Abstract—It has been observed from measurements that the spatially averaged channel impulse response for in-room scenarios exhibits an avalanche effect: The earliest signal components, which appear well separated in time, are followed by an avalanche of components arriving with increasing rate of occurrence, gradually merging into a diffuse tail with exponentially decaying power. A new approach is followed to design a model of the channel response which includes recursive scattering and thereby inherently accounts for the exponential power decay and the avalanche effect. The environment is modeled in terms of a propagation graph in which vertices represent transmitters, receivers, and scatterers, while edges represent propagation conditions between vertices. A closed form expression of the channel transfer function valid for any number of interactions is derived. We discuss an example where interactions are assumed to cause no time dispersion and thus delay occurs only due to propagation in between scatterers. For this example, a stochastic model of the propagation graph is stated based on which realizations of the channel transfer function and impulse response are generated for numerical evaluation. The results reveal that the graph’s recursive structure yields both an exponential power decay and an avalanche effect in the generated impulse responses.

I. INTRODUCTION

Engineering of modern indoor radio systems for communications and geolocation relies heavily on models for the time dispersion of the wideband and ultrawideband radio channels [3–5]. From measurement data, as exemplified in Fig. 1, it appears that the spatially averaged channel impulse response (or delay-power spectrum) for in-room scenarios exhibits an avalanche effect: The earliest signal components, which appear well separated in time, are followed by an avalanche of components arriving with increasing rate of occurrence, gradually merging into a diffuse tail with exponentially decaying power. A similar avalanche effect is well-known in room acoustics [7] where it is attributed to recursive scattering of sound waves. Indoor radio propagation environments are particularly exposed to recursive scattering as electromagnetic waves may be reflected back and forth in between walls, floor, and ceiling. Thus, in the present contribution we hypothesize that recursive scattering is the cause of both the observed avalanche effect and the exponential power decay.

Recursive scattering phenomena have been previously considered in a number of radio channel models. The works [8]–[11] use the analogy to acoustical reverberation theory to predict the exponential decay. As a matter of fact, there exists a well-developed theory of electromagnetic fields in cavities [12], [13], but in this field too the avalanche effect has received little attention. Recursive scattering between particles in a homogeneous medium is a well-known phenomenon studied by Foldy [14] and Lax [15], [16]. The solution, given by the so-called Foldy-Lax equation [17], has been applied in the context of time-reversal imaging by Shi and Nehorai [18]. The solution is, however, intractable for heterogeneous indoor environments. In [19] the radio propagation mechanism is modeled as a “stream of photons” performing continuous random walks in a homogeneously cluttered environment. The model predicts a delay power spectrum consisting of a single directly propagating “coherent component” followed by an incoherent tail. Time-dependent radiosity [20]–[23] accounting for delay dispersion has been recently applied to design a model for the received instantaneous power [24]. Thereby, the exponential power decay and the avalanche effect can be predicted.

Simulation studies of communication and localization systems commonly rely on synthetic realizations of the channel impulse response. A multitude of impulse response models exist [4], [5], [25], but only few account for the avalanche effect. Ray tracing methods can also be used, but to achieve tractable computational complexity, the maximum number of interactions considered is limited [25]. Thus the avalanche effect is discounted. The models [27]–[29] treat early components via a geometric model whereas the diffuse tail is generated via another stochastic process; the connection between the propagation environment and the diffuse tail is, however, not considered.

In this contribution, we model the channel response fol-
allowing a new approach which includes recursive scattering. The obtained model thus accounts inherently for the exponential power decay and the avalanche effect. We represent the environment in terms of a propagation graph, where vertices represent transmitters, receivers, and scatterers, while edges represent propagation conditions between vertices. This modelling approach allows for a closed form expression of the channel transfer function valid for any number of interactions. We assess the validity of the hypothesis by considering an example of a propagation graph suitable for Monte Carlo simulations. Scatterer interactions are assumed to cause no time dispersion and thus delay dispersion occurs only due to propagation in between vertices. We state a stochastic model of the propagation graph allowing for numerical evaluation of realizations of the channel impulse response and transfer function. The results reveal that the graph’s recursive structure yields both an exponential power decay and an avalanche effect in the generated impulse responses.

II. REPRESENTING RADIO CHANNELS AS GRAPHS

In a typical propagation scenario, the electromagnetic signal emitted by a transmitter propagates through the environment interacting with a number of objects called scatterers. The receiver, which is usually placed away from the transmitter, senses the electromagnetic signal. If a line of sight exists between the transmitter and the receiver, direct propagation occurs. Also, the signal may arrive at the receiver indirectly via one or more scatterers. In the following we represent recursive and non-recursive propagation mechanisms using graphs. First we state the necessary definitions of directed graphs and associated terms.

A. Directed Graphs

Following [39], we define a directed graph \( G = (V, E) \) of disjoint sets of vertices and edges. The two mappings \( \text{init} : E \to V \) and \( \text{term} : E \to V \) assign to edge \( e \in E \) an initial vertex \( \text{init}(e) \) and a terminal vertex \( \text{term}(e) \) respectively. We also say that the edge \( e \) is ingoing to vertex \( \text{term}(e) \) and outgoing from vertex \( \text{init}(e) \). Edges \( e \) and \( e' \) are parallel if \( \text{init}(e) = \text{init}(e') \) and \( \text{term}(e) = \text{term}(e') \). When a graph has no parallel edges, an edge \( e \) can be identified by the vertex pair \( (\text{init}(e), \text{term}(e)) \) \( \in V^2 \). With a slight abuse of notation we write in this case \( e = (e, e') \). With this identification, \( E \subseteq V^2 \). This notation also allows for the graph to have anti-parallel edges, i.e. if the edge \( e = (v, v') \) is in the graph, the edge \( e' = (v', v) \) can also exist. A walk (of length \( K \)) in \( G \) is a sequence \( (v_1, v_2, \ldots, v_{K+1}) \) of vertices in \( V \) such that \( (v_k, v_{k+1}) \in E, k = 1, \ldots, K \). A walk which fulfills \( v_1 = v_K \) is called a cycle. A loop is an edge \( e = (v, v) \), i.e., \( \text{init}(e) = \text{term}(e) \). Thus, by definition, a loop is a walk of length 1.

B. Propagation Graphs

We define a propagation graph as a directed graph \( G = (V, E) \) where vertices in \( V \) represent transmitters, receivers and scatterers. Edges in \( E \) represent the propagation conditions between the vertices. Thus, the vertex set of a propagation graph is a union of three disjoint sets: \( V = V_t \cup V_s \cup V_r \), where \( V_t = \{Tx_1, \ldots, Tx_M\} \) is the set of transmit vertices, \( V_r = \{Rx_1, \ldots, Rx_M\} \) is the set of receive vertices, and \( V_s = \{S_1, \ldots, S_N\} \) is the set of scatterer vertices. The transmit vertices are considered as sources with outgoing edges only. Likewise, the receivers are considered as sinks with only incoming edges. Thus for a propagation graph, the edge set can be partitioned into four disjoint sets as \( E = E_d \cup E_i \cup E_s \cup E_r \), where \( E_d = E \cap (V_t \times V_r) \) is the set of direct edges, \( E_s = E \cap (V_t \times V_s) \) is the set of transmitter-scatterer edges, \( E_i = E \cap (V_s \times V_r) \) is the set of scatterer-receiver edges, and \( E_r = E \cap (V_r \times V_s) \) is the set of inter-scatterer edges. Fig. 2 shows an example propagation graph.

The signals propagate in the graph in the following way. Each transmitter emits a signal that propagates via its outgoing edges. The signal observed by a receiver vertex is the sum of the signals arriving via the ingoing edges. A scatterer sums the signals arriving via the ingoing edges. Assuming these mechanisms to be linear and time-invariant, this effect can be represented as a convolution with an impulse response or, in the Fourier domain, as a multiplication with a transfer function. Let us for a moment consider the edge \( e = (v_n, v_{n'}) \) in \( E \). A filtered version of the signal \( C_n(f) \) emitted by vertex \( v_n \) is observed at vertex \( v_{n'} \). The signal observed at vertex \( v_{n'} \) via edge \( e \) reads \( A_e(f)C_n(f) \), where \( A_e(f) \) is the transfer function of edge \( e \). In other words, the transfer function \( A_e(f) \) describes the interaction the initial vertex \( v_n \) and the propagation from \( v_n \) to \( v_{n'} \).

C. Weighted Adjacency Matrix of a Propagation Graph

Propagation along the edges is described via a transfer matrix \( A(f) \) which can be viewed as an adjacency ma-

Fig. 2. A propagation graph \( G = (V, E) \) with four transmit vertices \( V_t = \{Tx_1, Tx_2, Tx_3, Tx_4\} \), three receive vertices \( V_r = \{Rx_1, Rx_2, Rx_3\} \), and six scatterer vertices \( V_s = \{S_1, S_2, S_3, S_4, S_5, S_6\} \). The edge set \( E \) is the union of the sets \( E_d = \{(Tx_1, Rx_1); (Tx_2, Rx_4); (Tx_3, Rx_1)\} \), \( E_i = \{(Tx_2, S_3); (Tx_3, S_3); (Tx_4, S_6); (Tx_4, S_1)\} \), \( E_s = \{(S_1, Rx_3); (S_3, Rx_3); (S_6, Rx_2)\} \), and \( E_r = \{(S_1, S_2); (S_2, S_1); (S_3, S_1); (S_2, S_4); (S_4, S_3); (S_4, S_5)\} \).
columns vertices have no outgoing edges.

where $Y_m(f)$ is the Fourier transform of the signal observed by receiver $R_{x+m}$. Similar, to $X(f)$ and $Y(f)$ we let $Z(f)$ denote the output signal vector of the scatterers:

$$Z(f) = [Z_1(f), \ldots, Z_N(f)]^T,$$

where the $n$th entry denotes the Fourier transform of the signal observed at scatterer vertex $S_n$. By the definition of the propagation graph, there are no other signal sources than the vertices in $V_i$. Assuming linear and time-invariant propagation mechanisms, the input-output relation in the Fourier domain reads

$$Y(f) = H(f)X(f),$$

where $H(f)$ is a $M_t \times M_t$ transfer matrix.

The structure of the propagation graph unfolds in the vector signal flow graph depicted in Fig. 3. The vertices of the vector signal flow graph represent the three sets $V_t$, $V_r$, and $V_s$ with the associated signals $X(f)$, $Y(f)$, and $Z(f)$. The edge transfer matrices of the vector signal flow graph are the submatrices of $A(f)$ defined in (4)-(7).

III. TRANSFER MATRIX OF A PROPAGATION GRAPH

In the following we derive the input-output relation of a propagation graph. In Subsection III-A we first discuss how the response of a graph is composed of signal contributions propagating via different propagation paths. This representation is, albeit intuitive, impractical for computation of the transfer function for graphs with cycles. Thus in Subsections III-B and III-C we give the transfer function and partial transfer matrices of a general propagation graph in closed form. Subsection III-D treats the graphical interpretation of reciprocal channels. The section concludes with a discussion of related results in the literature.

A. Propagation Paths and Walks

The concept of a propagation path is a corner stone in modeling multipath propagation. In the literature, this concept is most often defined in terms of the resulting signal components arriving at the receiver. A shortcoming of this definition is that it is often hard to relate to the propagation environment. The graph terminology offers a convenient alternative. We define a propagation path as a walk $\ell = (v_0, v_1, \ldots, v_K+1)$ in $\mathcal{G}$ such that the initial vertex $v_0$ is a transmitter and the terminal vertex $v_{K+1}$ is a receiver, i.e., $v_0 \in V_t$ and $v_{K+1} \in V_r$. A signal that propagates along propagation path $\ell$ traverses $K+1$ (not necessarily different) edges and undergoes $K$ interactions. We refer to such a propagation path as a $K$-bounce path. A zero-bounce propagation path $\ell = (v, v')$ is called a line-of-sight path, or direct path, from transmitter $v$ to receiver $v'$. As an example, referring to the graph depicted in Fig. 2 it is straightforward to verify that $\ell_1 = (Tx1, Rx1)$ is a direct path, $\ell_2 = (Tx4, S6, Rx2)$ is a single-bounce path, and $\ell_3 = (Tx4, S1, S2, Rx3)$ is a 3-bounce path.

We denote by $L_{v'v}$ the set of propagation paths in $\mathcal{G}$ from transmitter $v$ to receiver $v'$. The signal received at $v'$ originating from transmitter $v$ is the superposition of signal
components each propagating via a propagation path in \( L_{ov} \). Correspondingly, entry \((v, v')\) of \( H(f) \) reads

\[
H_{ov'}(f) = \sum_{\ell \in L_{ov'}} H_{\ell}(f),
\]

(12)

where \( H_{\ell}(f) \) is the transfer function of propagation path \( \ell \).

The number of terms in (12) equals the cardinality of \( L_{ov} \), which may, depending on the structure of the graph, be finite or infinite. As an example, the number of propagation paths is infinite if \( v \) and \( v' \) are connected via a directed cycle in \( G \). The graph in Fig. 2 contains two directed cycles which are connected to both transmitters and receivers.

In the case of an infinite number of propagation paths, computing \( H_{ov'}(f) \) directly from (12) is infeasible. This problem is commonly circumvented by truncating the sum in (12) to approximate \( H_{ov'}(f) \) as a finite sum. This approach, however, calls for a method for determining how many terms of the sum should be included in order to achieve reasonable approximation.

In the frequently used “\( K \)-bounce channel models”, propagation paths with more than \( K \) interactions are ignored. This approach is motivated by the rationale that at each interaction, the signal is attenuated, and thus terms in (12) resulting from propagation paths with a large number of bounces are weak and can be left out as they do not affect the sum much. This reasoning, however, holds true only if the sum of the components with more than \( K \) interactions is insignificant, which may or may not be the case. From this consideration, it is clear that the truncation criterion is non-trivial as it essentially necessitates computation of the whole sum before deciding whether a term can be ignored or not.

### B. Transfer Matrix for Recursive and Non-Recursive Propagation Graphs

As an alternative to the approximation methods applied to the sum (12) we now give an exact closed-form expression for the transfer function \( H(f) \). Provided that the spectral radius of \( B(f) \) is less than unity, the expression holds true for any number of terms in the sum (12) and thus holds regardless whether the number of propagation paths is finite or infinite.

**Theorem 1:** If the spectral radius of \( B(f) \) is less than unity, then the transfer matrix of a propagation graph reads

\[
H(f) = D(f) + R(f) [I - B(f)]^{-1} T(f),
\]

(13)

According to Theorem 1 the transfer matrix \( H(f) \) consists of the two following terms: \( D(f) \) representing direct propagation between the transmitters and receivers and \( R(f) [I - B(f)]^{-1} T(f) \) describing indirect propagation. The condition that the spectral radius of \( B(f) \) be less than unity implies that for any vector norm \( \| \cdot \| \), \( \| Z(f) \| > \| B(f) Z(f) \| \) for non-zero \( \| Z(f) \| \), cf. [13]. For the Euclidean norm in particular this condition implies the sensible physical requirement that the signal power decreases for each interaction.

**Proof:** Let \( H_k(f) \) denote the transfer matrix for all \( k \)-bounce propagation paths, then \( H(f) \) can be decomposed as

\[
H(f) = \sum_{k=0}^{\infty} H_k(f),
\]

(14)

where

\[
H_k(f) = \begin{cases} D(f), & k = 0 \\ R(f) B^{k-1}(f) T(f), & k > 0 \end{cases}
\]

(15)

Insertion of (15) into (14) yields

\[
H(f) = D(f) + R(f) \left( \sum_{k=1}^{\infty} B^{k-1}(f) \right) T(f).
\]

(16)

The infinite sum in (16) is a Neumann series converging to \( [I - B(f)]^{-1} \) if the spectral radius of \( B(f) \) is less than unity. Inserting this in (16) completes the proof.

The decomposition introduced in (14) makes the effect of the recursive scattering directly visible. The received signal vector is a sum of infinitely many components resulting from any number of interactions. The structure of the propagation mechanism is further exposed by (16) where the emitted vector signal is re-scattered successively in the propagation environment leading to the observed Neumann series. This allows for modeling of channels with infinite impulse responses by expression (13). It is possible to arrive at (13) in an alternative, but less explicit, manner:

**Proof:** It is readily observed from the vector signal flow graph in Fig. 3 that \( Z(f) \) can be expressed as

\[
Z(f) = T(f) X(f) + B(f) Z(f).
\]

(17)

Since the spectral radius of \( B(f) \) is less than unity we obtain for \( Z(f) \) the solution

\[
Z(f) = [I - B(f)]^{-1} T(f) X(f).
\]

(18)

Furthermore, according to Fig. 3 the received signal is of the form

\[
Y(f) = D(f) X(f) + R(f) Z(f).
\]

(19)

Insertion of (18) in this expression yields (13).

We remark that the above two proofs allow for propagation paths with any number of bounces. This is highly preferable, as the derived expression (13) is not impaired by approximation errors due to the truncation of the series into a finite number of terms as it occurs when using \( K \)-bounce models.

A significant virtue of the expression (13) is that propagation effects related to the transmitters and receivers are accounted for in the matrices \( D(f), T(f) \) and \( R(f) \), but do not affect \( B(f) \). Consequently, the matrix \([I - B(f)]^{-1}\) only needs to be computed once even though the configuration of transmitters and receivers changes. This is especially advantageous for simulation studies of e.g. spatial correlation as this leads to a significant reduction in computational complexity.

### C. Partial Transfer Matrices

The closed form expression (13) for the transfer matrix of a propagation graph accounts for propagation via an arbitrary number of scatterer interactions. For some applications it is, however, relevant to study only some part of the response according to a particular number of interactions. One case is where a propagation graph is used to generate only a part of the response and other techniques are used for the remaining
parts. Another case is when one must assess the approximation error when the infinite series is truncated. In the following we derive a few useful expressions for such partial transfer matrices.

We define the $K : L$ partial transfer matrix as

$$ H_{K:L}(f) = \sum_{k=K}^{L} H_k(f), \quad 0 \leq K \leq L, \quad (20) $$

i.e., we include only contributions from propagation paths with at least $K$, but no more than $L$ bounces. It is straightforward to evaluate (20) for $K = 0$, and $L = 0, 1, 2$:

$$ H_{0:0}(f) = D(f) \quad (21) $$

$$ H_{0:1}(f) = D(f) + R(f)T(f) \quad (22) $$

$$ H_{0:2}(f) = D(f) + R(f)T(f) + R(f)B(f)T(f). \quad (23) $$

This expansion of the truncated series is quite intuitive but the obtained expressions are increasingly complex for large $L$. Theorem 2 gives a closed form expression of the partial transfer function $H_{K:L}(f)$ for arbitrary $K$ and $L$:

**Theorem 2:** The partial response $H_{K:L}(f)$ is given by

$$ H_{K:L}(f) = \begin{cases} D(f) + R(f)[I - B^L(f)][I - B(f)]^{-1}T(f), & K = 0, L \geq 0 \\ R(f)[B^{K-1}(f) - B^L(f)][I - B(f)]^{-1}T(f), & 0 < K \leq L, \end{cases} $$

provided that the spectral radius of $B(f)$ is less than unity.

**Proof:** The partial transfer function for $0 \leq K \leq L$ reads

$$ H_{K:L}(f) = \sum_{k=K}^{L} H_k(f) - \sum_{k'=L+1}^{\infty} H_{k'}(f) $$

$$ = H_{K:\infty}(f) - H_{L+1:\infty}(f). \quad (24) $$

For $K = 0$ we have $H_{0:0}(f) = H(f)$ by definition; for $K \geq 1$ we have

$$ H_{K:\infty}(f) = R(f) \sum_{k=K-1}^{\infty} B^k(f)T(f) $$

$$ = R(f)B^{K-1}(f) \sum_{k=0}^{\infty} B^k(f)T(f) $$

$$ = R(f)B^{K-1}(f)[I - B(f)]^{-1}T(f). \quad (25) $$

Inserting (25) into (24) completes the proof.

Theorem 2 enables closed-form computation of $H_{K:L}(f)$ for any $K \geq L$. We have already listed a few partial transfer matrices in (21), (22), and (23). By definition the partial response $H_{0:K}(f)$ equals $H_K(f)$ for which an expression is provided in (13). The transfer function of the $K$-bounce approximation is equal to $H_{0:0}(f)$. Another special case worth mentioning is $H_{K+1:0}(f) = H(f) - H_{0:0}(f)$ available from (25), which gives the error due to the $K$-bounce approximation. Thus the validity of the $K$-bounce approximation can be assessed by evaluating some appropriate norm of $H_{K+1:0}(f)$.

### D. Reciprocity and Propagation Graphs

In most cases, the radio channel is considered reciprocal. As we shall see shortly, the graph terminology accommodates an interesting interpretation of the concept of reciprocity. For any propagation graph we can define the reverse graph in which the roles of transmitter and receiver vertices are swapped. The principle of reciprocity states that the transfer matrix of the reverse channel is equal to the transposed transfer matrix of the forward channel, i.e., a forward channel with transfer matrix $H(f)$ has a reverse channel with transfer matrix $\tilde{H}(f) = H^T(f)$. In the sequel we mark all entities related to the reverse channel with a tilde.

We seek the relation between the forward graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and its reverse $\tilde{\mathcal{G}} = (\mathcal{V}, \mathcal{E})$ under the assumption of reciprocity. More specifically, we are interested in the relation between the weighted adjacency matrix $A(f)$ of $\mathcal{G}$ and the weighted adjacency matrix $\tilde{A}(f)$ of $\tilde{\mathcal{G}}$. We shall prove the following theorem:

**Theorem 3:** A propagation graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with weighted adjacency matrix $A(f)$ has a reverse graph $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ with edge set $\tilde{\mathcal{E}} = \{(v, v') : (v', v) \in \mathcal{E}\}$ and weighted adjacency matrix $\tilde{A}(f) = A^T(f)$.

In words Theorem 3 identifies the graph of the reverse channel as the graph obtained by reversing the direction of all edges of $\mathcal{G}$ while maintaining the edge transfer functions.

**Proof:** We start by noting that the set of transmitters, receivers, and scatterers is maintained for the reverse channel, thus the vertex set of $\tilde{\mathcal{G}}$ is $\mathcal{V}$. Interchanging the roles of transmitters and receivers means that we admit no edges of the reverse graph $\tilde{\mathcal{G}}$ going into vertices in $\mathcal{V}_t$ and no outgoing edges from vertices in $\mathcal{V}_r$. Consequently, assuming the vertex indexing as in (2), the weighted adjacency matrix of $\tilde{\mathcal{G}}$ is of the form

$$ \tilde{A}(f) = \begin{bmatrix} 0 & \tilde{D}(f) & \tilde{T}(f) \\ 0 & 0 & 0 \\ 0 & \tilde{R}(f) & \tilde{B}(f) \end{bmatrix} \quad (26) $$

where the transfer matrices $\tilde{D}(f), \tilde{R}(f), \tilde{T}(f),$ and $\tilde{B}(f)$ are defined according to Fig. 3. The input-output relation of the reverse channel reads $\tilde{X}(f) = \tilde{H}(f)\tilde{Y}(f)$ where $\tilde{Y}(f)$ is the signal emitted by the vertices in $\mathcal{V}_t$ and $\tilde{X}(f)$ is the signal received by the vertices in $\mathcal{V}_r$. By inspection of Fig. 3 and by arguments similar to those presented in Section III-B we achieve for the reverse channel

$$ \tilde{H}(f) = \tilde{D}(f) + \tilde{R}(f)[I - \tilde{B}(f)]^{-1}\tilde{T}(f). \quad (27) $$

The reciprocity condition $\tilde{H}(f) = H^T(f)$ yields the alternative expression:

$$ \tilde{H}(f) = D^T(f) + T^T(f)[I - B^T(f)]^{-1}R^T(f). \quad (28) $$

Comparing (27) and (28) it is clear that $\tilde{D}(f) = D^T(f), \tilde{B}(f) = B^T(f), \tilde{T}(f) = R^T(f),$ and $\tilde{R}(f) = T^T(f)$. After inserting these four identities into (26) we obtain $\tilde{A}(f) = A^T(f)$. The relation between $\mathcal{E}$ and $\tilde{\mathcal{E}}$ now follows from the definition of the weighted adjacency matrix.
E. Related Recursive Scattering Models

We provide a few examples of recursive models to assist the reader in recognizing models which can be represented by the graphical structure.

In [18] Shi and Nehorai consider a model for recursive scattering between point scatterers in a homogeneous background. The propagation between any point in space is described by a scalar Green’s function. The transfer function obtained by applying the Foldy-Lax equation can also be obtained from a propagation graph by defining the sub-matrices of $A(f)$ as follows. The model does not include a directed term and thus $D(f) = 0$. The entry of $[T(f)]_{nm}$ is the Green’s function from transmit vertex $m_1$ to scatterer $n$ times the scattering coefficient of scatterer $n'$. Similarly, the entry $[R(f)]_{nm_2}$ is the Green’s function from the position of scatterer $n$ to receiver $m_2$. The entry $[B(f)]_{nn'}$, $n \neq n'$ is the Green’s function from the position of scatterer $n$ to the position of scatterer $n'$ times the scattering coefficient of scatterer $n$. Since a point scatterer does not scatter back on itself, the diagonal entries of $B(f)$ are all zero. As can be observed from the above definitions, the assumption of homogeneous background medium leads to the special case with $E_d = \emptyset$, $E_i = V_i \times V_s$, $E_r = V_s \times V_i$, and $E_r = V_s^2$.

Another modeling method that can be conveniently described using propagation graphs is (time-dependent) radiosity [24]. The time-dependent radiosity algorithms published in [20–24] are formulated in the delay domain. It appears, however, that no closed-form solution feasible for numerical evaluation is available in the literature. Thus [20–24] resort to iterative solutions which can be achieved after discretizing the inter-patch propagation delays. The time-dependent radiosity problem can be expressed in the Fourier domain in terms of a propagation graph where each patch is represented by a scatterer, and the entries of $A(f)$ are defined according to the Fourier transform of the delay-dependent form factor. Using this formulation, a closed form expression of the channel transfer function appears immediately by Theorem 11 with no need for quantization of propagation delays.

F. Revisiting Existing Stochastic Radio Channel Models

It is interesting to revisit existing radio channel models by means of the just defined framework of propagation graphs.

Such an effort may reveal some structural differences between models, which are not apparent merely from the mathematical formulation. It is, however, a fact that the interpretation of a transfer function as a propagation graph is not unique—different propagation graphs may yield the same transfer function. Therefore different equivalent graphical interpretations may be given for a particular model.

We first consider the structure of the seminal model [32] by Turin et al. This model can be expressed in the frequency domain as

$$H(f) = \sum_{\ell=0}^{\infty} \alpha_{\ell} \exp(-j2\pi f \tau_{\ell}), \quad (29)$$

where $\alpha_{\ell}$ is the complex gain and $\tau_{\ell}$ denotes the delay of the $\ell$th component. We assume that $\tau_0 = 0$. Thus $\{\tau_{\ell}, \alpha_{\ell}: \ell = 1, 2, \ldots \}$ is a marked Poisson point process of delays on $[0, \infty)$ with complex marks $\{\alpha_{\ell}: \ell = 1, 2, \ldots \}$. The reader is referred to [32] for further details. We represent this model as the graph depicted in Fig. 5(a). We construct the graph by identifying each term in (29) as corresponding to a specific propagation path from the transmitter to the receiver. It appears that the components $\alpha_{\ell} \exp(-j2\pi f \tau_{\ell})$, $\ell = 1, 2, \ldots$ are statistically independent. Therefore, we assign to each component a separate path which results in a graph with infinitely many single-bounce paths. This definition allows for direct interpretation of both the forward and reverse graphs. Since no scatterer-to-scatterer edges exist, $E_s = \emptyset$ and $B(f) = 0$. By blocking the propagation from the Tx to scatterer $\ell$, the edge $(\text{Tx}, \ell)$ is removed from the graph while the remaining paths are unaffected. The same happens if edge $(\ell, \text{Rx})$ is removed.

The celebrated model by Saleh and Valenzuela [33] is structured as a second-order Turin model:

$$H(f) = \sum_{\ell=0}^{\infty} \alpha_{\ell} \exp(-j2\pi f \tau_{\ell}) \sum_{\ell'=0}^{\infty} \alpha_{\ell\ell'} \exp(-j2\pi f \tau_{\ell\ell'}). \quad \quad (30)$$

Assuming for simplicity that $\tau_0 = 0$ and $\tau_{0\ell'} = 0$, $\ell' = 0, 1, \ldots$, the processes $\{\alpha_{\ell\tau_{\ell}}: \ell = 1, 2, \ldots \}$ and $\{\alpha_{\ell\ell'}: \ell = 1, 2, \ldots \}$ are independent marked Poisson processes on $[0, \infty)$ with complex marks. Details on these stochastic processes can be found in [33]. Again, considering that statistically independent terms in (30) stem from distinct propagation paths, we can construct the graph depicted in Fig. 5(b). The structure of the graph appears to be asymmetric in the sense that the transmitter is connected to a set of super-ordinate or “cluster” scatterers whereas the receiver is connected to the set of sub-ordinate scatterers. As a result, removing one of the outgoing edges from the transmitter makes a whole cluster disappear, but by removing one of the receiver’s ingoing edges only a single component vanishes in the double sum (30). This leads to an interesting effect in the reverse graph: After reversion of all edges the asymmetry is changed as the transmitting vertex is connected to the subordinate scatterers while the receiving vertex is connected to the cluster scatterers. This problem can be circumvented by making the graph symmetric as shown in Fig. 5(c), which in turn necessitates additional scatterers and edges.
IV. Example: Stochastic Model for In-Room Channel

The concept of propagation graph introduced until now can be used for describing a broad range of channel models. In this section we apply these general results to a specific example scenario where scatterer interactions are considered to be non-dispersive in delay. We specify a method feasible for generating such a graph in Monte Carlo simulations. The model discussed in this example is a variant of the model proposed in \[1\], \[2\].

A. Weighted Adjacency Matrix

We define the weighted adjacency matrix according to a geometric model of the environment. We consider a scenario with a single transmitter, a single receiver, and \(N\) scatterers, i.e., the vertex set reads \(V = V_t \cup V_s \cup V_n\) with \(V_t = \{Tx\}, V_s = \{Rx\}\), and \(V_n = \{S1, \ldots, SN\}\). To each vertex \(v \in V\) we assign a displacement vector \(r_v \in \mathbb{R}^3\) with respect to a coordinate system with arbitrary origin. To edge \(e = (v, v')\) we associate the Euclidean distance \(d_e = |r_v - r_{v'}|\), the gain \(g_e\), the phase \(\phi_e\), and the propagation delay \(\tau_e = d_e/c\) where \(c\) is the speed of light. The edge transfer functions are defined as

\[
A_e(f) = \begin{cases} g_e(f) \exp(j\phi_e - j2\pi \tau_e f); & e \in \mathcal{E} \\ 0; & e \notin \mathcal{E}. \end{cases}
\]

The edge gains \(\{g_e(f)\}\) are defined according to

\[
g_e^2(f) = \begin{cases} \frac{4\pi f \mu(\tau_e)}{\hbar} \frac{1}{S(\tau_e)} ; & e \in \mathcal{E}_d \\ \frac{4\pi f \mu(\tau_e)}{\hbar} \frac{1}{S(\tau_e)} ; & e \in \mathcal{E}_t \\ \frac{1}{\text{o}d_i(e)} ; & e \in \mathcal{E}_s \end{cases}
\]

where \(\text{o}d_i(e)\) denotes the number of edges from \(\text{init}(e)\) to other scatterers and for any \(\mathcal{E}' \subseteq \mathcal{E}\)

\[
\mu(\mathcal{E}') = \frac{1}{|\mathcal{E}'|} \sum_{e \in \mathcal{E}'} \tau_e \quad \text{and} \quad S(\mathcal{E}') = \sum_{e \in \mathcal{E}'} \tau_e^{-2},
\]

with \(|\cdot|\) denoting cardinality. The weight of the direct edge is selected according to the Friis equation \[34\] assuming isotropic antennas at both ends. The weights of edges in \(\mathcal{E}_t\) and \(\mathcal{E}_s\) also account for the antenna characteristics. They are computed at the average distance to avoid signal amplification when scatterers are close to a transmitter or receiver, namely when the far-field assumption is invalid.

B. Stochastic Generation of Propagation Graphs

We now define a stochastic model of the sets \(\{r_v\}, \mathcal{E},\) and \(\{\phi_v\}\) as well as a procedure to compute the corresponding transfer function and impulse response. The vertex positions are assumed to reside in a region \(R \subset \mathbb{R}^3\) corresponding to the region of interest. The transmitter and receiver positions are assumed to be fixed, while the positions of the \(N\) scatterers \(\{r_v : v \in V_n\}\) is a Bernoulli point process on \(\mathcal{R}\), i.e., the number \(N\) of scatterers is assumed constant, and the scatterer positions are drawn independently from a uniform distribution on \(\mathcal{R}\).

Edges are drawn independently such that a vertex pair \(e \in V^2\) is in the edge set \(\mathcal{E}\) with probability \(P_e = \text{Pr}[e \in \mathcal{E}]\) defined as

\[
P_e = \begin{cases} P_{\text{dir}}, & e = (\text{Tx}, \text{Rx}) \\ 0, & \text{term}(e) = \text{Tx} \\ 0, & \text{init}(e) = \text{Rx} \quad . \end{cases}
\]

The first case of \(\{34\}\) controls the occurrence of a direct component. If \(P_{\text{dir}}\) is zero, the direct term \(D(f)\) is zero with probability one. If \(P_{\text{dir}}\) is unity, the direct term \(D(f)\) is non-zero with probability one. The second and third cases of \(\{34\}\) exclude ingoing edges to the transmitter and outgoing edges from the receiver. Thus the generated graphs will have the structure defined in Section \[11\]. The fourth case of \(\{34\}\) excludes the occurrence of loops in the graphs. This is sensible as a specular scatterer cannot scatter a signal back to itself. A consequence of this choice is that any realization of the graph is loopless and therefore \(A(f)\) has zeros along its main diagonal. The last case of \(\{34\}\) assigns a constant probability \(P_{\text{vis}}\) of the occurrence of edges from \(V_t\) to \(V_n\), from \(V_s\) to \(V_n\) and from \(V_n\) to \(V_s\).

Finally, the phases \(\{\phi_v : e \in \mathcal{E}\}\) are drawn independently from a uniform distribution on the interval \([0; 2\pi)\).
Given the parameters $\mathcal{R}$, $r_{\text{TX}}$, $r_{\text{RX}}$, $N$, $P_{\text{dir}}$, $P_{\text{vis}}$ and $g$, realizations of the (partial) transfer function $H_{K,L}(f)$ and the corresponding (partial) impulse response $h_{K,L}(f)$ can now be generated for a preselected frequency range $[f_{\text{min}},f_{\text{max}}]$, using the algorithm stated in Fig. 6.

**C. Numerical Experiments**

The effect of the recursive scattering phenomenon can now be illustrated by numerical experiments. The parameter settings given in Table I are selected to mimic the experimental setup of [28] used to acquire the measurements reported in Fig. 1. The room size and positions of the transmitter and receiver are chosen as in [28]. We consider the case where direct propagation occurs and set $P_{\text{dir}}$ to unity. The probability of visibility $P_{\text{vis}}$ and the number of scatterers $N$ are chosen to mimic the observed avalanche effect. The value of $g$ is set to match the tail slope $\rho \approx -0.4 \text{ dB/ns}$ of the delay power spectrum depicted in Fig. 1. The value of $g$ can be related to the slope of the log delay power spectrum via the approximation $\rho \approx 20 \log_{10}(g)/\mu(\mathcal{E})$. This approximation arises by considering the power balance for a scatterer assuming the signal components arriving at a scatterer to be statistically independent, neglecting the probability of scatterers with outdegree zero, and approximating edge delays of edges in $\mathcal{E}_s$ by the average $\mu(\mathcal{E}_s)$ defined in [33].

Fig. 7 shows the amplitude of a single realization of the transfer function. Overall, the squared amplitude of the transfer function decays as $f^{-2}$ due to the definition of $\{g_e(f)\}$. Furthermore, the transfer function exhibits fast fading over the considered frequency band. The lower panel of Fig. 7 reports the corresponding impulse response for two different signal bandwidths. Both impulse responses exhibit an avalanche effect as well as a diffuse tail of which the power decays exponentially with $\rho \approx -0.4 \text{ dB/ns}$. As anticipated, the transition to the diffuse tail is most visible in the response obtained with the larger bandwidth.

The build up of the impulse response can be examined via the partial impulse responses given in Fig. 8. Inspection of the partial responses when $K=1$ reveals that the early part of the tail is due to signal components with a low $K$ while the late part is dominated by higher-order signal components. It can also be noticed that as $K$ increases, the delay at which the maximum of the $K$-bounce partial response occurs and the spread of this response are increasing.

Fig. 9 shows two types of delay-power spectra. The upper panel shows the ensemble average of squared amplitudes of 1000 independently drawn propagation graphs for the two signal bandwidths also considered in Fig. 7. Both spectra exhibit the same trend: A clearly visible peak due to the direct signal is followed by a tail with exponential power decay. As expected, the first peak is wider for the case with 1 GHz bandwidth than for the case with 10 GHz bandwidth. The tails differ by approximately 7 dB. This shift arises due to the $f^{-2}$ trend of the transfer function resulting in a higher received power for the lower frequencies considered in the 1 GHz bandwidth case.

The bottom panel shows spatially averaged delay-power spectra obtained for one particular realization of the propagation graph. The simulated spatial averaged delay-power spectra exhibit the avalanche effect similar to the one observed in Fig. 1. Indeed, for the 10 GHz bandwidth case the power level

![Fig. 6. Algorithm for generating full or partial transfer functions and impulse responses for a preselected bandwidth.](image)

![Fig. 7. Channel response for a specific realization of the propagation graph. Top: Transfer function in dB $|20 \log_{10}(H(f))|$ in the frequency range $[1,11]$ GHz. Bottom: Impulse responses in dB $|20 \log_{10}(h(\tau))|$ computed for two frequency ranges.](image)

![Fig. 8.](image)
Fig. 8. Partial responses obtained for one graph realization for the bandwidth [2, 3] GHz. The $K : L$ settings are indicated in each miniature. The full response is indicated in gray for comparison. Top row: responses of $K$-bounce approximations. Right-most column: error terms resulting from $(K - 1)$-bounce approximations. Main diagonal ($K = L$): $K$-bounce contributions.

Fig. 9. Simulated delay power spectra. Top panel: Ensemble average over 1000 Monte Carlo runs. Bottom panel: Spatial average of a single graph realization assuming the same grid as the one used in Fig. 1, i.e., 900 receiver positions on a 30×30 horizontal square grid with 1×1 cm$^2$ mesh centered at position $r_{Rx}$ given in Table I.

of diffuse tails of the delay power spectra agrees remarkably well with measurement in Fig. 1. The modest deviation of about 3 dB can be attributed to antenna losses in the measurement.

V. CONCLUSIONS

The outset for this work was the observation that in-room channels available in the literature are observed to exhibit an avalanche effect where separate signal components appear at increasing rate and gradually merge into a diffuse tail with an exponential decay of power. We hypothesized that this avalanche effect is due to recursive scattering. We propose a model which includes recursive scattering by modeling the propagation environment as a graph where vertices represent transmitters, receivers, and scatterers and edges represent propagation conditions between vertices. This general structure allows for the propagation graph’s full and partial transfer matrix to be derived in closed form. This expression can, by specifying the edge transfer functions, be directly used to perform numerical simulations.

We consider as an example a graph-based stochastic model where all interactions are non-dispersive in delay in a scenario similar to an experimentally investigated scenario where the
avalanche effect has been observed. The responses generated from the model also exhibit an avalanche effect. Thus we conclude: 1) the diffuse tail can be generated even when scatterer interactions are non-dispersive in delay and therefore the diffuse tail can be attributed to recursive scattering, and 2) the exponential decay of the delay-power spectra is caused by recursive scattering. As illustrated by the simulation results the proposed model, in contrast to existing models which treats dominant and diffuse components separately, provides a unified account for the avalanche effect.

APPENDIX

The transfer function $H(f)$ and impulse response $h(\tau)$ can be estimated as follows:

1) Compute $\Gamma$ samples of the transfer matrix within the bandwidth $[f_{\min}, f_{\max}]$

$$H[\gamma] = H(f_{\min} + \gamma \Delta f), \quad \gamma = 0, 1, \ldots, \Gamma - 1,$$

where $\Delta f = (f_{\max} - f_{\min})/(\Gamma - 1)$ and $H(\cdot)$ is obtained by Theorem.[1]

2) Estimate the received signal $y(\tau)$ via the inverse discrete Fourier transform:

$$y(i\Delta \tau) = \Delta f \sum_{\gamma=0}^{\Gamma-1} H[\gamma]X[\gamma]\exp(j2\pi i \gamma / \Gamma),$$

where $X[\gamma] = X(f_{\min} + \gamma \Delta f), \gamma = 0, 1, \ldots, \Gamma - 1$ and $\Delta \tau = 1/(f_{\max} - f_{\min})$.

The impulse response can be estimated by letting $X[f]$ be a window function of unit power

$$\int_{f_{\min}}^{f_{\max}} |X(f)|^2 df \approx \sum_{\gamma=0}^{\Gamma} |X[\gamma]|^2 \Delta f = 1.$$  \hspace{1cm}(36)

where $X[\gamma] = X(f_{\min} + \gamma \Delta f)$. The window function must be chosen such that its inverse Fourier transform exhibits a narrow main-lobe and sufficiently low side-lobes; $y(\tau)$ is then regarded as a good approximation of the impulse response of the channel and by abuse of notation denoted by $h(\tau)$.

Samples of the partial transfer matrix are obtained by replacing $H(\cdot)$ by $H_{\Gamma \times \Gamma}(\cdot)$ in $(35)$. The corresponding received partial impulse response is denoted by $h_{\Gamma \times \Gamma}(\tau)$.

REFERENCES


