D_{3,k}^T D_{3,k} + B_{3,k}^T M_{k+1} B_{2,k} M_{k+1} B_{3,k}$$

$$+ 2r_k^T [B_{3,k}^T M_{k+1} A_k + D_{3,k}^T C_k] x_k - E \{ J_k \}.$$ 

Completing next to squares for $u_k$ we obtain:

$$E \{ \phi_k \} = -\mathcal{E} (w_k - \tilde{w}_k)^T R_{k+1} (w_k - \tilde{w}_k) + (u_k - \tilde{u}_k)^T \Phi_k (u_k - \tilde{u}_k)$$

$$- E \{ J_k \} + x_k^T [A_k^T M_{k+1} A_k - (A_k^T M_{k+1} B_{2,k}) \Phi_k^{-1} (B_{2,k}^T M_{k+1} A_k) + B_{2,k}^T Q_k + B_{3,k}^T C_k - Q_k] x_k + r_k^T [D_{3,k}^T D_{3,k} + B_{3,k}^T M_{k+1} B_{2,k}] r_k + x_k^T D_{3,k}^T D_{3,k} + B_{3,k}^T M_{k+1} B_{2,k} M_{k+1} B_{3,k}$$

$$+ 2r_k^T [B_{3,k}^T M_{k+1} A_k + D_{3,k}^T C_k] x_k - E \{ J_k \}.$$ 

Similarly to [3], in order to get rid of the mixed terms of $r_k$ and $x_k$ we add $2(\theta_k^T x_k + \theta_k^T x_k)$ to $\phi_k$ before calculating $E \{ \phi_k \}$. Denoting

$$\hat{u}_k \triangleq u_k - \tilde{u}_k, \quad \hat{u}_k \triangleq u_k - \tilde{u}_k \quad \text{and} \quad \phi_k^* = \phi_k + 2(\theta_k^T x_k + \theta_k^T x_k)$$

we obtain, after completing the squares, the following expression

$$E \{ \phi_k^* \} = -\mathcal{E} (w_k - \tilde{w}_k)^T R_{k+1} (w_k - \tilde{w}_k) + (u_k - \tilde{u}_k)^T \Phi_k (u_k - \tilde{u}_k)$$

$$+ ||\hat{u}_k + \Phi_k^{-1} B_{2,k}^T M_{k+1} Q_{k+1} x_k ||^2_k - E \{ J_k \}$$

$$+ ||B_{2,k}^T \theta_{k+1}||^2_k - ||B_{2,k}^T M_{k+1} Q_{k+1} x_k ||^2_k + ||B_{3,k}^T \theta_{k+1}||^2_k$$

$$+ \gamma^2 (x_k^T A_k^T M_{k+1} A_k x_k + x_k^T B_{2,k}^T M_{k+1} B_{2,k} x_k) + x_k^T B_{3,k}^T M_{k+1} B_{3,k} x_k + x_k^T C_k x_k$$

$$+ \gamma^2 (x_k^T A_k^T M_{k+1} A_k x_k + x_k^T B_{2,k}^T M_{k+1} B_{2,k} x_k) + x_k^T B_{3,k}^T M_{k+1} B_{3,k} x_k + x_k^T C_k x_k$$

The performance index becomes:

$$J_E(r_k, u_k, w, x_0) = \sum_{k=0}^{N-1} \mathcal{E} (w_k - \tilde{w}_k)^T R_{k+1} (w_k - \tilde{w}_k) + \mathcal{J}(r)$$

$$+ \sum_{k=0}^{N-1} \mathcal{E} (\tilde{u}_k + \Phi_k^{-1} B_{2,k}^T M_{k+1} Q_{k+1} x_k ||^2_k - \gamma^2) ||x_0 - (\gamma^2 R_k - Q_k) - \theta_{k+1} ||^2_{R_k - \gamma^2 Q_k}.$$ 

(3.4) 

where

$$\mathcal{J}(r) \triangleq \sum_{k=0}^{N-1} \mathcal{E} (B_{2,k}^T M_{k+1} B_{2,k} - Q_{k+1}^1 x_k ||^2_k)$$

$$+ \gamma^2 (x_k^T A_k^T M_{k+1} A_k x_k + x_k^T B_{2,k}^T M_{k+1} B_{2,k} x_k) + x_k^T B_{3,k}^T M_{k+1} B_{3,k} x_k$$

$$+ \gamma^2 (x_k^T A_k^T M_{k+1} A_k x_k + x_k^T B_{2,k}^T M_{k+1} B_{2,k} x_k) + x_k^T B_{3,k}^T M_{k+1} B_{3,k} x_k$$

$$+ \gamma^2 (x_k^T A_k^T M_{k+1} A_k x_k + x_k^T B_{2,k}^T M_{k+1} B_{2,k} x_k) + x_k^T B_{3,k}^T M_{k+1} B_{3,k} x_k$$

(3.4) 

Remark 4: To recover the proof of Theorem 2 from the above derivation one should set $B_{2,k} \equiv 0$
and $D_{2,k} \equiv 0$. In this case $\tilde{J}(r)$ becomes $\tilde{J}(r, \epsilon)$ where $\epsilon$ is considered in Remark 3 and $J_B$ of Theorem 2 is recovered from (E.3) and becomes:

$$J_B = \sum_{k=0}^{N-1} -||\hat{w}_k - R_k^{-1} B_{k+1}^T \theta_{k+1}||_R^2 - \gamma^2 ||x_0 - (\gamma^2 R^{-1} - Q_0)^{-1} \theta_0||_{R_{k+1}^{-1} - \gamma^{-2} Q_0}.$$  

We note that $\tilde{Q}_k$ in Theorem 2 is denoted so to distinguish the Riccati solution of the BRL from that of the state-feedback solution.

References


