Stochastic Model on the Post-Fabrication Error for a Bragg Reflectors Based Photonic Allpass Filter

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Abstract—This paper presents an analysis of the post fabrication performance for a Bragg-mirrors-based photonic allpass filter. The development subsequently leads to a first definition of the stochastic measures in a realistic system implementation. A Digital Signal Processing (DSP) approach to examining photonic integrated circuitry is explored in depth. The physical origins of error that manifests in an unbalanced phase error is mathematically described in detail. We describe the details of error analysis and demonstrate a stochastic measure for the expected behavior of first-order allpass filter as well as high complexity systems. Our findings show that the frequency performance under error can be characterized by a convolution of the ideal response with a probability function based kernel.

Index Terms—Bragg mirrors, allpass filter, stochastic model, optical performance, fabrication error.

I. INTRODUCTION

Photonic systems have proven to be an integral element in the core areas of communication infrastructures such as Wavelength division multiplexing (WDM), Discrete time optical processing (DTOP), Optical time division multiplexing (OTDM), and Optical code division multiplexing (OCMD). An optical setup promises the possibility to sidestep the inflexibility of conventional electronic circuitry used in current telecommunications systems. Photonic signal processors possess several distinct advantages over their electronic counterparts, most notably superior bandwidth and sampling rates. Additionally, it is relevant to observe that optical devices benefit from very low transmission loss, and are impervious to the electromagnetic sources of interference that plague electronic devices [1]. With such solid foundations, the development of photonic integrated circuits have experienced a fruitful acceleration in recent years [2], [3].

While their presence in telecommunications is definitely undeniable and critical, research and development for photonic integrated circuits had been heavily self-contained. The vastly available communication theory and digital signal processing (DSP) techniques were decoupled from the photonic physics. A signal processing approach to photonic integrated circuits have been initiated in recent research with highly promising results [4]. A new breath of research that fuses the design techniques from DSP and the physics of photonic integrated circuitry is starting to take form. The DSP understanding of photonic systems is progressing forward from the basic connection between the delay element $z^{-1}$ and optical path length of a waveguide. The unyielding limitations from state-of-the-art photonic fabrications are starting to be acknowledged and addressed by the DSP design community.

Recent research has demonstrated innovative approaches to designing digital filters that are realizable using photonic components [5], [6]. However, the techniques are focused on providing architectures that can theoretically be realized with today’s technology. The necessary and critical question on the performance of the systems post fabrication has been left unaddressed. While DSP designs open up a new array of high complexity photonic systems, a realistic study on the characteristics of the structures under non-ideal processing has yet to be conducted.

A photonic system consists of basic building blocks that can be realized using nanoscale dielectric waveguide and resonant elements including rings [7], microdisks [8], and Bragg mirror pairs [9]. The recently developed Bragg mirror based waveguides have a clear advantage in system integration due to their compact size and resilience to errors in the optical sense [9]. In this paper we present the development of a mathematical model that clearly characterizes the performance of a Bragg mirrors based photonic allpass filter under error.

The allpass filter is an excellent building element for system realization, and has been shown to be an important building block in applications for optical communications such as WDM [10]. High-order filters of arbitrary functionality can be decomposed into a combination of allpass sub structures [11]. Allpass filters are highly scalable, in that complex allpass filters can be readily realized in latticed or cascaded assemblies of lower ordered sections. Allpass filters are desirable as building blocks of a photonic filter implementation [12] because they do not intrinsically attenuate the optical signal and do not require subsequent amplification of it, thus keeping power consumption and power dissipation to a minimum.

Although significant efforts have been put into allpass analysis to demonstrate its utility and versatility [13] and robustness to coefficient perturbations [14] in DSP filter design, its error resilience in photonic filtering has never been studied. In this paper we consider a unique form of phase error that arises only in the optical setup and has remained as a challenge in research on error analysis. The fabrication process changes the ideal allpass transfer function $A(z)$ into $A(e^{j\delta}z)$ where $\delta$ is a random variable based on the imperfection. It can be readily observed that the error term $e^{-j\delta}$ is interconnected with both the feed-forward and feed-back paths of an infinite impulse response (IIR) allpass filter $A(z)$. The coupling with two dependent signal paths causes statistical performance measures on the filter difficult to obtain. We present an innovative approach to extracting the error term, and subsequently present a first definition of the stochastic model on the magnitude response of the error corrupted system. Specifically, we demonstrate the procedures to evaluating $E[|A(e^{j\delta}z)|^2]$ where $E[\cdot]$ denotes the expectation on the error term $\delta$.

The rest of the paper is structured as follow: Section II provides the background on an ideal allpass filter’s behavior and operation. This section also briefly demonstrates how a high-order allpass filter can be modularized into first-order sub structures. Section III details the characteristics of a Bragg mirrors based allpass filter. The physical origins of the unique form of phase error is explained in Section IV. Section V describes in detail our approach to deriving the stochastic model on the performance under error. Simulation results are shown in Section VI, and Section VII concludes the paper.

II. IDEAL ALLPASS TRANSFER FUNCTION

In the context of digital filter design, a system is completely described by its transfer function. A transfer function $H(z)$ describes
the system behavior under different frequencies, and is characterized by the complex variable \( z = re^{j\omega} \), where \( r \) is the magnitude of the complex variable and \( \omega \) denotes the frequency. A particularly robust structure in filter design is the allpass filter with a transfer function of the form [15]

\[
A(z) = -\frac{a_N^* + a_{N-1}^*z^{-1} + \cdots + z^{-N}}{1 + a_1z^{-1} + \cdots + a_Nz^{-N}}
\]  

(1)

where \( N \) is the filter order, \( a_i \) are the filter coefficients, and the operator \(^*\) represents complex conjugation. High-order allpass filters can be decomposed into a product form:

\[
A(z) = \prod_{i=1}^{N} \left( k_i^* + z^{-1} \right)
\]

where \( k_i \) is the real filter coefficient for the \( i^{th} \) first-order section.

Notice that the phase response of the filter therefore depends on the placement of the poles and zeros governed by the \( k_i \)'s, as well as the filter order \( N \). The magnitude response, however, will be unity regardless of either of the design criteria. For an ideal first-order (\( N = 1 \)) allpass filter, the magnitude response can be found by

\[
|A(z)|^2 = A(z)A^*(1/z^*) = \frac{k^* + z^{-1}}{1 + k^*z^{-1}} = 1
\]

(2)

Therefore, the allpass filter processes all frequency components without any attenuation yet provides a change in the phase response. Such attributes of the allpass structure has been harnessed for designing filters of arbitrary frequency response. An arbitrary frequency selective IIR filter \( H(z) \) can be decomposed as the sum of two allpass filters [11]:

\[
H(z) = \frac{1}{2} [A_1(z) + A_2(z)]
\]

(4)

This allows us to further simplify the form by empirically choosing one of the allpass sections as a delay:

\[
H(z) = \frac{1}{2} [z^{-(N-1)} + A(z)]
\]

(5)

where \( N \) is the order of the allpass filter \( A(z) \). This form simplifies the design procedure by reducing the number of design parameters, and reduces the complexity of the analysis. In terms of photonic fabrication, (5) presents an ideal scenario for implementing high complexity systems. The form simply requires the design and fabrication of an allpass filter that can be readily realized using a variety of photonic components.

III. PHOTONIC UNIT CELL MODEL

Since a high-order allpass filter can be decomposed into a cascade of single-coefficient sections [16], the fabrication challenge reduces to the realization of a first-order unit cell. The Bragg mirror based topology, illustrated in Figure 1, has received recent attention due to several clear advantages over more traditional photonic allpass structures.

A Bragg mirror is implemented as a periodic perturbation, for example, a small periodic modulation of waveguide width (Figure 2). The periodic structure reflects optical signals whose in-guide wavelength matches the modulation period. In the context of a DTDCOP [1] application, the reflection bandwidth is wide enough to include the optical carrier along with any sidebands, but narrow enough to help suppress some of the optical noise at different wavelengths. This wavelength specificity represents one advantage of the Bragg topology over rings and microdisks. A second advantage, clearly seen in Figure 1, is a more compact footprint, allowing devices to be placed close to one another to minimize relative fabrication error between them.

\[
\text{Fig. 1: Photonic Allpass Unit Cell. L indicates the optical path length, s represents phase delays and } \rho \text{ is the amplitude reflection coefficient of the first Bragg mirror.}
\]

The unit cell includes a pair of Bragg mirrors as its main components, one with an amplitude reflection coefficient \( \rho \) and one with perfect amplitude reflection. The unit cell transfer function is

\[
A(z) = e^{j\rho -js} = \frac{-\alpha z^{-1}}{1 - \alpha e^{j\phi}z^{-1}}
\]

(6)

where \( \alpha \) is the waveguide amplitude propagation loss, and \( s \) represents the phase delay that can be controlled by small changes in the optical path length [17]. The optical path \( L \) between the mirrors is related to the filter free spectral range (FSR) by \( L = \frac{2\pi}{c} \cdot n_g \), where \( n_g \) is the group index and \( c \) is the speed of light in vacuum. The waveguide loss \( \alpha < 1 \) reflects the imperfection of photonic fabrication technology. Although this error imposes constraint on the placement of the poles and zeros of the allpass filter, the effect is deterministic and has been addressed [4].

The phase delay \( s \), however, is a product of engineering and it is stochastic by nature. For general applications, the phase delay is engineered to be \( s = n_\delta \), translating (6) into a real-coefficient filter. Small optical path errors lead to large errors in the phase factor in the form of \( e^{j(s+\delta)} \) according to

\[
\delta = \frac{2\pi\Delta L\cdot n_{eff}}{\lambda}
\]

(7)

where \( \Delta L \) is the optical path length error, \( n_{eff} \) is the waveguide effective index and \( \lambda \) is the free space wavelength. Neglecting the effect of \( \alpha \), a fabrication imperfection on the device’s length will therefore yields a transfer function

\[
A'(z) = \frac{k_i^* - e^{j\delta}z^{-1}}{1 - k_i e^{j\delta}z^{-1}} = A(z)\text{e}^{-js}
\]

(8)

where \( k_i \) is a composite parameter representing the fabricated device values. The error \( \delta \) therefore exists in both the feed-forward and feedback signal paths within the Bragg mirror structure shown in Figure 1.

IV. PHYSICAL ORIGINS OF ERROR

The process of creating a sample via electron beam lithography may be divided into the following general steps [18]. To begin, the wafer on which the pattern will be etched must be spin coated with a layer of resist. The wafer is then secured to a mount and inserted into the lithography apparatus. Prior to writing, the electron beam must be aligned, focused, and corrected for aberrations such as astigmatism. The desired pattern is then transferred to the resist by scanning the beam across the surface of the wafer. For small areas, this is accomplished by deflecting the beam, while larger areas require the wafer to be physically repositioned. The wafer is then removed from the apparatus and the resist is chemically developed. For negative resists, the development process removes the portion of resist that was not exposed to the electron beam (vice versa for
positive resists). The wafer is then subjected to an etching process which selectively removes the portions of it that are not protected by resist.

Error can be introduced to the sample during each of the above steps. The alignment and focusing of the electron beam are performed at a limited number of points on the wafer. Any mounting irregularity or imperfections in the wafer and resist that cause the sample to deviate from being perfectly planar will introduce distortion into the written pattern. The beam itself is subject to drift that interferes with beam alignment and resist exposure. Uncertainty associated with deflecting the beam during writing can lead to geometrical distortion of the pattern. Physically repositioning the sample during stitching can lead to error where adjacent write fields are stitched together. Finally, roughness introduced during the etching process introduces additional error.

Although there are many distinct sources of error during the fabrication of a photonic filter, they are all ultimately manifested as error in the physical dimensions of the waveguide. Error in dimension parallel to the direction of signal propagation translates into error in the waveguide effective index. Error in dimension perpendicular to the direction of signal propagation translates into error in the physical dimensions of the waveguide. Error in our fabrication process, we fabricated sets of waveguide Bragg mirror and waveguide resonators with both 100 µm and 200 µm cavity lengths as shown in Figures 3a and 3b. The observed standard deviation in relative optical path length for the samples with 100 µm cavity length is 0.030 µm. The observed standard deviation in relative optical path length for the samples with 200 µm cavity length is 0.031 µm, which can be readily converted into phase measures through (7).

V. STOCHASTIC ERROR ANALYSIS

To characterize the effect of phase error on the transfer function in (5), we first consider a first-order allpass filter

\[ A(z) = \frac{k^z + z^{-1}}{1 + k z^{-1}} = \sum_{n=0}^{\infty} a(n) z^{-n} \]  

where \( a(n) \triangleq Z^{-1}\{A(z)\} \) is the time domain response of the allpass filter obtained through inverse z-transform. The expected value of \( A(z) \) under error is

\[ E\left[A(e^{-j\delta}z)\right] = E\left[\sum_{n=0}^{\infty} a(n) e^{jn\delta} z^{-n}\right] \]

Since the expectation is taken only on the random variable \( \delta \), all other parameters in (10) become deterministic. Therefore

\[ E\left[A(e^{-j\delta}z)\right] = A(z) \Phi(z) \]

where \( \Phi(z) \triangleq \mathbb{E}\{e^{jn\delta}\} \) can be recognized as the characteristic function of \( \delta \) with parameter \( n \). The right hand side of (11) is z-transform of the product of two sequences, \( a(n) \) and \( \phi(n) \), which can be re-expressed as the convolution of the respective z-transforms,

\[ E[A(e^{-j\delta}z)] = A(z) * \Phi(z) \]

Finally, the magnitude response error of the order-N frequency selective filter in (5) is

\[ |H(z)|^2 = E\left[\sum_{i=1}^{N} A_i(e^{-j\delta_i}z)\right]^2 \]

where \( \phi(\delta) \triangleq \mathbb{E}\{|e^{-j\delta}|\} \) is the bilatera z-transform of the characteristic function of \( \delta \). We now apply this result to find the expected magnitude response error of the order-N frequency selective filter in (5). Let \( \delta = [\delta_1, \delta_2, \ldots, \delta_N] \) be the vector of phase errors of the \( N \) allpass stages, and let \( \gamma \) be the phase error of the single delay element \( z^{-(N-1)} \). The expectation of \( |H(z)|^2 \) under error is

\[ E\left[|H(z, \delta)|^2\right] = E[1] + e^{-j\gamma z^{-(N-1)}} \prod_{i=1}^{N} A_i(e^{-j\delta_i}z) \]

It is immediately verified by inspection that \( |A_i(e^{-j\delta_i}z)|^2 \equiv 1 \) independent of \( \delta_i \) (i.e. the error does not corrupt the allpass magnitude response). In the second and third terms of (13), since by assumption \( \gamma \) and \( \delta_i \) are all statistically independent. The expectation then can
be taken separately on each factor in the product and yields the form
\[
E [ |H(z, d)|^2 ] = \frac{1}{4} [ 1 + E[e^{-j\gamma}] z^{-\delta_1} ] \prod_{i=1}^{N} E \left[ A_i(e^{-j\delta_i} z) \right] \\
+ \frac{1}{4} [ 1 - E[e^{j\gamma}] z^{-\delta_1} ] \prod_{i=1}^{N} E \left[ A_i^*(e^{-j\delta_i} z) \right] \\
+ 1 \tag{14}
\]

Notice that \(E[e^{-j\gamma}] = \psi(-1)\) is the characteristic function of \(\gamma\) with a parameter \(-1\), and similarly \(E[e^{j\gamma}] = \psi(1)\). Therefore,
\[
E [ |H(z, d)|^2 ] = \frac{1}{4} [ 2 + \psi(-1) z^{-\delta_1} ] \prod_{i=1}^{N} \Phi(z) * A_i(z) + \psi(1) z^{-\delta_1} \prod_{i=1}^{N} \Phi(z) * A_i^*(z) \tag{15}
\]

where the error terms \(\delta_i\)'s are assumed to be independent and identically distributed (i.i.d.) to yield \(\Phi(z) = \Phi(z)\).

VI. RESULTS

We first demonstrate that the allpass magnitude response is immune to the effects of the phase error \(\delta\), since \(E[|A(e^{-j\delta} z)|^2] = 1\) regardless of the pole and zero displacements. Figure 4 shows the magnitude response and pole zero map for a first-order allpass filter corrupted by random error of Gaussian distribution \(\delta \sim \mathcal{N}(0, \frac{\pi}{4})\), and the corresponding error-free responses.

While the magnitude response of the allpass filter is not affected, the effect becomes much more prominent in a frequency selective architecture. The displacements of the poles and zeros of the allpass filter directly translate to an offset in the transition band of the frequency selective filter. An examination in the \(z\)-plane on the average of these displacements would result in the probability density function (p.d.f.) of the random variables \(\delta_i\)'s. From here it can be deduced that the transition bands of a frequency selective filter would reflect the p.d.f. of the random error, and is readily demonstrated by our model. The effect can be immediately observed in a \(N = 1\) frequency selective filter
\[
H(z) = \frac{1}{2} [ 1 + A(z) ] \tag{16}
\]
where \(A(z)\) is a first-order allpass filter. Notice that there is no fabrication error coupled with the delay path, since the source signal is available. The expected performance under error is
\[
E[|H(z, d)|^2] = \frac{1}{4} [ 2 + \Phi(z) * A_i(z) + \Phi(z) * A_i^*(z) ] \tag{17}
\]
The smoothing kernel \(\Phi(z)\) has unit area, or \(\int_{-\infty}^{\infty} \Phi(z) dz = 1\) because it is the transform of a distribution for a random variable. We can then readily express the constant 1 as \(1 * \Phi(z)\). Therefore the expected performance of the filter can be observed as
\[
E[|H(z, d)|^2] = \Phi(z) * \left[ \frac{1}{2} (1 + A(z)) \right]^2 \tag{18}
\]

It now becomes obvious that the kernel \(\Phi(z)\) has a smoothing effect on the frequency selective filters. Figure 5 shows the effect on the location of the transition band on a first-order system. Figures 6 and 7 subsequently demonstrate the performance of our error model for the extreme cases of \(\delta \sim \mathcal{N}(0, \frac{\pi}{4})\) and \(\delta \sim \mathcal{N}(0, \frac{2\pi}{3})\). In each result, the ensemble average of 1000 independent trials is calculated and plotted along with the theoretical model. We extend the intuition by demonstrating the results for a \(3^{rd}\) order system
\[
H(z) = \frac{1}{2} [ z^{-2} + A(z) ] \tag{19}
\]
where \(A(z)\) is a \(3^{rd}\) order allpass filter with coefficients \(k_1, k_2, k_3\) arbitrarily chosen as \(-0.7291, 0.546\), and 0.988. Note that we are ignoring the error \(\gamma\) on the delay term \(z^{-n}\) to emphasize the effect from fabricating the Bragg mirror allpass filter. The ideal magnitude response and the effect of the error are shown in Figure 8. Figures 9 and 10 subsequently demonstrate the performance of our error model in comparison to the ensemble average of 1000 independent trails with \(d \sim \mathcal{N}(0, \frac{\pi}{4})\) and \(d \sim \mathcal{N}(0, \frac{2\pi}{3})\).

VII. CONCLUSION

We present a statistical model that directly reflects the behaviors of a Bragg mirror based allpass filter under fabrication error. Our analysis carefully considers the realistic photonic manufacturing limitations, and demonstrates an approach to decouple the error analysis from the ideal response of the waveguide. We have shown that our result directly applies to allpass filters of any order, and can be readily extended to frequency selective filters based on allpass substructures. The model on the expected performance of the Bragg mirrors based allpass building block can readily provide mathematical insights to the analysis and optimization of high complexity photonic filters.

\[\text{Fig. 4: Ideal response of a first-order allpass filter } A(z) \text{ and the effect of fabrication error with } \delta \sim \mathcal{N}(0, \frac{\pi}{4}) \text{. The pole zero pair displace by the same degree under each trial.}\]

\[\text{Fig. 5: Ideal performance of } H(z) = \frac{1}{2}(1 + A(z)) \text{ and the effect of fabrication error. } A(z) \text{ is a first-order allpass filter with filter coefficient arbitrarily chosen at } -0.99.\]

\[\text{Fig. 6: Magnitude response and pole zero map for a first-order allpass filter. Notice that there is no fabrication error coupled with the delay path, since the source signal is available. The expected performance under error is}\]
\[\text{Fig. 7: Magnitude response and pole zero map for a first-order allpass filter with error. Notice that there is no fabrication error coupled with the delay path, since the source signal is available. The expected performance under error is}\]
Fig. 6: Calculated model and the ensemble average of 1000 independent trials with $\delta \sim N(0, \frac{\pi}{3})$.

Fig. 7: Calculated model and the ensemble average of 1000 independent trials with $\delta \sim N(0, \frac{\pi}{32})$.

Fig. 8: Ideal performance of $H(z) = \frac{1}{2}(z^{-2} + A(z))$ and the effect of fabrication error. $A(z)$ is a third-order allpass filter.

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